2012

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Tian-Xiao He, *Illinois Wesleyan University*
P. J.-S. Shiue

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Tian-Xiao He[1] and Peter J.-S. Shiue [2]
[1]Department of Mathematics and Computer Science
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
[2]Department of Mathematical Sciences, University of Nevada, Las Vegas
Las Vegas, NV 89154-4020, USA

Abstract

Here we present an application of Horner's method in evaluating the sequence of Stirling numbers of the second kind. Based on the method, we also give an efficient way to calculate the difference sequence and divided difference sequence of a polynomial, which can be applied in the Newton interpolation. Finally, we survey all of the results in Proposition 1.4.

AMS Subject Classification: 05A15, 65B10, 33C45, 39A70, 41A80.

Key Words and Phrases: Horner's method, Stirling numbers of the second kind, divided difference, Newton interpolation.

1 Introduction

The number of ways of partition a set of n elements into k nonempty subsets is called the Stirling number of the second kind, denoted by $S(n,k)$. In other words, $S(n,k)$ is the number of equivalence relations with k classes on a finite set with n elements. From [3], $S(n,k)$ equals

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n = \frac{1}{k!}\Delta^k t^n\Big|_{t=0}. \quad (1)$$

As a division algorithm, Horner's method is a nesting technique requiring only n multiplications and n additions to evaluate an arbitrary nth-degree polynomial, which can be surveyed by Horner's theorem (see, for example, [1]).

Theorem 1.1 *Let*

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0.$$

If $b_d = a_d$ *and*

$$b_k = a_k + b_{k+1}x_0, \quad k = n-1, n-2, \ldots, 1, 0,$$

then $b_0 = P(x_0)$*, and* $P(x)$ *can be written as*

$$P(x) = (x - x_0)Q(x) + b_0,$$

where

$$Q(x) = b_d x^{d-1} + b_{d-1}x^{d-2} + \cdots + b_2 x + b_1.$$

The theorem can be proved using a direct calculation. An additional advantage of Horner's method is the differentiation of $P(x)$:

$$P'(x) = Q(x) + (x - x_0)Q'(x).$$

Hence, $P'(x_0) = Q(x_0)$, which is very convenient when applying Newton's method to find roots of a polynomial.

Example 1 As an example, we use Horner's method to evaluate $P(x) = x^4 - 2x^2 + 3x - 4$ at $x_0 = -1$. First we construct the synthetic division as follows.

$x_0=-1$	$a_4=1$	$a_3=0$	$a_2=-2$	$a_1=3$	$a_0=-4$
		$b_4x_0=-1$	$b_3x_0=1$	$b_2x_0=1$	$b_1x_0=-4$
	$b_4=1$	$b_3=-1$	$b_2=-1$	$b_1=4$	$b_0=-8$

Hence,

$$x^4-2x^2+3x-4=(x+1)(x^3-x^2-x+4)-8.$$

In [6], Pathan and Collyer present an excellent survey on Horner's method and its application in solving polynomial equations by determining the location of roots. In this note, we shall give other applications of Horner's method in the calculation of Stirling numbers of the second kind, the difference sequences, and the divided difference sequences (or equivalently, the coefficients of Newton interpolation) of polynomials. There are numerous ways to evaluate a Stirling number sequence or Stirling matrix. For example, in [4], El-Mikkawy gives an algorithm based on Newton's divided difference interpolating polynomials. In [2], Cheon and Kim present a method based on the relationship between the Stirling matrix and other combinatorial sequences such as the Vandermonde matrix, the Bernoulli numbers, and Eulerian numbers. However, our algorithm of calculating Stirling number sequences based on Horner's method is different and efficient, which contains an idea suitable for constructing algorithms in calculation of many sequences. This general idea will be presented in Proposition 1.4.

From Proposition 1.4.2 of [7], if the polynomial $f(n)$ of degree $\leq d$ is expanded in terms of the basis $\binom{n}{k}$, $0\leq k\leq d$, then the coefficients are $\Delta^k f(0)$, namely,

$$f(n)=\sum_{k=0}^{d}\Delta^k f(0)\binom{n}{k}=\sum_{k=0}^{d}\frac{\Delta^k f(0)}{k!}(n)_k, \tag{2}$$

where $(n)_k=n(n-1)\cdots(n-k+1)$ are the falling factorial polynomials. In particular, for $f(n)=n^d$, we have $\Delta^0 f(0)=f(0)=0$ and

$$n^d=\sum_{k=1}^{d}\Delta^k 0^d\binom{n}{k}=\sum_{k=1}^{d}S(d,k)(n)_k, \tag{3}$$

where the rightmost equation comes from (1). Therefore, we may give the following algorithm to find out the kth order difference of f at 0 and Stirling numbers of the second order from (2) and (3) respectively.

Algorithm 1.2 *Write (2) as*

$$
\begin{aligned}
f(n) &= (n-0)\left(\Delta^0 f(0) + (n-1)\left(\frac{\Delta^1 f(0)}{1!} + (n-2)\left(\frac{\Delta^2 f(0)}{2!} + \cdots \right.\right.\right. \\
&\qquad \left.\left.\left. +(n-d+1)\frac{\Delta^d f(0)}{d!}\right)\right)\cdots\right).
\end{aligned}
$$

Use synthetic division to obtain $f(n)/(n-0)$, a polynomial of degree $d-1$, with the constant term $\Delta^0 f(0)$. Then, evaluate $(f(n)/(n-0) - \Delta^0 f(0))/(n-1)$ to find the quotient polynomial of degree $d-2$ including its constant term $\Delta^1 f(0)$. Continue this process until a single constant is left, which is $\Delta^d f(0)/d!$. Or equivalently, Use Horner's method to find

$$f(n) = (n-0)f_1(n), \quad deg\ f_1(n) \le d-1,$$

where the constant term of $f_1(n)$ is $\Delta^0 f(0)$. Then, use Horner's method again to evaluate

$$f_1(n) = (n-1)f_2(n), \quad deg\ f_2(n) \le d-2,$$

which contain the constant term $\Delta^1 f(0)/1!$. Continue the process and finally obtain

$$f_{d-1} = (n-d+1)f_d(n), \quad f_d(n) = \Delta^d f(0)/d!.$$

When $f(n) = n^d$, from (3) it can be seen that the above algorithm provides a way to evaluate the Stirling numbers of the second kind $S(d,1)$, $S(d,2), \ldots,$ $S(d,d)$ defined by (1).

Example 2 Consider $f(n) = n^4 - 2n^2 + 3n - 4$. We use the following synthetic division to find out its difference sequence from order 0 to 4.

$$
\begin{array}{c|ccccc}
0 & 1 & 0 & -2 & 3 & -4 \\
 & & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 0 & -2 & 3 & \underline{-4} \\
 & & 1 & 1 & -1 & \\ \hline
2 & 1 & 1 & -1 & \underline{2} & \\
 & & 2 & 6 & & \\ \hline
3 & 1 & 3 & \underline{5} & & \\
 & & 3 & & & \\ \hline
 & \underline{1} & \underline{6} & & &
\end{array}
$$

Hence, $\Delta^0 f(0) = f(0) = -4$, $\Delta^1 f(0) = 2$, $\Delta^2 f(0) = 5(2!) = 10$, and $\Delta^3 f(0) = 6(3!) = 36$, and $\Delta^4 f(0) = 1(4!) = 24$, which can be read on the diagonal from the top right to the bottom line.

Example 3 From expansion (see, for examples, [3] and [7])

$$n^4 = \sum_{k=1}^{4} S(4,k)(n)_k,$$

or equivalently,

$$n^3 = S(4,1) + (n-1)(S(4,2) + (n-2)(S(4,3) + (n-3)S(4,4))),$$

we may use the following division to evaluate $S(4,k)$ $(k = 1,2,3,4)$.

$$
\begin{array}{c|cccc}
1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 \\ \hline
2 & 1 & 1 & 1 & \underline{1} \\
 & & 2 & 6 & \\ \hline
3 & 1 & 3 & \underline{7} & \\
 & & 3 & & \\ \hline
 & \underline{1} & \underline{6} & &
\end{array}
$$

Hence, we can read $S(4,1)=1$, $S(4,2)=7$, $S(4,3)=6$, and $S(4,4)=1$ diagonally from the top right to the bottom line. In addition, the first calculation gives $\{1,0,0,0,0\}$, the second calculation $\{1,1,1,1,\}$, the third calculation $\{1,3,7\}$, and the fourth calculation $\{1,6\}$, which are respectively the first, second, third, and fourth row of the table of the Stirling numbers of the second kind. In other words, the division of x^d by $x-j$ generates $\{S(d,j), S(d-1,j),\cdots, S(j,j)\}$.

From (3) we immediately know that $S(d,d)=1$ because it is the coefficient of n^d. Using our method, one may calculate the matrices related to Stirling numbers easily, for example, matrices T_n and W_n defined by (2) and (16) in [8].

Algorithm 1.2 can also be used to evaluate non-centeral Stirling numbers of the second kind (*cf.* [5]) defined by

$$(x-a)^d=\sum_{k=0}^{d} S_a(d,k)(x)_k$$

with a parameter a. In fact, a similar argument can be used to calculate $\{S_a(d,k)\}$ by using the transformation $x-a\mapsto x$ in Algorithm 1.2.

From Theorem 1.1 we also know that Horner's method provides simple algorithms to evaluate divided differences and derivatives of a polynomial, and the former can be used to find the coefficients of the Newton interpolation while the latter can be used to approximate the zeros of the polynomial with any required significant digits.

Let $X=\{x_0,x_1,\ldots,x_d\}$ be a set of $d+1$ distinct points, and let $f(x)$ be a polynomial of degree d. Then we can write $f(x)$ in terms of its Newton interpolation form on the set X as

$$\begin{aligned} f(x) &= f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\cdots \\ &\quad +f[x_0,x_1,\ldots,x_d](x-x_0)(x-x_1)\cdots(x-x_{d-1}), \end{aligned} \tag{4}$$

where $f[x_0]=f(x_0)$ and $f[x_0,x_1,\ldots x_k]$ is the kth order divided difference of f at $\{x_0,x_1,\ldots,x_k\}$ defined by

$$f[x_0,x_1,\ldots x_k]=\frac{1}{x_k-x_0}(f[x_1,x_2,\ldots,x_k]-f[x_0,x_1,\ldots,x_{k-1}])$$

for $k=1,2,\ldots,d$, and can be evaluated using the following algorithm.

Algorithm 1.3 *Write (4) as*

$$\begin{aligned} f(x) &= f[x_0] + (x - x_0)\left(f[x_0, x_1] + (x - x_1)\left(f[x_0, x_1, x_2] + \cdots \right.\right.\\ &\quad \left.\left. +(x - x_{d-1}) f[x_0, x_1, \ldots, x_d]\right)\right), \end{aligned}$$

where $f[x_0] = f(x_0)$*. Use synthetic division to obtain* $(f(x) - f(x_0))/(x - x_0)$*, a polynomial of degree* $d-1$*, with the constant term* $f[x_0, x_1]$*. Then, evaluate* $(f(x) - f(x_0))/(x - x_0) - f[x_0, x_1])/(x - x_1)$ *to find the quotient polynomial of degree* $d-2$ *and its constant term* $f[x_0, x_1, x_2]$*. Continue this process until a single constant is left, which is* $f[x_0, x_1, \ldots, x_d]$*. Or equivalently, Use Horner's method to find*

$$f(x) - f(x_0) = (x - x_0) f_1(x), \quad deg\ f_1(x) \le d - 1,$$

where the constant term of $f_1(x)$ *is* $f[x_0, x_1]$*. Then, use Horner's method again to evaluate*

$$f_1(x) = (x - x_1) f_2(x), \quad deg\ f_2(x) \le d - 2,$$

which contains the constant term $f[x_0, x_1, x_2]$*. Continue the process and finally to obtain*

$$f_{d-1} = (x - x_{d-1}) f_d(x), \quad f_d(n) = f[x_0, x_1, \ldots, x_d].$$

Example 4 To find the divided differences of $f(x) = x^4 - 2x^2 + 3x - 4$ on the set $\{-1, 0, 1, 3, 4\}$, we consider $f(x) - f(-1) = x^4 - 2x^2 + 3x + 4$ and use the following synthetic division to obtain its divided difference at the given knot points.

$$\begin{array}{r|rrrrr}
-1 & 1 & 0 & -2 & 3 & 4 \\
 & & -1 & 1 & 1 & -4 \\
\hline
0 & 1 & -1 & -1 & 4 & 0 \\
 & & 0 & 0 & 0 & \\
\hline
1 & 1 & -1 & -1 & \underline{4} & \\
 & & 1 & 0 & & \\
\hline
3 & 1 & 0 & \underline{-1} & & \\
 & & 3 & & & \\
\hline
 & \underline{1} & \underline{3} & & &
\end{array}$$

Hence, $f[-1] = f(-1) = -8$, $f[-1,0] = 4$, $f[-1,0,1] = -1$, $f[-1,0,1,3] = 3$, and $f[-1,0,1,3,4] = 1$. It can be seen that the new method is much easier than the traditional method.

Let r be a real number, and let $f(x)$ be a polynomial of degree d. Then, using the Taylor expansion of $f(x)$ yields

$$f(x) = f(r) + f'(r)(x-r) + \frac{f''(r)}{2!}(x-r)^2 + \cdots + \frac{f^{(d)}(r)}{d!}(x-r)^d, \quad (5)$$

which can written recursively as

$$f(x) - f(r) = (x-r)f_1(x), \quad f_k(x) = (x-r)f_{k+1}(x), \quad k = 1, 2, \ldots, d-1,$$

and the constant term of $f_k(x)$ is $f^{(k)}(r)/k!$ $(k = 1, 2, \ldots, d)$. Thus we may apply Horner's method to find all derivatives of f at r. Obviously, for polynomial

$$g(x) = f(r) + f'(r)x + \frac{f''(r)}{2!}x^2 + \cdots + \frac{f^{(d)}(r)}{d!}x^d, \quad (6)$$

the roots of $g(x) = 0$ are the roots of equation $f(x) = 0$, each diminished by r. We can use the process to diminish a root of the proposed equation by its first digit. Then we apply it again to diminish the corresponding root of the resulting equation by its first digit, which is the second digit of the required root of the original equation. Using this process continuously, we finally approximate the root of the original equation $f(x) = 0$ to the required significant digits. More details can be found in [9]. Here is an example.

Example 5 Consider equation $f(x) = x^4 - 2x^2 + 3x - 4 = 0$, which has a root in the interval $(1, 2)$. We may use (6) to find $g(x)$, where $r = 1$. The process can be shown in the following synthetic division.

$$
\begin{array}{c|ccccc}
1 & 1 & 0 & -2 & 3 & -4 \\
  &   & 1 & 1 & -1 & 2 \\
\hline
  & 1 & 1 & -1 & 2 & \underline{-2} \\
  &   & 1 & 2 & 1 & \\
\hline
  & 1 & 2 & 1 & \underline{3} & \\
  &   & 1 & 3 & & \\
\hline
  & 1 & 3 & \underline{4} & & \\
  &   & 1 & & & \\
\hline
  & \underline{1} & \underline{4} & & &
\end{array}
$$

Hence, we obtain

$$f(1) = -2, f'(1) = 3, \frac{f''(1)}{2!} = 4, \frac{f'''(1)}{3!} = 4, \frac{f^{(4)}(1)}{4!} = 1,$$

and the corresponding

$$g(x) = -2 + 3x + 4x^2 + 4x^3 + x^4.$$

Therefore the new equation is $g(x) = 0$. Multiply the root by 10 and change the equation to be

$$x^4 + 40x^3 + 400x^2 + 3000x - 20000 = 0.$$

It is easy to see that $g(x)$ has a root between 3 and 4. Thus we may use (6) again to generate a new polynomial and solve the corresponding equation.

3	1	40	400	3000	−20000
		3	129	1587	13761
	1	43	529	4587	<u>−6239</u>
		3	138	2001	
	1	46	667	<u>6588</u>	
		3	147		
	1	49	<u>814</u>		
		3			
	<u>1</u>	<u>52</u>			

The above table shows that an approximation of the original polynomial equation to its second significant digit is 1.3, and the third significant digit can be found using the polynomial equation

$$x^4 + 52x^3 + 814x^2 + 6588x - 6239 = 0.$$

Multiply the root by 10 to change the equation to be

$$x^4 + 520x^3 + 81400x^2 + 6588000x - 62390000 = 0,$$

which has a root in the interval $(8, 9)$. Thus, the original polynomial equation $f(x) = x^4 - 2x^2 + 3x - 4 = 0$ has a root of approximately 1.38, and its better approximation with more significant digits can be found from the equation

$$x^4 + 552x^3 + 94264x^2 + 7992288x - 4206064 = 0$$

generated by using the following table. Since $x^4 + 552x^3 + 94264x^2 + 7992288x - 4206064 = 0$ has a root between 5 and 6, we obtain the root of original equation with four significant digits as 1.385.

8	1	520	81400	6588000	−62390000
		8	4224	684992	58183936
	1	528	85624	7272992	$\underline{-4206064}$
		8	4288	719296	
	1	536	89912	$\underline{7992288}$	
		8	4352		
	1	544	$\underline{94264}$		
		8			
	$\underline{1}$	$\underline{552}$			

From Example 5, one may find Horner's method is not an efficient way to evaluate the roots of polynomial equations, but it is a faster way to find out the coefficients of the expansions of polynomials in terms of nested bases formed by products of linear polynomials.

Proposition 1.4 *Let $\phi_k(x) = a_k x - b_k$, $k = 1, 2, \ldots$, and let $f(x)$ be a polynomial of degree d. Then*

$$f(x) = c_0 + \sum_{k=1}^{d} c_k \Pi_{j=1}^{k} \phi_j(x), \tag{7}$$

where c_k $(k = 0, 1, \ldots, d)$ can be found using the synthetic division based on Horner's method.

One may see the examples of Proposition 1.4 from the algorithms applied to the expansions (2)-(5). Interested readers may also construct examples for any polynomial expansion defined by (7). For instance, we may calculate the binomial sequence $\binom{n}{k}$ $(k = 0, 1, \ldots, n)$ for $n \in \mathbb{N}$ by applying Horner's method to the expansion

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k.$$

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