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Tian-Xiao He, *Illinois Wesleyan University*

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Tian-Xiao He

Department of Mathematics and Computer Science
Illinois Wesleyan University, Bloomington, IL61702-2900
`the@iwu.edu`

Dedicated to Roger B. Eggleton on the occasion of his 70th birthday

Abstract

Here we present a characterization of Sheffer-type polynomial sequences based on the isomorphism between the Riordan group and Sheffer group and the sequence characterization of Riordan arrays. We also give several alternative forms of the characterization of the Riordan group, Sheffer group and their subgroups. Formulas for the computation of the generating functions of Riordan arrays and Sheffer-type polynomial sequences from the characteristics are shown. Furthermore, the applications of the characteristics to lattice walks and recursive construction of Sheffer-type polynomial sequences are also given.

1 Introduction

In the recent literatures, special emphasis has been given to the concept of *Riordan arrays*, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called the *Riordan group* (see Shapiro et al. [18]). Sheffer-type polynomial sequences are infinite polynomial sequences defined by the expansions of exponential functions of formal power series (FPS). They also form a group, called the *Sheffer group* (see Roman and Rota [13, 14]). In [7], the isomorphism between the Riordan group and Sheffer group is established.

Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli [20, 21], on subgroups of the Riordan group in Peart and Woan [11] and Shapiro [15, 16, 17], on some characterizations of Riordan arrays in He and Sprugnoli [8], Merlini et al. [9], and Rogers [12], and on many interesting related results in He et al. [7], Nkwanta [10], Shapiro [16, 17], Cheon et al. [2] and [3], author's [5], and so forth.

More precisely, let us consider the set of formal power series (FPS) $\mathcal{F} = \mathbb{R}[[t]]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$, is the minimal number $r \in \mathcal{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of FPS of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* FPS and $\mathcal{F}_1$ is the set of *compositionally invertible* FPS (see for example, [4] and [23]), that is, for a FPS $f(t) \in \mathcal{F}_1$ its compositional inverse $\overline{f}(t)$ exists such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$.

Definition 1 *Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$. We call the infinite matrix $D = [d_{n,k}]_{n,k \geq 0}$ a Riordan array if its kth column satisfies*

$$\sum_{n \geq 0} d_{n,k} t^n = d(t)(h(t))^k; \qquad (1.1)$$

or equivalently, $d_{n,k} = [t^n]d(t)(h(t))^k$.

If $(d(t), h(t))$ and $(f(t), g(t))$ are Riordan arrays, then

$$(d(t), h(t)) * (f(t), g(t)) := (d(t)f(h(t)), g(h(t))) \qquad (1.2)$$

is called the matrix multiplication, i.e., for $(d(t), h(t)) = [d_{n,k}]_{n \geq k \geq 0}$ and $(f(t), g(t)) = [c_{n,k}]_{n \geq k \geq 0}$ we have

$$(d(t), h(t)) * (f(t), g(t)) = [d_{n,k}][c_{n,k}]. \qquad (1.3)$$

And the set of all Riordan arrays is a group under the matrix multiplication.

The Riordan array $I = (1, t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it is easily seen that I acts as an identity for this product, that is, $(1, t) * (d(t), h(t)) = (d(t), h(t)) * (1, t) = (d(t), h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t), h(t))^{-1} = \left(\frac{1}{d(\overline{h}(t))}, \overline{h}(t) \right) \qquad (1.4)$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of Riordan arrays is a group. Particular subgroups of $\mathcal{R}$ are important and have been considered in this paper:

- the set $\mathcal{A}$ of *Appell arrays*, that is the set of Riordan arrays $D = (d(t), h(t))$ for which $h(t) = t$; it is an invariant subgroup and is isomorphic to the group $\mathcal{F}_0$, with the usual product as group operation; in addition, the collection of all Riordan arrays $D_c = (d(t), ct)$ is a subgroup called the *c-Appell subgroup* for a constant $c \neq 0$.

- the set $\mathcal{L}$ of *Lagrange arrays*, that is the set of Riordan arrays $D = (d(t), h(t))$ for which $d(t) = 1$; it is also called the *associated subgroup*; it is isomorphic with the group $\mathcal{F}_1$, with composition as group operation;

- the set $\mathcal{B}$ of *Bell* or *renewal arrays*, that is the set of Riordan arrays $D = (d(t), h(t))$ for which $h(t) = td(t)$; it is a subgroup of $\mathcal{R}$, called the *Bell subgroup*, and is the set originally considered by Rogers in [12];

It is clear that there exists a semidirect product decomposition for Riordan group $\mathcal{R}$.

$$\mathcal{R} \simeq \mathcal{A} \rtimes \mathcal{B} \; since \; (d(t), h(t)) = \left(\frac{td(t)}{h(t)}, t \right) \left(\frac{h(t)}{t}, h(t) \right).$$

It can also be seen there is the decomposition $\mathcal{R} = \mathcal{A} \otimes \mathcal{L}$ due to $(d(t), h(t)) = (d(t), t)(1, h(t))$.

If $r(t)$ is the generating function (GF) of number sequence $\{r_k\}$, i.e. $r_k = [t^k]r(t)$ for all $k \geq 0$, then

$$(d(t), h(t)) \begin{pmatrix} r_0 \\ r_1 \\ \vdots \end{pmatrix} = (d(t), h(t)) * r(t) = d(t)r(h(t)), \qquad (1.5)$$

which is called the fundamental theorem for Riordan arrays (see [15]).

The convolution $f * g$ of two sequences $f = \{f_n\}$, $g = \{g_n\}$, $n \geq 0$, is defined by

$$f * g := \left\{ \sum_{k=0}^{n} f_k g_{n-k} \right\}_{n \geq 0},$$

or equivalently, the Cauchy multiplication of f and g. The r-fold convolution $f^{(r)} = \{f_n^{(r)}\}_{n \geq 0}$ $(r \geq 0)$ may then be defined recursively by

$$f^{(r)} = f * f^{(r-1)},$$

where $r \geq 1$, $f_n^{(0)} = \delta_{n,0}$, and thus $f^{(1)} = f$. The renewal array $\{b_{n,m}\}_{0 \leq m \leq n}$ is the triangular array generalized by a sequence $\{b_n\}_{n \geq 0}$ as follows:

$$\begin{cases} b_{n,m} := b_n^{(m+1)}, & 0 \leq m \leq n, \\ b_{n,m} := 0, & m < 0 \text{ or } m > n. \end{cases} \tag{1.6}$$

Thus by denoting the generating functions

$$B^{(m)}(x) = \sum_{n \geq 0} b_{n,m} x^n, \qquad B(x) = \sum_{n \geq 0} b_n x^n \tag{1.7}$$

we have

$$B^{(m)}(x) = (B(x))^{m+1}, \quad m \geq 1, \tag{1.8}$$

where $(B(x))^{m+1}$ is the $m+1$-fold convolution of $B(x)$. A natural question is raised: Is $[b_{n,m}]_{0 \leq m \leq n}$ a Riordan array? The answer is given in [12] by using the so-called A-sequence $A = \{a_n\}_{n \geq 0}$ defined in (2.13) in Section 2. We will show more details in the section.

Rota and Roman give the definition of the Sheffer group (see [13, 14]) with the $n!$-umbral calculus, which is reformulated as follows (see [7]) with 1-umbral calculus.

Definition 2 *Let $d(t)$ and $h(t)$ be defined as Definition 1. Then the polynomials $p_n(x)$ $(n = 0, 1, 2, \cdots)$ defined by the GF*

$$d(t)e^{xh(t)} = \sum_{n \geq 0} p_n(x) t^n \tag{1.9}$$

are called Sheffer-type polynomials with $p_0(x) = 1$. The set of all Sheffer-type polynomial sequences with the polynomial sequence multiplication, defined by

$$\left\{ p_n(x) = \sum_{k=0}^{n} p_{n,k} x^k \right\} \# \left\{ q_n(x) = \sum_{k=0}^{n} q_{n,k} x^k \right\}$$

$$= \left\{ r_n(x) = \sum_{k=0}^{n} (\sum_{\ell=k}^{n} \ell! p_{n,\ell} q_{\ell,k}) x^k \right\}, \tag{1.10}$$

forms a group, called the Sheffer group.

From Definitions 1 and 2 we have (see also [7])

$$\begin{aligned} p_n(x) &= [t^n] d(t) e^{xh(t)} = [t^n] \sum_{k \geq 0} \frac{1}{k!} d(t) (h(t))^k x^k \\ &= \sum_{k=0}^{n} d_{n,k} \frac{x^k}{k!}, \end{aligned} \tag{1.11}$$

where we use $d_{n,k} = 0$ for all $k > n$. From Definition 1 we also have

$$[x^k]p_n(x) = p_{n,k} = \frac{d_{n,k}}{k!} = \frac{1}{k!}[t^n]d(t)(h(t))^k.$$

Therefore, with a constant multiple, $1/(k!)$, of the kth column, the rows of the Riordan array present the coefficients of the Sheffer-type polynomial sequences. In [7], the isomorphism between the Sheffer group and the Riordan group is established. Indeed, there exists a one-to-one and onto mapping $\theta : [d_{n,k}]_{n \geq k \geq 0} \mapsto \{p_n(x)\}_{n \geq 0}$ or simply, $\theta : (d(t),\, h(t)) \mapsto \{p_n(x)\}$ since

$$p_n(x) = [t^n]d(t)e^{xh(t)} \quad if \ and \ only \ if \quad d_{n,k} = [t^n]d(t)(h(t))^k.$$

Furthermore, for $\theta((d(t),\, h(t)) = \{p_n(x)\}$ and $\theta((f(t),\, g(t)) = \{q_n(x)\}$ we clearly have

$$\{p_n(x)\} \# \{q_n(x)\} = \theta((d(t)\, h(t)) * (f(t),\, g(t)))$$

because of (1.10) and (1.11), which presents the isomorphism between the Riordan group and Sheffer group. Since the rows of the Riordan array are finite, based on this feature and the isomorphism, we may derive the properties of the Riordan group from those of the Sheffer group. Conversely, many properties including the characterization of Riordan arrays can be transfered to the Sheffer-type polynomial sequences. For instance, we may find the characterizations of the subgroups of Sheffer-type polynomial sequences formed by the following sets.

- the set $\mathcal{A}$ of *Appell polynomials*, that is the set of Sheffer-type polynomials generated by (1.9) with $D = (d(t),\, h(t))$ for which $h(t) = t$; in addition, the set of Sheffer-type polynomials generated by (1.9) with $D = (d(t),\, h(t))$ for which $h(t) = ct$, $c \neq 0$, is the *c-Appell subgroup*.

- the set $\mathcal{L}$ of *Lagrange polynomials*, that is the set of Sheffer-type polynomials generated by (1.9) with $D = (d(t),\, h(t))$ for which $d(t) = 1$.

- the set $\mathcal{B}$ of *Bell-type* or *renewal polynomials*, that is the set of Sheffer-type polynomials generated by (1.9) with $D = (d(t),\, h(t))$ for which $h(t) = td(t)$.

From the isomorphism between the Riordan group and Sheffer group, we know all the above sets are subgroups of the Sheffer group, called the *Appell subgroup*, the *Lagrange subgroup*, and the *Bell subgroup*, respectively.

Remark 1.1 Definition (1.9) can be considered as $(d(t), h(t)) * e^{xt}$ using the fundamental theorem for Riordan arrays or the operation defined as in (1.5).

A generalized set of Sheffer-type polynomial sequences can be found in [6], in which the exponential function $exp(xh(t))$ is replaced by $B(xh(t))$ with $B(t) \in \mathcal{F}_0$:

Definition 3 *Let $d(t)$, $B(t)$, and $h(t)$ be any formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $d(t), B(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$. Then the polynomials $p_n(x)$ $(n = 0, 1, 2, \cdots)$ defined by the generating function (GF)*

$$d(t)B(xh(t)) = \sum_{n \geq 0} \tilde{p}_n(x)t^n \qquad (1.12)$$

are called generalized Sheffer-type polynomials associated with $(d(t), h(t))_{B(t)}$. In particular, we denote the polynomial sequence generated by $(d(t), h(t))_{1/(1-t)}$ by $\{\tilde{p}_n(x)\}$.

It is obvious that the generalized Sheffer-type polynomial sequence generated by $(d(t), h(t))_{1/(1-t)}$ can also be written as $\{\tilde{p}_n(x) = \sum_{k=0}^n d_{n,k} x^k\}$, where $[d_{n,k}]_{n \geq k \geq 0} = (d(t), h(t))$ generates $\{p_n(x)\}$ shown as (1.9). Clearly, the collection of all such $\{\tilde{p}_n(x)\}$ form a group, and the corresponding group operation of two generalized Sheffer-type polynomial sequences is

$$\left\{\tilde{p}_n(x) = \sum_{k=0}^n p_{n,k} x^k\right\} \# \left\{\tilde{q}_n(x) = \sum_{k=0}^n q_{n,k} x^k\right\}$$
$$= \left\{\tilde{r}_n(x) = \sum_{k=0}^n (\sum_{\ell=k}^n p_{n,\ell} q_{\ell,k}) x^k\right\},$$

It can be seen the groups $\{p_n(x)\}$ and $\{\tilde{p}_n(x)\}$ defined above with corresponding group operations are isomorphic.

The main results will be presented in the next sections. In Section 2, we shall discuss the characterization of Riordan arrays and Sheffer-type polynomial sequences and give its reformulated compact expression using GFs. In addition, several formulas for the computation of the generating functions of Riordan arrays and Sheffer-type polynomial sequences from the characteristics are presented. Some applications of the characterization shown as in Section 2 to the Riordan group and subgroups such as lattice walks and the generalized recursive formula of Sheffer-type polynomials are presented with some examples in Section 3.

2 Fundamental characterization of proper Riordan arrays

A Riordan array defined in Definition 1 is called a proper Riordan array, and the corresponding Sheffer-type polynomial sequence defined by Definition 2 is a regular Sheffer-type polynomial sequence. A non-proper Riordan array $(d(t), h(t))$ with $h(t) = t^s \bar{h}(t)$, $h(0) = 0$, and $h'(0) \neq 0$ can be reduced to a proper one, $(d(t), \bar{h}(t))$, by moving every column k up $k + s$ position, which will be discussed in another paper. Denote by $\mathcal{R}$ and $\mathcal{S}$ the Riordan group of proper Riordan arrays and the Sheffer group of regular Sheffer-type polynomial sequences, respectively. The following fundamental characterization of proper Riordan arrays is a comprehensive results shown in [12], [20], [9], [22], and [8]: $(d(t), h(t)) = [d_{n,k}]_{0 \leq k \leq n} \in R$ if and only if there is a unique A-sequence, $A = \{a_j\}_{j \geq 0}$ with $a_0 \neq 0$, and a unique Z-sequence, $Z = \{z_j\}_{j \geq 0}$, such that

$$d_{n,k} = \sum_{j \geq 0} a_j d_{n-1,k+j-1}, \ \ d_{n,0} = \sum_{j \geq 0} z_j d_{n-1,j} \qquad (2.13)$$

for all $n, k \geq 0$. (2.13) also answer the question when the array with entries $b_{n,m}$ defined by (1.6) is a Riordan array. We now extend the above characterization to the Sheffer group using the isomorphism between the Riordan group and Sheffer group shown in [7].

Theorem 1 *Let $(d(t), h(t)) = [d_{n,k}]_{0 \leq k \leq n} \in \mathcal{R}$. Then $\{p_n(x) = \sum_{n \geq k \geq 0} d_{n,k} x^k / k!\} \in S$ if and only if there is a unique A-sequence, $A = \{a_j\}_{j \geq 0}$ with $a_0 \neq 0$, and a unique Z-sequence, $Z = \{z_j\}_{j \geq 0}$, such that (2.13) hold for all $n, k \geq 0$; or equivalently,*

$$p_n(x) = \sum_{j \geq 0} a_j \sum_{k=j}^{n} d_{n-1,k-1} \frac{x^{k-j}}{(k-j)!}, \ \ p_n(0) = \sum_{j \geq 0} z_j \frac{d^j}{dx^j} \left. p_{n-1}(x) \right|_{x=0}$$

$$(2.14)$$

for all $n, k \geq 0$.

Remark 2.1 Since a Riordan Array is a lower-triangular array, if the summations (2.13) hold for $[d_{n,k}]_{o \leq k \leq n} = (d(t), h(t))$, then the summations must be finite for fixed n, namely,

$$d_{n,k} = \sum_{j=0}^{n-k} a_j d_{n-1,k+j-1}, \ \ d_{n,0} = \sum_{j=0}^{n-1} z_j d_{n-1,j}.$$

Similarly, the summations held for $p_n(x) = \sum_{n \geq k \geq 0} d_{n,k} x^k / k!$ can be written as

$$p_n(x) = \sum_{j=0}^{n} a_j \sum_{k=j}^{n} d_{n-1,k-1} \frac{x^{k-j}}{(k-j)!}, \quad p_n(0) = \sum_{j=0}^{n-1} z_j \frac{d^j}{dx^j} \left. p_{n-1}(x) \right|_{x=0}.$$

Proof. To show (2.14), we substituting the first expression of (2.13) into $p_n(x)$ and interchange the order of summation:

$$p_n(x) = \sum_{k=0}^{n} d_{n,k} \frac{x^k}{k!} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_j d_{n-1,k+j-1} \frac{x^k}{k!}$$

$$= \sum_{j=0}^{n} a_j \sum_{k=0}^{n-j} d_{n-1,k+j-1} \frac{x^k}{k!}$$

$$= \sum_{j=0}^{n} a_j \sum_{k=j}^{n} d_{n-1,k-1} \frac{x^{k-j}}{(k-j)!}.$$

Conversely, the first equation of (2.14) is equivalent to

$$p_n(x) = \sum_{k=0}^{n} d_{n,k} \frac{x^k}{k!} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_j d_{n-1,k+j-1} \frac{x^k}{k!}.$$

By comparing the coefficients of x^k term on the two sides of the last equation, one may obtain the first formula of (2.13). The second expression of (2.14) comes from the second equation of (2.13) and the observation

$$d_{n,j} = \frac{d^j}{dx^j} \sum_{k=0}^{n} d_{n,k} \frac{x^k}{k!} \bigg|_{x=0} = \frac{d^j}{dx^j} p_n(x) \bigg|_{x=0}.$$

Hence, we establish the equivalence between (2.13) and (2.14). The remaining thing can be proved by using the isomorphism between $\mathcal{R}$ and $\mathcal{S}$.

∎

Remark 2.2 We now give an alternative form of the first expression of (2.14). The Sheffer-type polynomial sequence $\{p_n(x)\}$ defined in (1.9) can be written as follows.

$$p_n(x) = \sum_{j=0}^{n-1} d_{n-1,j} \left(\sum_{k=0}^{j+1} a_{j-k+1} \frac{x^k}{k!} \right). \tag{2.15}$$

In fact, substituting the first expression of (2.13) into $p_n(x) = \sum_{n \geq k \geq 0} d_{n,k} x^k / k!$ yields

$$p_n(x) = \sum_{k=0}^{n} d_{n,k} \frac{x^k}{k!} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_j d_{n-1,k+j-1} \frac{x^k}{k!}$$

$$= \sum_{k=0}^{n} \sum_{j=k-1}^{n-1} a_{j-k+1} d_{n-1,j} \frac{x^k}{k!}$$

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} a_{j-k+1} d_{n-1,j} \frac{x^k}{k!}.$$

Similarly, for the generalized Sheffer-type polynomial sequence $\tilde{p}_n(x) = \sum_{n \geq k \geq 0} d_{n,k} x^k$, where $[d_{n,k}]_{0 \leq k \leq n} = (d(t), h(t))$, we have

$$\tilde{p}_n(x) = \sum_{j=0}^{n} a_j \sum_{k=j}^{n} d_{n-1,k-1} x^{k-j}, \ \ \tilde{p}_n(x) = \sum_{j=0}^{n-1} d_{n-1,j} x^j \left(\sum_{k=0}^{j+1} a_{j-k+1} x^{k-j} \right). \tag{2.16}$$

Secondly, we recall from [8] the following alternative expressions of the A-sequence and Z-sequence by using their GF's, which are called respectively the *A-series (or polynomial)* and *Z-series (or polynomial)*.

Theorem 2 *[8] $(d(t), h(t)) = [d_{n,k}]_{0 \leq k \leq n} \in \mathcal{R}$ if and only if a unique A-series $A(t) = \sum_{j \geq 0} a_j t^j$ $(a_0 \neq 0)$ and a unique Z-series $Z(t) = \sum_{j \geq 0} z_j t^j$ such that there hold*

$$h(t) = tA(h(t)) \ \text{and} \ d(t) = d_{0,0} + td(t)Z(h(t)), \tag{2.17}$$

or equivalently,

$$A(y) = \frac{y}{t} \ \text{and} \ Z(y) = \frac{d(t) - d_{0,0}}{td(t)}, \quad t = \bar{h}(y), \tag{2.18}$$

where $d_{0,0} = d(0)$, $\bar{h}(y)$ is the compositional inverse of $h(y)$, and $A(t)$ and $Z(t)$ are respectively the GF of A-sequence and Z-sequence as mentioned above.

Proof. A proof of the first formula of (2.17) can be found in [8]. Here, we only discuss the proof of the second formula. Some different proofs of the second formula of (2.17) can be found from [19] and [8]. We now give a simpler proof. From the second condition in (2.13), for all $n \geq 0$ we have

$$[t^{n+1}]d(t) = \sum_{j \geq 0} z_j [t^n] d(t) (h(t))^j = [t^n]d(t) \sum_{j \geq 0} z_j (h(t))^j,$$

which implies

$$[t^n]d(t)t^{-1} = [t^n]d(t)\sum_{j\geq 0} z_j\,(h(t))^j\,.$$

Thus,

$$(d(t) - d_{0,0})/t = d(t)\sum_{j\geq 0} z_j\,(h(t))^j = d(t)Z(h(t)). \qquad (2.19)$$

Note that the first formula of (2.17) can also be re-proved using a similar argument. The equivalence between (2.17) and (2.18) is obvious.

The uniqueness of the solution for $h(t)$ of the first equation in (2.17) has been obtained in [8]. Therefore, from (2.18) we have

$$Z(y) = A(y)\frac{d(t) - d_{0,0}}{yd(t)}, \qquad (2.20)$$

where $t = \bar{h}(y)$, which implies the uniqueness of the solution for $d(t)$ of the second equation. This completes the proof of the theorem.

■

Let us consider the characterization of some subgroups of $\mathcal{R}$ and $\mathcal{S}$.

Corollary 2.1 *With the same notation shown as in Theorem 2, a Riordan array $(d(t), h(t))$ is in the Appell (c-Appell, $c \neq 0$,) subgroup, or equivalently, a Sheffer-type polynomial sequence $\{p_n(x)\}$ generated by $(d(t), h(t))$ is an Appell (c-Appell, $c \neq 0$) polynomial sequence if and only if its A-series is $A(t) = 1$ $(A(t) = 1/c)$, or equivalently,*

$$p_n(x) = \sum_{k=0}^{n} d_{n-k}\frac{x^k}{k!}, \qquad (2.21)$$

where $d_j = [t^j]d(t)$.

A Riordan array $(d(t), h(t))$ is an element in the Lagrange subgroup, or equivalently, a Sheffer-type polynomial sequence $\{p_n(x)\}$ generated by $(d(t), h(t))$ is a Lagrange polynomial sequence if and only if its Z-series is $Z(t) = 0$, or equivalently,

$$p_0(x) \equiv 1 \quad and \quad p_n(0) = 0 \qquad (2.22)$$

for all $n \geq 1$.

A Riordan array $(d(t), h(t))$ is in the Bell subgroup, or equivalently, a Sheffer-type polynomial sequence $\{p_n(x)\}$ generated by $(d(t), h(t))$ is a Bell-type polynomial sequence if and only if $A(t) = d_{0,0} + tZ(t)$, or equivalently,

$$p_n(x) = \sum_{k=0}^{n} d_{n-k}^{(k+1)}\frac{x^k}{k!}, \qquad (2.23)$$

where $d_j^{(k+1)} = [t^j](d(t))^{(k)}$ *and* $(d(t))^{(k)}$ *is defined by (1.8) and (1.7).*

In addition, $(1, t)$ *is the only element in* $\mathcal{R}$ *that satisfies both* $A(t) = 1$ *and* $Z(t) = 0$, *and its corresponding Sheffer-type polynomial sequence is* $\{p_n(x) = x^n/n!\}$.

Proof. It is easy to see that $(d(t), h(t)) = (d(t), t) \in \mathcal{R}$, $d(t) = \sum_{j \geq 0} d_j t^j$, if and only if

$$d_{n,k} = d_{n-1,k-1},$$

or equivalently, from (2.18), $A(y) = y/\bar{h}(y) = 1$. Hence, the corresponding Appell polynomials

$$p_n(x) = [t^n]d(t)e^{xt} = [t^n]\sum_{k \geq 0} d(t)t^k \frac{x^k}{k!}$$

$$= [t^n]\sum_{k \geq 0}\sum_{j \geq 0} d_j t^{k+j} \frac{x^k}{k!} = [t^n]\sum_{k \geq 0}\sum_{j \geq k} d_{j-k} t^j \frac{x^k}{k!}$$

$$= [t^n]\sum_{j \geq 0} t^j \sum_{k=0}^{j} d_{j-k} \frac{x^k}{k!},$$

which implies (2.21). Conversely, (2.21) implies $[t^k]p_n(x) = [t^n](d(t),\, t)/k!$.

Similarly, $(d(t), h(t)) = (1, h(t)) \in \mathcal{R}$ if and only if

$$d_{0,0} = 1 \quad and \quad d_{n,0} = 0$$

for all $n \geq 1$, or equivalently, from (2.18), $Z(y) = 0$. Hence, from (1.11), we obtain (2.22).

If $(d(t), h(t))$ is in the Bell subgroup, then $h(t) = td(t)$ and $d(\bar{h}(t)) = t/\bar{h}(t)$. Hence, substituting $t = \bar{h}(t)$ into the characterization shown in Theorem 2, we obtain $A(t) = t/\bar{h}(t) = d(\bar{h}(t))$ and $Z(t) = (d(\bar{h}(t)) - d_{0,0})/t$, which implies $A(t) = d_{0,0} + tZ(t)$. It is easy to verify that the last equation also implies $h(t) = td(t)$. If $\{p_n(x)\}$ is a Bell-type polynomial sequence generated by $(d(t), h(t))$ with $h(t) = td(t)$ and $d(t) = \sum_{j \geq 0} d_j t^j$, then using (1.8) and (1.7) yields

$$p_n(x) = [t^n]d(t)e^{xtd(t)} = [t^n]\sum_{k \geq 0} t^k (d(t))^{k+1} \frac{x^k}{k!}$$

$$= [t^n]\sum_{k \geq 0}\sum_{j \geq 0} d_j^{(k+1)} t^{k+j} \frac{x^k}{k!} = [t^n]\sum_{k \geq 0}\sum_{j \geq k} d_{j-k}^{(k+1)} t^j \frac{x^k}{k!}$$

$$= [t^n]\sum_{j \geq 0} t^j \sum_{k=0}^{j} d_{j-k}^{(k+1)} \frac{x^k}{k!},$$

which implies (2.23). Conversely, (2.23) implies $[t^k]p_n(x) = [t^n](d(t), td(t))/k!$.

Finally, it is clear that $(d(t), h(t)) = (1, t)$ if only if $A(t) = 1$ and $Z(t) = 0$, and the Sheffer-type polynomial sequence generated by $(1, t)$ is $\{p_n(x) = x^n/n!\}$.

∎

Remark 2.3 To find the A-sequence $\{a_j\}_{j\geq 0}$ for $(d(t), h(t)) \in \mathcal{R}$, we may use either (2.18) or (2.13). If the compositional inverse of $h(t) = \sum_{j\geq 1} h_j t^j$, denoted by $\bar{h}(t)$, is found as $\bar{h}(t) = \sum_{j\geq 1} \bar{h}_j t^j$, then

$$A(t) = \frac{t}{\bar{h}(t)} = \frac{1}{\bar{h}_1 + \bar{h}_2 t + \bar{h}_3 t^2 + \cdots}.$$

Thus, from the formula of the reciprocal power series shown as on page 31 in [14], A-sequence can be written as $a_0 = 1/\bar{h}_1 = h_1$, and

$$a_n = -h_1 \sum_{k=1}^{n} \bar{h}_{k+1} a_{n-k} \tag{2.24}$$

for all $n \geq 1$.

An explicit formula of A-sequence for $(d(t), h(t)) \in G$ can be found recursively from (2.13) as follows:

$$
\begin{aligned}
a_{n-k} &= \frac{1}{d_{n,n}} \left(d_{n+1,k+1} - \sum_{j=0}^{n-k-1} a_j d_{n,k+j} \right) \\
&= \frac{1}{d_{0,0} g_1^n} \left(d_{n+1,k+1} - \sum_{j=0}^{n-k-1} a_j d_{n,k+j} \right). \tag{2.25}
\end{aligned}
$$

We now consider the converse process of evaluating $h(t)$ from the A-sequence for an element $(d(t), h(t)) \in G$.

Theorem 3 *Let* $(d(t), h(t)) = [d_{n,k}]_{0\leq k\leq n}$ *be a lower triangle array. If* (2.13) *holds for* $(d(t), h(t)) = [d_{n,k}]$, *then the compositional inverse of* $h(t)$, *denoted by* $\bar{h}(t) = \sum_{j\geq 1} \bar{h}_j t^j$ *and the exponential power series of* $d(t)$, *denoted by* $\tilde{d}(t)$, *can be evaluated as*

$$\bar{h}_1 = 1/a_0, \quad \bar{h}_n = -\frac{1}{a_0} \sum_{k=1}^{n-1} a_k \bar{h}_{n-k} \ n \geq 2 \tag{2.26}$$

and

$$\tilde{d}(t) = e^{\int Z(h(t))dt}. \tag{2.27}$$

Proof. From Theorem 2, we also have the expression of the GF of $\{a_j\}$ as

$$A(y) \;=\; \sum_{j\geq 0} a_j y^j = \frac{y}{h(y)} = \frac{1}{\bar{h}_1 + \bar{h}_2 y + \bar{h}_3 y^2 + \cdots}.$$

Thus,

$$\left(\sum_{j\geq 0} a_j y^j\right)\left(\bar{h}_1 + \bar{h}_2 y + \bar{h}_3 y^2 + \cdots\right) = 1,$$

from which we obtain the expression of $\bar{h}_j$ (2.26).

From the second expression in (2.13) we immediately have

$$\frac{d}{dt}\tilde{d}(t) = \sum_{j=0}^{n} z_j \tilde{d}(t) \left(h(t)\right)^j = \tilde{d}(t) Z(h(t)),$$

which has a solution as (2.27).

$\blacksquare$

Example 2.1 Using (2.26) we obtain

$$h_1 = a_0^{(1)} = a_0, \;\; h_2 = \frac{1}{2}a_1^{(2)} = a_0 a_1,$$

$$h_3 = \frac{1}{3}a_2^{(3)} = a_0^2 a_2 + a_0 a_1^2.$$

Another method to evaluate $h(t)$ based on the Lagrange inversion formula is given in [8].

Since the expression of $\{h_n\}$ given in Theorem 3 is not an explicit one, we establish the following formulas for $\{h_n\}$.

Theorem 4 *Let lower triangle array* $(d(t), h(t)) = [d_{n,k}]_{0\leq k\leq n}$ *satisfy condition (2.13). Then the coefficients* h_n $(n \geq 1)$ *of function* $h(t)$ *can be evaluated by* $h_1 = a_0$, $h_2 = a_0 a_1$ *and for* $n \geq j \geq 3$

$$h_j = a_{i_0} \sum_{i_1=0}^{1-i_0} a_{i_1} \sum_{i_2=0}^{2-i_1} a_{i_2} \cdots \sum_{i_{j-2}=0}^{j-2-i_{j-3}} a_{i_{j-2}} a_{j-1-i_{j-2}-i_{j-3}}, \tag{2.28}$$

where $i_0 = 0$.

Proof. Assume condition (2.13) holds for $(d(t), h(t))$. Then $(d(t), h(t)) = [d_{n,k}]_{n \geq k \geq 0}$ is a Riordan array. Where $d(t) = \sum_{k \geq 0} d_{k,0} t^k$. From the definition of Riordan array, the entry $d_{n,1}$ of $(d(t), h(t))$ can be written as

$$d_{n,1} = [t^n] d(t) h(t) = \sum_{k=1}^{n} h_k d_{n-k,0} \qquad (2.29)$$

Since (2.13) is assumed, we may write the entry $d_{n,1}$ $(n \geq 0)$ as

$$d_{n,1} = \sum_{k \geq 0}^{n-1} a_k d_{n-1,k}. \qquad (2.30)$$

Based on the above two equations, we now construct the expression of h_k with respect to the coefficients a_j $(n - 1 \geq j \geq 0)$ as follows. Noting that the function $d(t) = \sum_{k \geq 0} d_{k,0} t^k$ is arbitrarily chosen and condition (2.13) holds for all $n \geq k \geq 0$, we will re-write the right-hand side of (2.30) in terms of $d_{n-k,0}$ $(n \geq k \geq 1)$ by substituting condition (2.13) consecutively into the sum and compare the coefficients of the term $d_{n-k,0}$ in the resulting sum with the same term in the right-hand sum of (2.29).

First, the only term with $d_{n-1,0}$ on the right-hand sum of (2.30) is $a_0 d_{n-1,0}$, which implies $h_1 = a_0$. Secondly, there is only one term, $a_1 d_{n-1,1}$, in the sum of (2.30) that contains $d_{n-2,0}$ since $d_{n-1,1} = \sum_{k \geq 0} a_k d_{n-2,k}$ and other $d_{n-1,k}$ $(k \geq 2)$ do not contain $d_{n-2,0}$. Thus the coefficient of term $d_{n-2,0}$ is $a_0 a_1$. And comparing with the same term, $h_2 d_{n-2,0}$, in the right-hand sum of (2.29) we obtain $h_2 = a_0 a_1$. In general, in order to find the expression of h_j, we now collect all terms with factor $d_{n-j,0}$ in the sum of (2.30) by using the following process. We consider condition (2.13) as a walk from entry $d_{n+1,k+1}$ of the Riordan array $(d(t), h(t))$ to entries $d_{n,k}$, $d_{n,k+1}$, ..., till $d_{n,n}$ with weights a_0, a_1, ..., and a_n, respectively. The only entry on the $n - j + 1$st row to walk to entry $d_{n-j,0}$ in $(d(t), h(t))$ (by using (2.13)) is $d_{n-j+1,1}$ with weight a_0. Similarly, the only two entries walk to $d_{n-j+1,1}$ from the row $n - j + 2$ are $d_{n-j+2,1}$ and $d_{n-j+2,2}$ with weights a_1 and a_0, respectively. Hence, continuing this process, we have the following tree of the indexes of the weights a_j for the walk:

$$\begin{cases} (2), \\ (k)_i \to (0)(1) \cdots (i - k), \end{cases}$$

where $i \geq k \geq 0$, in which (2) means that the root (or axiom) in the $n - j + 1$st row has 2 sons and then that for all k, any node in the $n - j + i$th row labeled k (i.e., the index of weight a_k from $n - j + i$th row to $n - j + i - 1$st row), denoted by $(k)_i$, will have $i - k + 1$ descendants and they have labels 0, 1, ..., and $i - k$. Here, all labels 0, 1, ..., and $i - k$ are the indexes of

the corresponding weights a_j from the $n - j + i + 1$st row to the $n - j + i$th row. The above tree is a kind of ECO-system (see [1]). We construct tree till row $n - 1$. Finally, from the entry $d_{n,1}$ we walk to the entries $d_{n-1,k}$ $(j - 1 \geq k \geq 1)$ with weights a_k $(j - 1 \geq k \geq 1)$. Note that all other walks from $d_{n,1}$ to row $n - 1$ do not contribute the walks from $d_{n,1}$ to $d_{n-j,0}$. Therefore, considering the walks from $d_{n,1}$ to $d_{n-1,k}$ $(j - 1 \geq k \geq 1)$ with weights a_k $(j - 1 \geq k \geq 1)$, we obtain the total weights during the walk from $d_{n,1}$ to $d_{n-j,0}$ along all different ways based on the style (2.13) is

$$a_0 \sum_{i_1=0}^{1} a_{i_1} \sum_{i_2=0}^{2-i_1} a_{i_2} \sum_{i_3=0}^{3-i_2} a_{i_3} \cdots \sum_{i_{j-2}=0}^{j-2-i_{j-3}} a_{i_{j-2}} a_{j-1-i_{j-2}-i_{j-3}}.$$

The above weight expression is the coefficients of term $d_{n-j,0}$ in the resulting sum generated from the sum in (2.30) by applying substitution (2.30) consecutively from row $n - j$ to row n. Comparing with the coefficient of $d_{n-j,0}$, i.e., h_j, in the sum in (2.29), we obtain (2.28) and complete the proof of the theorem.

∎

Example 2.2 For $n = 4$, Applying (2.28) with $j = 3$ and $j = 4$, we easily get

$$h_3 = a_0 \sum_{i_1=0}^{1} a_{i_1} a_{2-i_1} = a_0(a_0 a_2 + a_1^2)$$

$$h_4 = a_0 \sum_{i_1=0}^{1} a_{i_1} \sum_{i_3=0}^{2-i_1} a_{i_3} a_{3-i_2-i_3}$$

$$= a_0 \left[a_0(a_0 a_3 + a_1 a_2 + a_2 a_1) + a_1(a_0 a_2 + a_1 a_1) \right]$$

$$= a_0(3 a_0 a_1 a_2 + a_0^2 a_3 + a_1^3).$$

Since a Riordan array $(d(t), h(t))$ is characterized by a pair series A-series and Z-series, which describe an entries of the array using a linear combination of other entries lie on the previous row, we establish the following notation.

Definition 4 *Let $(d(t), h(t)) \in \mathcal{R}$, and $A(t)$ and $Z(t)$ be the A-series (or polynomial) and Z-series (or polynomial). Then $(A(t), Z(t))$ is called the characteristic pair of the Riordan array $(d(t), h(t))$.*

At the end of this section, we establish a characterization of the generalized Sheffer-type polynomial sequence $\{\tilde{p}_n(x)\}$ defined by $(d(t), h(t))_{1/(1-t)}$ in Definition 3.

Theorem 5 *Let $(d(t), h(t)) = [d_{n,k}]_{n \geq k \geq 0}$ be a lower triangle array that satisfies condition (2.13) with $d(0) = 1$. Then, $\{\tilde{p}_n(x) = \sum_{k \geq 0} d_{n,k} x^k\}_{n \geq 0}$ is the corresponding Sheffer-type polynomial sequence satisfying*

$$x^n \tilde{p}_n \left(\frac{1}{x} \right) = (A(x))^n, \tag{2.31}$$

where $A(x)$ is the A-series, i.e., the GF of the A-sequence defined by condition (2.13).

Proof. Formula (2.31) is obviously true for $n = 1$. Indeed,

$$x\tilde{p}_1 \left(\frac{1}{x} \right) = d_{1,0} x + d_{1,1} = a_1 d_{0,0} x + a_0 d_{0,0} = A(x),$$

because $d_{0,0} = f_0 = 1$.

Since

$$
\begin{aligned}
x^n \tilde{p}_n \left(\frac{1}{x} \right) &= \sum_{k \geq 0} d_{n,k} x^{n-k} \\
&= \sum_{k \geq 0} d_{n,n-k} x^k = \sum_{k \geq 0} \sum_{j=0}^{k} d_{n-1,n-k+j-1} x^k \\
&= \sum_{j \geq 0} a_j x^j \sum_{k \geq j} d_{n-1,n-1-(k-j)} x^{k-j} \\
&= A(x) \sum_{k \geq 0} d_{n-1,n-1-k} x^k,
\end{aligned}
$$

where the third equation is from the substitution of condition (2.13). By using the induction assumption for $n - 1$, we immediately obtain formula for all $n \in \mathbb{N}$.

∎

3   Applications and examples

Condition (2.13) can be considered as a special (lattice) walk. A general lattice walk that starts at the origin $(0,0)$ and ends at (n, k) is of the form

$$(0,0) \to (1, k_1) \to (2, k_2) \to \cdots \to (n-1, k_{n-1}) \to (n, k).$$

The step $(i, k_i) \rightarrow (i+1, k_{i+1})$ is assigned the weight $w_{k_{i+1}-k_i}$. The weight of a walk is the product of the weights of its steps. For example, $(0,0) \rightarrow (1,2) \rightarrow (2,2) \rightarrow (3,3)$ has weight $w_2 w_0 w_1$, while the walk $(0,0) \rightarrow (1,-1) \rightarrow (2,-3) \rightarrow (3,-5)$ has weight $w_{-1} w_{-2} w_{-2}$. Let the step weights be integers and satisfy $w_k = 1$ for all $k \leq 1$ and $w_k = 0$ for all $k \geq 2$. We denote by $a_{n,k}$ the sum of the weights of all walks from $(0,0)$ to (n,k).

Theorem 6 *The weight summation array $[a_{n,k}]_{n \geq k \geq 0}$ defined as before is a Riordan array $(d(t), h(t))$, where $h(t)$ is the unique solution of the functional equation*

$$h(t) = tA(h(t)),$$

where $A(y) = \sum_{k \geq -1} w_{-k} x^{k+1}$.

Proof. It is easy to find out that

$$a_{1,k} = w_k \text{ and } a_{n,k} = \sum_{\ell=0}^{n-k} w_{1-\ell} a_{n-1,\ell-1+k}.$$

Thus the conclusion is a corollary of Theorem 2.

∎

[11] establishes a one-to-one correspondence between the weighted lattice walks and a subgroup of Riordan arrays whose entries share the divisibility proper displayed by the entries of the Pascal matrix. More precisely, [11] identifies a subgroup, called the hitting time subgroup, of the Riordan group G composed by the Riordan arrays $(d(t), h(t))$ for which $d(t) = th'(t)/h(t)$. This subgroup contains some important Riordan arrays such as the Pascal, Catalan and Motzkin triangles, as well as many other triangular arrays originated by lattice walks with weighted (or coloured) steps. [11] shows that all the Riordan arrays in the subgroup share this common divisibility property: if $D = \{d_{n,k}\}_{n,k \in N}$, then n divides $kd_{n,k}$ for every $0 < k < n$. Obviously this fact implies that n divides $d_{n,k}$ whenever n is a prime number. Thus, we can say that the collection of all elements $(d(t), h(t)) \in \mathcal{R}$ satisfy the divisibility property is the Hitting time subgroup, and the corresponding set of all Sheffer-type polynomial sequences generated by those $(d(t), h(t))$ are hitting time polynomial sequences.

Theorem 7 *Let $\mathcal{S}$ be the Sheffer group defined as in Definition 2, and let $\{p_n(x)\}_{n \geq 0}$ be a Sheffer-type polynomial sequence generated by $(d(t), h(t))$.*

Then there exists the following recursive relation of the coefficients of $p_n(x) = \sum_{k=0}^n d_{n,k} x^k / k!$:

$$d_{n,k} = \sum_{j \geq 0} a_j d_{n-1,k+j-1}, \qquad (3.32)$$

where

$$a_0 = h_1, \quad a_j = -h_1 \sum_{k=1}^j \bar{h}_{k+1} a_{j-k} \qquad (3.33)$$

and $\{\bar{h}_j\}_{j \geq 1}$ are $[t^j]\bar{h}(t)$. Here, $\bar{h}(t)$ is the compositional inverse of $h(t)$.

Proof. Theorem 1 implies the existence of the recursive relation (3.32) with respect to a certain A-sequence. A similar argument of the proof of (2.26) can be applied here to prove formula (3.33), which is omitted.

■

We now give some examples as the applications of Theorem 7.

Example 3.1 Bernoulli polynomial sequence is generated by $(d(t), h(t)) = (t/(e^t - 1), t)$. Thus the GF of the corresponding A-sequence is

$$A(y) = \frac{y}{\bar{h}(y)} = \frac{y}{y} = 1,$$

which implies the recurrence relation of the coefficients of the Bernoulli polynomials $B_n(x)/n! = \sum_{k=0}^n d_{n,k} x^k / k!$ satisfy $d_{n,k} = d_{n-1,k-1}$. It is indeed true because of $d_{n,k} = B_{n-k}/(n-k)!$, where B_j are the Bernoulli numbers.

Example 3.2 Consider the Angelescu polynomial sequence (see, for example, [18]) $\{A_n(x) = \sum_{k \geq 0} a_{n,k} x^k\}$ generated by the Riordn array $(d(t), h(t)) = (1/(1+t), t/(t-1))$. The corresponding A-series and Z-series are

$$A(y) = \frac{y}{\bar{h}(y)} = \frac{y}{y/(y-1)} = -1 + y$$

and

$$Z(y) = \frac{1/(1+t) - 1}{t/(1+t)} = -1,$$

respectively. Hence, we have the A-sequence $\{-1, 1\}$ and Z-sequence $\{-1\}$, which generates the recursive formula of the coefficients of the polynomials $A_n(x)$ as follows:

$$a_{n,k} = -a_{n-1,k-1} + a_{n-1,k}$$

for all $n \geq k \geq 1$ and $a_{n,0} = -a_{n-1,0}$. Thus, the recursive formula of the coefficients of $A_n(x)$ generates the following recursive formula for $A_n(x)$ by using the expression (2.16):

$$A_n(x) = 2(-1)^n + (1-x)A_{n-1}(x).$$

Acknowledgments

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