2013

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August 2, 2012

Abstract

Here presented is a unified expression of Stirling numbers and their generalizations by using generalized factorial functions and generalized divided difference. Three algorithms for calculating the Stirling numbers and their generalizations based on our unified form are also given, which include a comprehensive algorithm using the characterization of Riordan arrays.

AMS Subject Classification: 05A15, 65B10, 33C45, 39A70, 41A80.

Key Words and Phrases: Stirling numbers of the first kind, Stirling numbers of the second kind, factorial polynomials, generalized factorial, divided difference, k-Gamma functions, Pochhammer symbol and k-Pochhammer symbol.

1 Introduction

The classical Stirling numbers of the first kind and the second kind, denoted by $s(n,k)$ and $S(n,k)$, respectively, can be defined via a pair of inverse relations

$$[z]_n = \sum_{k=0}^{n} s(n,k) z^k, \quad z^n = \sum_{k=0}^{n} S(n,k)[z]_k, \tag{1.1}$$

with the convention $s(n,0) = S(n,0) = \delta_{n,0}$, the Kronecker symbol, where $z \in \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the falling factorial polynomials $[z]_n =$

$z(z-1)\cdots(z-n+1)$. $|s(n,k)|$ presents the number of permutations of n elements with k disjoint cycles while $S(n,k)$ gives the number of ways to partition n elements into k nonempty subsets. The simplest way to compute $s(n,k)$ is finding the coefficients of the expansion of $[z]_n$. [20] gives a simple way to evaluate $S(n,k)$ using Horner's method.

Another way of introducing classical Stirling numbers is via their exponential generating functions

$$\frac{(\log(1+x))^k}{k!} = \sum_{n\geq k} s(n,k)\frac{x^n}{n!}, \quad \frac{(e^x-1)^k}{k!} = \sum_{n\geq k} S(n,k)\frac{x^n}{n!}, \tag{1.2}$$

where $|x| < 1$ and $k \in \mathbb{N}_0$. In [26], Jordan said that, "Stirling's numbers are of the greatest utility. This however has not been fully recognized." He also thinks that, "Stirling's numbers are as important or even more so than Bernoulli's numbers."

Besides the above two expressions, the Stirling numbers of the second kind has the following third definition (see [11] and [26]), which is equivalent to the above two definitions but makes a more important rule in computation and generalization.

$$\begin{aligned} S(n,k) &:= \frac{1}{k!}\,\Delta^k z^n\big|_{z=0} = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n \\ &= \frac{1}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(k-j)^n. \end{aligned} \tag{1.3}$$

Expressions (1.1) - (1.3) will be our starting points to extend the classical Stirling number pair and the Stirling numbers.

Denote $\langle z\rangle_{n,\alpha} := z(z+\alpha)\cdots(z+(n-1)\alpha)$ for $n = 1, 2, \ldots$, and $\langle z\rangle_{0,\alpha} = 1$, where $\langle z\rangle_{n,\alpha}$ is called the generalized factorial of z with increment α. Thus, $\langle z\rangle_{n,-1} = [z]_n$ is the classical falling factorial with $[z]_0 = 1$, and $\langle z\rangle_{n,0} = z^n$. More properties of $\langle z\rangle_{n,\alpha}$ will be presented below.

With a closed observation, Stirling numbers of two kinds defined in (1.1) can be written as a unified Newton form:

$$\langle z\rangle_{n,-\alpha} = \sum_{k=0}^{n} S(n,k,\alpha,\beta)\langle z\rangle_{n,-\beta}, \tag{1.4}$$

with $S(n,k,1,0) = s(n,k)$, the Stirling numbers of the first kind and $S(n,k,0,1) = S(n,k)$. the Stirling numbers of the second kind. Inspired by (1.4) and many extensions of classical Stirling numbers or Stirling number pairs introduced by [6], [23], [46], [24], etc. Inspired with (1) and (2) in [24], the author defines a unified the following *generalized Stirling numbers* $S(n,k,\alpha,\beta,r)$ in [18].

Definition 1.1 *Let $n \in \mathbb{N}$ and $\alpha,\beta,r \in \mathbb{R}$. A generalized Stirling number denoted by $S(n,k,\alpha,\beta,r)$ is defined by*

$$\langle z\rangle_{n,-\alpha} = \sum_{k=0}^{n} S(n,k,\alpha,\beta,r)\langle z-r\rangle_{k,-\beta}. \tag{1.5}$$

In particular, if $(\alpha,\beta,r) = (1,0,0)$, $S(n,k,1,0,0)$ is reduced to the unified form of Classical Stirling numbers defined by (1.4).

Each $\langle z\rangle_{n,-\alpha}$ does have exactly one such expansion (1.5) for any given z. Since *deg* $\langle z-r\rangle_{k,-\beta} = k$ for all k, which generates a graded basis for $\Pi \subset \mathbb{F} \to \mathbb{F}$, the linear spaces of polynomials in one real (when $\mathbb{F} = \mathbb{R}$) or complex (when $\mathbb{F} = \mathbb{C}$), in the sense that, for each n, $\{\langle z-r\rangle_{n,-\beta}\}$ is a basis for $\Pi_n \subset \Pi$, the subspace of all polynomials of degree $< n$. In other wards, the column map

$$W_z : \mathbb{F}_0^N \to \Pi : s \mapsto \sum_{k\geq 0} S(n,k,\alpha,\beta,r)\langle z\rangle_{k,-\beta},$$

from the space $\mathbb{F}_0^N$ of scalar sequences with finitely many nonzero entries to the space Π is one-to-one and onto, hence invertible. In particular, for each $n \in \mathbb{N}$, the coefficient $c(n)$ in the Newton form (1.5) for $\langle z\rangle_{n,-\alpha}$ depends linearly on $\langle z\rangle_{n,-\alpha}$, i.e., $\langle z\rangle_{n,-\alpha} \mapsto s(n) = (W_z^{-1}\langle z\rangle_{n,-\alpha})(n)$, the set of $S(n,k,\alpha,\beta,r)$, is a well-defined linear functional on Π, and vanishes on $\Pi_{<n-1}$.

Similarly to (1.1), from Definition 1.1 a Stirling-type pair $\{S^1, S^2\} = \{S^1(n,k),\ S^2(n,k)\} \equiv \{S(n,k;\ \alpha,\beta,r),\ S(n,k;\beta,\alpha,-r)\}$ (see also in [24]) can be defined by the inverse relations

$$\begin{aligned}
\langle z\rangle_{n,-\alpha} &= \sum_{k=0}^{n} S^1(n,k)\langle z-r\rangle_{k,-\beta} \\
\langle z\rangle_{n,-\beta} &= \sum_{k=0}^{n} S^2(n,k)\langle z+r\rangle_{k,-\alpha},
\end{aligned} \tag{1.6}$$

where $n \in \mathbb{N}$ and the parameter triple $(\alpha,\beta,r) \neq (0,0,0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. Hence, we may call S^1 and S^2 an (α,β,r) and a $(\beta,\alpha,-r)-$ pair. Obviously,

$$S(n,k;0,0,1) = \binom{n}{k}$$

because $z^n=\sum_{k=0}^n\binom{n}{k}(z-1)^k$. In addition, the classical Stirling number pair $\{s(n,k),S(n,k)\}$ is the $(1,0,0)-$ pair $\{S^1,S^2\}$, namely,

$$s(n,k)=S^1(n,k;1,0,0)\quad S(n,k)=S^2(n,k;1,0,0).$$

For brevity, we will use $S(n,k)$ to denote $S(n,k,\alpha,\beta,r)$ if there is no need to indicate α, β, and r explicitly. From (1.5), one may find

$$S(0,0)=1,\quad S(n,n)=1,\quad S(1,0)=r,\quad \textit{and}\quad S(n,0)=\langle r\rangle_{n,-\alpha}. \tag{1.7}$$

Evidently, substituting $n=k=0$ into (1.5) yields the first formula of (1.7). Comparing the coefficients of the highest power terms on the both sides of (1.5), we obtain the second formula of (1.7). Let $n=1$ in (1.5) and noting $S(1,1)=1$, we have the third formula. Finally, substituting $z=r$ in (1.5), one can establish the last formula of (1.7). The numbers $\sigma(n,k)$ discussed by Doubilet et al. in [15] and by Wagner in [47] is $k!S(n,k;0,1,0)$. More special cases of the generalized Stirling numbers and Stirling-type pairs defined by (1.5) or (1.6) are surveyed below in Table 1.

The classical falling factorial polynomials $[z]_n=z(z-1)\cdots(z-n+1)$ and classical rising factorial polynomials $[z]^n=z(z+1)\cdots(z+n-1)$, $z\in\mathbb{C}$ and $n\in\mathbb{N}$, can be unified to the expression

$$\langle z\rangle_{n,\pm1}:=z(z\pm1)\cdots(z\pm(n-1)),$$

using the *generalized factorial polynomial* expression

$$\langle z\rangle_{n,k}:=z(z+k)\cdots(z+(n-1)k)=\langle z+(n-1)k\rangle_{n,-k}\quad (z\in\mathbb{C},n\in\mathbb{N}). \tag{1.8}$$

Thus $\langle z\rangle_{n,1}=[z]^n$ and $\langle z\rangle_{n,-1}=[z]_n$.

In next section, we will present the unified expression and some properties of the generalized Stirling numbers of integer orders. Two algorithms based on the unified expression will be given. The third algorithm of the computation of the generalized Stirling numbers, including the classical Stirling numbers as a special case, will be shown using the characterizations of their Riordan arrays in the last section.

(α,β,r)	$S(n,k)$	*Name of Stirling numbers*
$(-1,1,0)$	$n!\binom{n-1}{k-1}/k!$ $(-1)^{n-k}n!\binom{n-1}{k-1}/k!$	*Lah number pair*[25]
$(-1,0,0)$	$\lvert s(n,k)\rvert$ $(-1)^{n-k}S(n,k)$	*signless Stirling numbers*[37]
$(1,\theta,0)(\theta\neq 0)$	$S(n,k,1,\theta,0)$ $S(n,k,\theta,1,0)$	*Carlitz's degenerate Stirling number pair*[5]
$(1,0,-\lambda)$	$S(n,k,1,0,-\lambda)$ $S(n,k,0,1,\lambda)$	*Carlitz's weighted Stirling number pair*[6]
$(1,\theta,-\lambda)$	$S(n,k,1,\theta,-\lambda)$ $S(n,k,\theta,1,\lambda)$	*Howard's weighted degenerate Stirling number pair*[23]
$(0,1,-a+b)$	$S(n,k,0,1,-a+b)$ $S(n,k,1,0,-b+a)$	*Gould − Hopper's non − central Lah number pair*[17]
$(1/s,1,-a+b)$	$S(n,k,1/s,1,-a+b)$ $S(n,k,1,1/s,-b+a)$	*Charalambides − Koutras's non− central C number pair*[7, 8]
$(1,0,b-a)$	$S(n,k,1,0,b-a)$ $S(n,k,0,1,a-b)$	*Riordan's non − central Stirling number pair*[34]
$(\alpha,\beta,0)$	$A_{\alpha\beta}(r,m)$ $B_{\alpha\beta}(r,m)$	*Tsylova's Stirling number pair*[46]
(α,β,r)	$S(n,k,\alpha,\beta,r)$ $S(n,k,\beta,\alpha,-r)$	*Hsu − Shiue's Stirling number pair*[24]
$(1,x,0)$	$a_{nk}(x)$	*Todorov's Stirling numbers*[45]
$(-1/r,1,0)$	$B(n,r,k)$	*Ahuja − Enneking's associated Lah numbers*[31]
$(-1,0,r)$	$S(n-r,k-r,-1,0,r)$	*Broder's r − Stirling numbers*[3]

Table 1. Some generalized Stirling Numbers and Stirling Number pairs

2 Expressions of generalized Stirling numbers

First, we give an equivalent form of the generalized Stirling numbers $S(n,k)$ defined by (1.5) by using the *generalized difference operator* in terms of β ($\beta\neq 0$) defined by

$$\Delta_\beta^k f=\Delta_\beta(\Delta_\beta^{k-1}f)\quad (k\geq 2)\quad and\quad \Delta_\beta f(t):=f(t+\beta)-f(t). \tag{2.1}$$

It can be seen that $\Delta_\beta^k \langle z\rangle_{j,-\beta}\Big|_{z=0} = \beta^k k!\delta_{k,j}$, where $\delta_{k,j}$ is the Kronecker delta symbol; i.e., $\delta_{k,j} = 1$ when $k = j$ and 0 otherwise. Evidently, from (**??**) there holds

$$\Delta_\beta^k \langle z\rangle_{j,-\beta}\big|_{z=0} = \Delta_\beta^k \beta^j \left[\frac{t}{\beta}\right]_j\Bigg|_{z=0} = \beta^j \Delta^k [t]_j\big|_{z=0} = \beta^k k!\delta_{k,j}. \tag{2.2}$$

Denote the divided difference of $f(t)$ at $t+i$, $i = 0, 1, \ldots, k$, by $f[t, t+1, \ldots, t+k]$, or $[t, t+1, \ldots, t+k]f(t)$. Using the well-known forward difference formula, it is easy to check that

$$\frac{1}{k!}\Delta^k f(t) = f[t, t+1, \ldots, t+k] = [t, t+1, \ldots, t+k]f(t)$$

and

$$\frac{1}{\beta^k k!}\Delta_\beta^k f(t) = f[t, t+\beta, t+2\beta, \ldots, t+k\beta] = [t, t+\beta, \ldots, t+k\beta]f(t).$$

We now give the following definition of the *generalized divided differences.*

Definition 2.1 *We define* $\underline{\triangle}_\beta^k f(t)$ *by*

$$\underline{\triangle}_\beta^k f(t) = \begin{cases} \frac{1}{\beta^k k!}\Delta_\beta^k f(t) = f[t, t+\beta, \ldots, t+k\beta] & if\ \beta \neq 0 \\ \frac{1}{k!}D^k f(t) & if\ \beta = 0 \end{cases}, \tag{2.3}$$

where $\Delta_\beta^k f(t)$ *is shown in (2.1),* $f[t, t+\beta, \ldots, t+k\beta] \equiv [t, t+\beta, \ldots, t+k\beta]f$ *is the* k*th divided difference of* f *in terms of* $\{t, t+\beta, \ldots, t+k\beta\}$*, and* $D^k f(t)$ *is the* k*th derivative of* $f(t)$*.*

From the well-known formula

$$f[t, t+\beta, t+2\beta, \ldots, t+k\beta] = \frac{D^k f(\xi)}{k!},$$

where ξ is between t and $t+k\beta$, it is clear that

$$D^k f(t) = \lim_{\beta\to 0} \frac{1}{\beta^k}\Delta_\beta^k f(t), \tag{2.4}$$

which shows the generalized divided difference is well defined.

We now give a unified expression of the generalized Stirling numbers in terms of the the generalized divided differences.

Theorem 2.2 *Let $n, k \in \mathbb{N}_0$ and the parameter triple $(\alpha, \beta, r) \neq (0,0,0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. For the generalized Stirling numbers defined by (1.5), there holds*

$$S(n,k,\alpha,\beta,r) = \underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r}$$
$$= \begin{cases} \frac{1}{\beta^k k!}\ \Delta_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} = [r, r+\beta, \ldots, r+k\beta]\langle z\rangle_{n,-\alpha} & if\ \beta \neq 0 \\ \frac{1}{k!}\ D^k \langle z\rangle_{n,-\alpha}\big|_{z=r} & if\ \beta = 0. \end{cases} \tag{2.5}$$

In particular, for the generalized Stirling number pair defined by (1.6), we have the expressions

$$S^1(n,k) \equiv S^1(n,k,\alpha,\beta,r) = \underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r}$$
$$= \begin{cases} \frac{1}{\beta^k k!}\ \Delta_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} = [r, r+\beta, \ldots, r+k\beta]\langle z\rangle_{n,-\alpha}, & if\ \beta \neq 0 \\ \frac{1}{k!} D^k \langle z\rangle_{n,-\alpha}\big|_{z=r}, & if\ \beta = 0 \end{cases} \tag{2.6}$$
$$S^2(n,k) \equiv S^2(n,k,\beta,\alpha,-r) = \underline{\triangle}_\alpha^k \langle z\rangle_{n,-\beta}\Big|_{z=-r}$$
$$= \begin{cases} \frac{1}{\alpha^k k!}\ \Delta_\alpha^k \langle z\rangle_{n,-\beta}\big|_{z=-r} = [-r, -r+\alpha, \ldots, -r+k\alpha]\langle z\rangle_{n,-\beta}, & if\ \alpha \neq 0 \\ \frac{1}{k!} D^k \langle z\rangle_{n,-\beta}\big|_{z=-r}, & if\ \alpha = 0 \end{cases} \tag{2.7}$$

Furthermore, if $(\alpha, \beta, r) = (1,0,0)$, then (2.5) is reduced to the classical Stirling numbers of the first kind defined by (1.1) with the expression

$$s(n,k) = S(n,k,1,0,0) = \frac{1}{k!}\ D^k [z]_n\big|_{z=0}.$$

If $(\alpha, \beta, r) = (0,1,0)$, then (2.5) is reduced to the classical Stirling numbers of the second kind shown in (1.3) with the following divided difference expression form:

$$S(n,k) = S(n,k,0,1,0) = [0,1,2,\ldots,k]z^n\big|_{z=0}. \tag{2.8}$$

Proof. If $\beta \neq 0$, taking forward kth differences in terms of β on the both sides of (1.5) and letting $z = r$, from formula (2.2) we have

$$\Delta_\beta^k \langle z\rangle_{n,-\alpha}\big|_{z=r} = \Delta_\beta^k \sum_{j=0}^n S(n,j,\alpha,\beta,r)\langle z-r\rangle_{j,-\beta}\big|_{z=r}$$
$$= \sum_{j=0}^n S(n,j,\alpha,\beta,r)\ \Delta_\beta^k \langle z-r\rangle_{j,-\beta}\big|_{z=r} = \beta^k k! S(n,k,\alpha,\beta,r).$$

which implies the expression of $S(n,k,\alpha,\beta,r)$ in (1.5) for the case of $\beta\neq 0$. If $\beta=0$, we take kth derivative in terms of z on the both sides of (1.5) and let $z=r$, which yields

$$D^k\langle z\rangle_{n,-\alpha}\Big|_{z=r}=\Delta^k_\beta\sum_{j=0}^n S(n,j,\alpha,\beta,r)(z-r)^j\Big|_{z=r}=k!S(n,k,\alpha,0,r),$$

completing the proof of (2.5).

Similarly, if $\beta\neq 0$, taking forward kth difference in terms of β and α on the both sides of two equations of (1.6), respectively, and letting $z=r$ and $z=-r$, respectively, we immediately obtain

$$\Delta^k_\beta\langle z\rangle_{n,\alpha}\Big|_{z=r}=\beta^k k!S^1(n,k,\alpha,\beta,r)$$
$$\Delta^k_\alpha\langle z\rangle_{n,\beta}\Big|_{z=-r}=\alpha^k k!S^1(n,k,\beta,\alpha,-r).$$

which imply two first formulas of (2.6) for $\beta\neq 0$. Two formulas for the case of $\beta=0$ in (2.6) can be obtained by using kth differentiation and a similar argument in the proof of their unified form (2.5).

■

The following corollary is obvious due to the expansion formula of the divided differences generated from their definition.

Corollary 2.3 *Let $n,k\in\mathbb{N}_0$ and the parameter triple $(\alpha,\beta,r)\neq(0,0,0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. If $\beta\neq 0$, for the generalized Stirling numbers defined by (1.5), there holds*

$$S(n,k)\equiv S(n,k,\alpha,\beta,r)=\frac{1}{\beta^k k!}\sum_{j=0}^n(-1)^j\binom{k}{j}\langle r+(k-j)\beta\rangle_{n,-\alpha}\quad(n\neq 0),\tag{2.9}$$

and $S(0,k)=\delta_{0k}$.

Remark 2.1 It can be seen from (2.9) that

$$S(n,0)\equiv S(n,0,\alpha,\beta,r)=\langle r\rangle_{n,-\alpha},\tag{2.10}$$

which is independent of β and has been shown in (1.7). The difference is deriving (2.10) from (2.9) needs $(\alpha,r)\neq(0,0)$ when $\beta=0$. However, we have seen from (1.7) that the condition is not necessary. Another way to derive (2.10) using the characterization of the Riordan arrays of the generalized Stirling numbers will be presented in the Algorithm 3.3 in Section 4.

Remark 2.2 If $\alpha\beta \neq 0$, by taking the nth forward differences in terms of α and β on the both sides of two equations of (1.6), respectively, one may obtain identities

$$\begin{aligned} n!\alpha^n &= \sum_{k=0}^{n} S^1(n,k)\ \Delta_\alpha^n \langle z-r\rangle_{k,-\beta}\big|_{z=0} \\ n!\beta^n &= \sum_{k=0}^{n} S^2(n,k)\ \Delta_\beta^n \langle z+r\rangle_{k,-\alpha}\big|_{z=0}. \end{aligned}$$

The above two identities can be unified to be one:

$$n!\alpha^n = \sum_{k=0}^{n} S(n,k,\alpha,\beta,r)\Delta_\alpha^n \langle z-r\rangle_{k,-\beta}\big|_{z=0}.$$

When $\alpha = 0$, the above identity turns to

$$n! = \sum_{k=0}^{n} S(n,k,0,\beta,r)\ D^n \langle z-r\rangle_{k,-\beta}\big|_{z=0}.$$

Remarker 2.3 There exists another expression of the divided difference $\underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r}$ in terms of *Peano kernel of B-spline*. Assume that the set $\tau := \{t, t+\beta, \ldots, t+k\beta\}$ lies in the interval $[a,b]$. Then on the interval, we have Taylor's identity

$$\langle z\rangle_{n,-\alpha} = \sum_{j<k} \frac{(z-a)^j}{j!}\ D^j \langle z\rangle_{n,-\alpha}\big|_{z=a} + \int_a^b \frac{(x-y)_+^{k-1}}{(k-1)!} \langle y\rangle_{n,-\alpha} dy.$$

If $\beta > 0$, then $\underline{\triangle}_\beta^k$ is a weighted sum of values of derivatives of order $< k$, hence commutes with the integral in the above Taylor's expansion, which annihilates any polynomial of degree $< k$. Therefore,

$$\underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} = \int_a^b \frac{M(y|\tau)}{k!} \langle y\rangle_{n,-\alpha} dy,$$

where

$$M(y|\tau) := k[r, r+\beta, \ldots, r+k\beta](\cdot - y)_+^{k-1}$$

is the Curry-Schoenberg B-spline (see [12]) with the knot set τ and normalized to have integral 1. In particular,

$$S(n,n,\alpha,\beta,r) = \underline{\triangle}_\beta^k \langle z\rangle_{n,-\alpha}\Big|_{z=r} = \int_a^b M(y|r, r+\beta,\ldots,r+n\beta)dy = 1.$$

We now present two algorithms for calculating generalized Stirling numbers. If $\beta \neq 0$, we denote

$$\underline{\triangle}_\beta^j f(t+\ell\beta) := f[t, t+\ell\beta, t+(\ell+1)\beta,\ldots,t+j\beta] \tag{2.11}$$

Thus, from (2.5) in Theorem 2.2, based on the recursive definition of the divided difference with respect to β (see Definition 2.1)

$$\underline{\triangle}_\beta^j f(t+\ell\beta) = \frac{1}{j\beta}(\underline{\triangle}_\beta^{j-1} f(t+(\ell+1)\beta) - \underline{\triangle}_\beta^{j-1} f(t+\ell\beta)), \tag{2.12}$$

we obtain an algorithm shown below.

Algorithm 2.4 *This algorithm of evaluating the generalized Stirling numbers is based on the construction of the following lower triangle array by using (2.11) and (2.12).*

$$\begin{array}{lllll}
\langle z\rangle_{n,-\alpha}\big|_{z=r} & & & & \\
\langle z+\beta\rangle_{n,-\alpha}\big|_{z=r} & \underline{\triangle}_\beta\langle z\rangle_{n,-\alpha}\Big|_{z=r} & & & \\
\langle z+2\beta\rangle_{n,-\alpha}\big|_{z=r} & \underline{\triangle}_\beta\langle z+\beta\rangle_{n,-\alpha}\Big|_{z=r} & \underline{\triangle}_\beta^2\langle z\rangle_{n,-\alpha}\Big|_{z=r} & & \\
\vdots & \vdots & \vdots & \ddots & \\
\langle z+k\beta\rangle_{n,-\alpha}\big|_{z=r} & \underline{\triangle}_\beta\langle z+(k-1)\beta\rangle_{n,-\alpha}\Big|_{z=r} & \underline{\triangle}_\beta^2\langle z+(k-2)\beta\rangle_{n,-\alpha}\Big|_{z=r} & \cdots & \underline{\triangle}_\beta^k\langle z\rangle_{n,-\alpha}\Big|_{z=r}
\end{array}$$

Table 2. The generalized Stirling numbers

Thus, the diagonal of the above lower triangle array gives $S(n,i,\alpha,\beta,r) = \underline{\triangle}_\beta^i\langle z\rangle_{n,-\alpha}\Big|_{z=r}$ *for* $i = 0,1,\ldots,k$.

Example 2.1 We now use Algorithm 2.4 shown in Table 2 to evaluate the classical Stirling numbers of the second kind $S(4,k) = S(4,k,0,1,0)$ $(k = 1,2,3,4)$, which are re-expressed by (2.8). Thus,

$$\begin{array}{lllll}
0 & & & & \\
1 & 1 & & & \\
2^4 = 16 & 15 & 7 & & \\
3^4 = 81 & 65 & 25 & 6 & \\
4^4 = 256 & 175 & 55 & 10 & 1
\end{array}$$

From the diagonal of the above lower triangular matrix, we may read $S(4,0)=0$, $S(4,1)=1$, $S(4,2)=7$, $S(4,3)=6$, and $S(4,4)=1$. Meanwhile, the subdiagonal gives $S(5,1)=1$, $S(5,2)=15$, $S(5,3)=25$, and $S(5,4)=10$.

Example 2.2 For the Howard's weighted degenerate Stirling numbers $S(4,k)=S(4,k,1,1,-1)$, a similar argument of Example 2.1 yields

$$\begin{array}{lllll}
\langle z\rangle_{4,-1}\big|_{z=-1}=24 & & & & \\
\langle z+1\rangle_{4,-1}\big|_{z=-1}=0 & -24 & & & \\
\langle z+2\rangle_{4,-1}\big|_{z=-1}=0 & 0 & 12 & & \\
\langle z+3\rangle_{4,-1}\big|_{z=-1}=0 & 0 & 0 & -4 & \\
\langle z+4\rangle_{4,-1}\big|_{z=-1}=0 & 0 & 0 & 0 & 1
\end{array}$$

Thus, $S(4,0)=24$, $S(4,1)=-24$, $S(4,2)=12$, $S(4,3)=-4$, and $S(4,4)=1$.

Example 2.3 For the Howard's weighted degenerate Stirling numbers $S(4,k)=S(4,k,1,2,-1)$, using Algorithm 2.4, we obtain $S(4,0)=24$, $S(4,1)=-12$, $S(4,2)=3$, $S(4,3)=2$, and $S(4,4)=1$ reading from the following table.

$$\begin{array}{lllll}
\langle z\rangle_{4,-1}\big|_{z=-1}=24 & & & & \\
\langle z+2\rangle_{4,-1}\big|_{z=-1}=0 & -12 & & & \\
\langle z+4\rangle_{4,-1}\big|_{z=-1}=0 & 0 & 3 & & \\
\langle z+6\rangle_{4,-1}\big|_{z=-1}=120 & 60 & 15 & 2 & \\
\langle z+8\rangle_{4,-1}\big|_{z=-1}=840 & 360 & 75 & 10 & 1
\end{array}$$

Remark 2.4 Obviously, Algorithm 2.4 is not limited to the case of $\beta\neq 0$ since when $\beta=0$, $\underline{\triangle}_{\beta}^{k}\langle z\rangle_{n,-\alpha}\Big|_{z=r}$ $(k=0,1,\ldots,n)$ on the diagonal of the lower triangle matrix in Table 1 are simply the $1/k!$ multiply of the derivatives $D^k\langle z\rangle_{n,-\alpha}\big|_{z=r}$ (see Theorem 2.2).

Another algorithm based on the Horner's method can be established using a modified argument in the computation of the classical Stirling numbers of the second kind shown in [20]. More precisely, we have the following algorithm.

Algorithm 2.5 *First, we may write the generalized Stirling numbers $S(n,k)=S(n,k,\alpha,\beta,r)$ defined by (1.5) (see Definition 1.1) as*

$$\begin{aligned}
& \langle z\rangle_{n,-\alpha}=\sum_{k=0}^{n}S(n,k)\langle z-r\rangle_{k,-\beta} \\
= \ & S(n,0)+(z-r)(S(n,1)+(z-r-\beta)(S(n,2)+(z-r-2\beta)(S(n,3)+\cdots \\
& (z-r-(n-1)\beta)S(n,n)))).
\end{aligned} \tag{2.13}$$

Secondly, Use synthetic division to obtain $\langle z\rangle_{n,-\alpha}/(z-r)$*, a polynomial of degree* $\le n-1$*, with the remainder* $S(n,0)$*. Then, evaluate* $(\langle z\rangle_{n,-\alpha}/(z-r)-S(n,0))/(z-r-\beta)$ *to find the quotient polynomial of degree* $\le n-2$ *as well as the remainder* $S(n,1)$*. Continue this process until a polynomial of degree* ≤ 1 *left, which is* $S(n,n-1)+(z-r-(n-1)\beta)S(n,n)$*. A equivalent description of the above process can be presented as follows. Use Horner's method to find*

$$f(r) \equiv \langle z\rangle_{n,-\alpha} = S(n,0) + (z-r)f_1(z), \quad deg\ f_1(z) \le n-1,$$

where the remainder is $S(n,0))$*. Then, use Horner's method again to evaluate*

$$f_1(z) = S(n,1) + (z-r-\beta)f_2(z), \quad deq\ f_2(z) \le d-2,$$

which generates the remainder $S(n,1)$*. Continue the process and finally obtain*

$$f_{n-1} = S(n,n-1) + (z-r-(n-1)\beta)S(n,n).$$

In short, we obtain $S(n,0) = \langle z\rangle_{n,-\alpha}\big|_{z=r}$*,* $S(n,1) = (\langle z\rangle_{n,-\alpha} - S(n,0))/(z-r)\big|_{z=r+\beta}$*, etc.*

Algorithm 2.5 can be demonstrated by the following examples.

Example 2.4 For the classical Stirling numbers of the second kind in the case of $n=5$ and $(\alpha,\beta,r)=(0,1,0)$, from expansion (2.13) we have

$$z^5 = S(5,0)+z(S(5,1)+(z-1)(S(5,2)+(z-2)(S(5,3)+(z-3)(S(5,4)+(z-4)S(5,5))))),$$

which implies $S(5,0)=0$ and

$$z^4 = S(5,1)+(z-1)(S(4,2)+(z-2)(S(4,3)+(z-3)(S(4,4)+(z-4)S(5,5)))).$$

Thus, we may use the following division to evaluate $S(5,k)$ $(k=1,2,3,4,5)$.

1	1	0	0	0	0
		1	1	1	1
2	1	1	1	1	**1**
		2	6	14	
3	1	3	7	**15**	
		3	18		
4	1	6	**25**		
		4			
	1	**10**			

Hence, $S(5,1) = 1$, $S(5,2) = 15$, $S(5,3) = 25$, $S(5,4) = 10$, and $S(5,5) = 1$.

From (2.13) we also immediately know that $S(n,n) = 1$ because it is the coefficient of z^n on the right-hand side while the coefficient on the left-hand side is 1.

Example 2.5 Consider the Howard's weighted degenerate Stirling numbers with $n = 4$ and $(\alpha, \beta, r) = (1, 2, -1)$, we now calculate $S(n,k) = S(n,k,1,2,-1)$ using Horner's method based on expansion (2.13), which can be reduced to

$$
\begin{aligned}
& \langle z \rangle_{4,-1} = z^4 - 6z^3 + 11z^2 - 6z \\
= \quad & S(4,0) + (z+1)(S(4,1) + (z-1)(S(4,2) + (z-3)(S(4,3) + (z-5)(S(4,4))))).
\end{aligned}
$$

Therefore, we have synthetic division scheme as

-1	1	-6	11	-6	0
		-1	7	-18	24
1	1	-7	18	-24	**24**
		1	-6	12	
3	1	-6	12	**-12**	
		3	-9		
5	1	-3	**3**		
		5			
	1	**2**			

Thus, $S(4,0)=24$, $S(4,1)=-12$, $S(4,2)=3$, $S(4,3)=2$, and $S(4,4)=1$, which yield the same results obtained in Example 2.3 by a different method.

Similarly, for the case of $n=5$ and $(\alpha,\beta,r)=(1,2,-1)$, we may establish the following expansion

$$
\begin{aligned}
& \langle z\rangle_{5,-1}=z^5-10z^4+35z^3-50z^2+24z \\
= \; & S(5,0)+(z+1)(S(5,1)+(z-1)(S(5,2)+(z-3)(S(5,3) \\
& +(z-5)(S(5,4)+(z-7)S(5,5))))).
\end{aligned}
$$

Thus, we may also read $S(5,0)=-120$, $S(5,1)=60$, $S(5,2)=-15$, $S(5,3)=5$, $S(5,4)=5$, and $S(5,5)=1$ from the table:

-1	1	-10	35	-50	24	0
		-1	11	-46	96	-120
1	1	-11	46	-96	120	**-120**
		1	-10	36	-60	
3	1	-10	36	-60	**60**	
		3	-21	45		
5	1	-7	15	**-15**		
		5	-10			
7	1	-2	**5**			
		7				
	1	**5**				

Let $\{t_j\}_{j=1}^n$ be a strictly increasing n-sequence, and let $\sigma = \{\sigma(j)\}_{j=1}^k$ be any strictly increasing integer sequence in $[1, n]$. There holds the following well-known refinement formula of divided difference (see, for example, [2])

$$f[t, t - t_{\sigma(1)}, \ldots, t - t_{\sigma(k)}] = \sum_{j=\sigma(1)-1}^{\sigma(k)-k} c(j) f[t, t_{j+1}, \ldots, t - t_{j+k}],$$

where $c(j) = c_{t,\sigma} > 0$. Using this refinement formula one may obtain the refinement formula of the generalized Stirling numbers defined by (1.5).

Proposition 2.6 *Let $n, k \in \mathbb{N}_0$ and the parameter triple $(\alpha, \beta, r) \neq (0, 0, 0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. Then there holds refinement formula,*

$$\underline{\triangle}(\beta_{\sigma(1:k)}) \left. \langle z \rangle_{n,-\alpha} \right|_{z=r} = \sum_{j=\sigma(1)-1}^{\sigma(k)-k} c(j) \underline{\triangle}(\beta_{j+1:j+k}) \left. \langle z \rangle_{n,-\alpha} \right|_{z=r},$$

where

$$\underline{\triangle}(\beta_{\ell:j}) f := f[t, t + \ell\beta, t + (\ell + 1)\beta, \ldots, t + j\beta]$$

3 A comprehensive method of computation of generalized Stirling numbers

Let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[[t;\{c_k\}]]$ or $\mathbb{C}[[t;\{c\}]]$ (where $c = (c_0, c_1, c_2, \ldots)$ satisfies $c_0 = 1$, $c_k > 0$ for all $k = 1, 2, \ldots$); the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k/c_k$, is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s.'s $f(t)$ for which the compositional inverse $\overline{f}(t)$ exists such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$. We call the element $g \in \mathcal{F}$ with the form $g(x) = \sum_{k\geq 0} \frac{x^k}{c_k}$ a generalized power series (GPS) associated with $\{c_n\}$ or, simply, a (c)-GPS, and $\mathcal{F}$ the GPS set associated with $\{c_n\}$. In particular, when $c = (1, 1, \ldots)$, the corresponding $\mathcal{F}$ and $\mathcal{F}_r$ denote the classical formal power series and the classical formal power series of order r, respectively.

In the recent literature, special emphasis has been given to the concept of *Riordan arrays*, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [42]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli [43, 44], on subgroups of the Riordan group in Peart and Woan [33] and Shapiro [39], on some characterizations of Riordan matrices in Rogers [36], Merlini et al. [28] and He et al. [21], and on many interesting related results in Cheon et al. [9, 10], He et al. [19], Nkwanta [32], Shapiro [40, 41], and so forth. We now generalize the Riordan arrays associated with classical power series to those associated with (c)-GPS, where $c = \{c_k = k!\}_{k\geq 0}$. The Riordan arrays associated with other (c)-GPS can be found in author's later paper. More precisely, let $c = \{c_k = k!\}_{k\geq 0}$. The (c)-Riordan array generated by $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$ with respect to $\{c_k\}_{k\geq 0}$ is an infinite complex matrix $[d_{n,k}]_{0\leq k\leq n}$, whose bivariate generating function has the form

$$F(t,x) = \sum_{n,k} d_{n,k} \frac{t^n}{n!} x^k = d(t) e^{xh(t)}, \tag{3.1}$$

which is called a *Sheffer type Riordan array.*

Thus, the (n,k) entry of (c)-Riordan array $[d_{n,k}]$ is

$$d_{n,k} = \left[\frac{t^n}{n!}\right] d(t) \frac{(h(t))^k}{k!} = [t^n] \frac{n!}{k!} d(t)(h(t))^k \tag{3.2}$$

for all $0 \leq k \leq n$ and $d_{n,k} = 0$ otherwise. It is easy to see that a lower triangular array $[d_{n,k}]$ is a (c)-Riordan array if and only if the array

$(k!d_{n,k}/n!)$ is a (1)-Riordan array, i.e., a classical Riordan array. Evidently, $[d_{n,k}] = (d(t), h(t))$ can be written as

$$[d_{n,k}] = D[[t^n]d(t)(h(t))^k]_{n\geq k\geq 0}D^{-1}, \tag{3.3}$$

where $D = diag(1, 1, 2!, \ldots)$.

Rogers [36] introduced the concept of the *A-sequence* for the classical Riordan arrays; Merlini et al. [28] introduced the related concept of the *Z-sequence* and showed that these two concepts, together with the element $d_{0,0}$, completely characterize a proper classical Riordan array. In [21], Sprugnoli and the author consider the characterization of Riordan arrays, their multiplications, and their inverses by means of the A- and Z-sequences.

In [36], Rogers states that for every proper Riordan array $D = (d(t),\ h(t))$ there exists a sequence $A = (a_k)_{k\in\mathbb{N}}$ such that for every $n, k \in \mathbb{N}$ we have:

$$\begin{aligned} & [t^{n+1}]d(t)(h(t))^{k+1} \\ = \; & a_0[t^n]d(t)(h(t))^k + a_1[t^n]d(t)(h(t))^{k+1} + a_2[t^n]d(t)(h(t))^{k+2} + \cdots \\ = \; & \sum_{j=0}^{\infty} a_j[t^n]d(t)(h(t))^{k+j} \end{aligned} \tag{3.4}$$

where the sum is actually finite since $d_{n,k} = 0,\ \forall k > n$. We can reformulate it to the generalized (c)-Riordan array as follows.

Theorem 3.1 *An infinite lower triangular array $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ is a (c)-Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ relation*

$$\frac{c_{k+1}}{c_{n+1}}d_{n+1,k+1} = \frac{c_0}{c_n}a_0d_{n,k} + \frac{c_1}{c_n}a_1d_{n,k+1} + \frac{c_2}{c_n}a_2d_{n,k+2} + \cdots = \sum_{j=0}^{\infty}\frac{c_{k+j}}{c_n}a_jd_{n,k+j} \tag{3.5}$$

holds. In addition, the generating function $A(t)$ of $A-$ sequence is uniquely determined by $tA(h(t)) = h(t)$.

Proof. Using expression (3.2) and expression (3.4), we obtain formula (3.5) immediately. From a similar argument of the proof of Theorem 2.1 in [21], we have $tA(h(t)) = h(t)$, where $A(t)$ is the generating function of $A-$ sequence.

The sequence $A = (a_n)_{n\in\mathbb{N}_0}$ is the A-sequence of the Riordan array $D = (d(t),\ h(t))$ and it only depends on $h(t)$. In fact, as we have shown during the proof of the theorem, we have:

$$h(t) = tA(h(t)) \qquad \text{or} \qquad A(y) = \left[\frac{h(t)}{t} \;\middle|\; y = h(t)\right] = \left[\frac{y}{t} \;\middle|\; y = h(t)\right] \tag{3.6}$$

and this uniquely determines A when $h(t)$ is given and vice versa, $h(t)$ is uniquely determined when A is given.

■

We now use Theorem 3.1 to establish a new recursive relationship of generalized Stirling numbers. From expression (12) in Theorem 2 of [24] with $\alpha\beta \neq 0$, we have the generating function of the generalized Stirling numbers shown below:

$$\frac{1}{k!}(1+\alpha z)^{r/\alpha}\left(\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^k = \sum_{n\geq 0} S(n,k)\frac{z^n}{n!}. \tag{3.7}$$

Theorem 3.2 *Let $\alpha\beta \neq 0$. The $A-$ sequence $(a_n)_{n\in\mathbb{N}_0}$ of the Riordan array of the generalized Stirling number array $[d_{n,k} = k!S(n,k)/n!]_{0\leq k\leq n}$ satisfies*

$$a_0 = 1, \quad a_n = -\frac{1}{\alpha}\sum_{k=1}^{n} a_{n-k}\frac{\langle\alpha\rangle_{k+1,-\beta}}{(k+1)!} \tag{3.8}$$

for all $n \geq 1$.

Proof. Denote the compositional inverse of $h(z) = ((1+\alpha z)^{\beta/\alpha} - 1)/\beta$ by $\bar{h}(z)$. Thus,

$$\bar{h}(z) = \frac{(1+\beta z)^{\alpha/\beta}-1}{\alpha}.$$

Thus, the generating function, $A(z) = \sum_{k\geq 0} a_k z^k$, of the A-sequence characterized the Riordan array of (3.7), $[d_{n,k}]_{0\leq k\leq n}$, satisfies $zA(h(z)) = h(z)$, or equivalently,

$$A(z) = \frac{z}{\bar{h}(z)} = \frac{\alpha z}{(1+\beta z)^{\alpha/\beta}-1} = \frac{\alpha z}{\sum_{k\geq 1}\binom{\alpha/\beta}{k}(\beta z)^k} = \frac{\alpha}{\sum_{k\geq 0}\langle\alpha\rangle_{k+1,-\beta}z^k/(k+1)!}, \tag{3.9}$$

which implies

$$\left(\sum_{k\geq 0} a_k z^k\right)\left(\sum_{k\geq 0}\frac{\langle\alpha\rangle_{k+1,-\beta}}{(k+1)!}z^k\right) = \sum_{n\geq 0} z^n\left(\sum_{k=0}^{n} a_{n-k}\frac{\langle\alpha\rangle_{k+1,-\beta}}{(k+1)!}\right) = \alpha$$

using the Cauchy multiplication formula. Comparing the coefficients of powers z^n, we obtain a system that can be solved to obtain the solution of $(a_n)_{n\in\mathbb{N}}$ shown in (3.8).

■

To find the fist column of the array $[d_{n,k}]_{0\le k\le n}$, we consider (3.7) for $k=0$ and have

$$(1+\alpha z)^{r/\alpha}=\sum_{n\ge 0}\frac{S(n,0)}{n!}z^n.$$

On the other hand,

$$(1+\alpha z)^{r/\alpha}=\sum_{n\ge 0}\binom{r/\alpha}{n}(\alpha z)^n.$$

Comparing the right-hand sides of the last two equations, we obtain

$$S(n,0)\equiv S(n,0,\alpha,\beta,r)=n!\binom{r/\alpha}{n}\alpha^a=\langle r\rangle_{n,-\alpha}. \tag{3.10}$$

Formula (3.10) was given in (1.7) and also in (2.9), which are derived by different approaches.

From (3.7) we have

$$[d_{n,k}]_{0\le k\le n}=\left[\frac{k!}{n!}S(n,k)\right]_{0\le k\le n}, \tag{3.11}$$

where $S(n,k)\equiv S(n,k,\alpha,\beta,r)$ $(\alpha\beta\neq 0)$. Therefore, surveying the above process, we obtain an algorithm to evaluate generalized Stirling numbers $S(n,k)\equiv S(n,k,\alpha,\beta,r)$ with $\alpha\beta\neq 0$.

Algorithm 3.3 *Denote $d(t)=(1+\alpha z)^{r/\alpha}$ and $h(z)=((1+\alpha z)^{\beta/\alpha}-1)/\beta$ $(\alpha\beta\neq 0)$. Let $n,k\in\mathbb{N}_0$ and $\alpha\beta\neq 0$. Then we may find A-sequence $(a_n)_{n\in\mathbb{N}_0}$ shown in (3.8) and establish the array (3.11) except its first column by using the recursive relation (3.5) shown in Theorem 3.1, i.e.,*

$$\frac{k!}{n!}S(n,k)=\sum_{j\ge 0}a_j\frac{(k+j-1)!}{(n-1)!}S(n-1,k+j-1) \tag{3.12}$$

for all $1\le k\le n$. The first column of array (3.11) can be constructed by using (3.10). Thus, the nth entry of the first column is

$$\frac{1}{n!}S(n,0)=\frac{\langle r\rangle_{n,-\alpha}}{n!}. \tag{3.13}$$

Finally, all $S(n,k)\equiv S(n,k,\alpha,\beta,r)$ $(0\le k\le n)$ can be read from a modification of array (3.11); namely from

$$\left[\frac{n!}{k!}d_{n,k}\right]_{0\le k\le n}=[S(n,k)]_{0\le k\le n},$$

where $S(n,k)=n\sum_{j\geq 0}a_j[k+j-1]_{j-1}S(n-1,k+j-1)$ *when* $1\leq k\leq n$, *and* $S(n,0)$ *can be obtained from (3.13) or (3.10).*

Remark 3.1 The condition $\alpha\beta\neq 0$ in Theorem 3.2 and Algorithm 3.3 is not necessary. Algorithm 3.3 can be modified to adapt some of cases when $\alpha\beta=0$. We will show the application of Algorithm 3.3 to the calculations of the classical Stirling numbers of the second and the first kind, i.e., $S(n,k,\alpha,\beta,r))=S(n,k,0,1,0)$ and $S(n,k,\alpha,\beta,r)=S(n,k,1,0,0)$, in Examples 4.2 and 4.3, respectively.

Example 3.1 For the Howard's weighted degenerated Stirling numbers $S(n,k)\equiv S(n,k,1,1,-1)$. From Algorithm 3.3 or Theorem 3.2, we immediately have generating function of the corresponding A-sequence $A(z)=1$. Then, using (3.12) and (3.13) we obtain the Riordan array $[d_{n,k}]_{0\leq k\leq n}=\left[\frac{k!}{n!}S(n,k)\right]_{0\leq k\leq n}$ as

$$\left[\frac{k!}{n!}S(n,k)\right]_{0\leq k\leq n}=\left[\begin{array}{rrrrr}1&&&&\\-1&1&&&\\1&-1&1&&\\-1&1&-1&1&\\1&-1&1&-1&1\end{array}\right].$$

Therefore,

$$[S(n,k)]_{0\leq k\leq n}=\left[\begin{array}{rrrrr}1&&&&\\-1&1&&&\\2&-2&1&&\\-6&6&-3&1&\\24&-24&12&-4&1\end{array}\right],$$

which gives $S(0,0)=1$; $S(1,0)=-1$, $S(1,1)=1$; $S(2,0)=2$, $S(2,1)=-2$, $S(2,2)=1$; $S(3,0)=-6$, $S(3,1)=6$, $S(3,2)=-3$, $S(3,3)=1$; and $S(4,0)=24$, $S(4,1)=-24$, $S(4,2)=12$, $S(4,3)=-4$, and $S(4,4)=1$ row by row.

Example 3.2 As we have presented in Remark 4.1, the condition $\alpha\beta\neq 0$ in Theorem 3.2 and Algorithm 3.3 is not necessary. Here, we demonstrate how to modify Algorithm 3.3 for the case of $(\alpha,\beta,r)=(0,1,0)$. The generating function of the corresponding classical Stirling numbers $\{S(n,k)\equiv S(n,k,0,1,0)\}_{0\leq k\leq n}$ of the second kind is

$$\frac{1}{k!}(e^z-1)^k=\sum_{n\geq 0}S(n,k)\frac{z^n}{n!}.$$

Thus the corresponding Riordan array has generating functions $d(z)=1$ and $h(z)=e^z-1$. Since the compositional inverse of $h(z)$ is $\bar{h}(z)=\ln(1+$

z), the A-sequence characterization of the Riordan array has generating function

$$A(z) = \frac{z}{\ln(1+z)} = \frac{z}{\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} z^k} = \frac{1}{\sum_{k\geq 0} \frac{(-1)^k}{k+1} z^k},$$

which coefficients $\{a_n\}_{n\geq 0}$, i.e., the elements of A-sequence, can be solved from the above equation as

$$a_0 = 1, \quad a_n = -\sum_{k=1}^{n} a_{n-k} \frac{(-1)^k}{k+1} = \sum_{k=2}^{n+1} a_{n-k+1} \frac{(-1)^k}{k} \quad (n \geq 1).$$

Thus, we obtain the first few a_n:

$$a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = -\frac{1}{12},\ a_3 = \frac{1}{24},\ a_4 = -\frac{19}{720},\ etc.$$

Similar to Algorithm 3.3, we may find the Riordan array

$$[d_{n,k}]_{0\leq k\leq n} = \left[\frac{k!}{n!} S(n,k)\right]_{0\leq k\leq n} = \left[\frac{k!}{n!} S(n,k)\right]_{0\leq k\leq n} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & \frac{1}{2} & 1 & & \\ 0 & \frac{1}{6} & 1 & 1 & \\ 0 & \frac{1}{24} & \frac{7}{12} & \frac{3}{2} & 1 \end{bmatrix}.$$

The Riordan Stirling array of the Stirling numbers of the second kind is

$$[S(n,k)]_{0\leq k\leq n} = \left[\frac{k!}{n!} S(n,k)\right]_{0\leq k\leq n} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 3 & 1 & \\ 0 & 1 & 7 & 6 & 1 \end{bmatrix},$$

which gives all $S(n,k) = S(n,k,0,1,0)$ for $0 \leq k \leq 4$. For instance, $S(4,0) = 0$, $S(4,1) = 1$, $S(4,2) = 7$, $S(4,3) = 6$, and $S(4,4) = 1$.

Example 4.3 For $(\alpha, \beta, r) = (1,0,0)$, we can also applied a modification of Algorithm 3.3 to evaluate the classical Stirling numbers of the first kind $s(n,k) \equiv S(n,k,1,0,0)$ as follows. In this case, we have the corresponding Riordan array $(d(z), h(z)) = (1, \ln(1+z))$. Thus the compositional inverse of $\bar{h}(z) = e^z - 1$. Thus the A-sequence $\{a_n\}_{n\geq 0}$ has its generating function

$$A(z) = \frac{z}{\bar{h}(z)} = \frac{z}{\sum_{k\geq 1} \frac{z^k}{k!}} = \frac{1}{\sum_{k\geq 0} \frac{z^k}{(k+1)!}}.$$

Solve the above equation to obtain

$$a_0 = 1,\ a_1 = -\frac{1}{2},\ a_2 = \frac{1}{12},\ a_3 = 0,\ a_4 = -\frac{1}{720}\ etc.,$$

which brings us the Riordan array

$$[d_{n,k}]_{0\le k\le n} = \left[\frac{k!}{n!}s(n,k)\right]_{0\le k\le n} = \left[\frac{k!}{n!}s(n,k)\right]_{0\le k\le n} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & -\frac{1}{2} & 1 & & \\ 0 & \frac{1}{3} & -1 & 1 & \\ 0 & -\frac{1}{4} & \frac{11}{12} & -\frac{3}{2} & 1 \end{bmatrix}.$$

The Riordan Stirling array of the signed Stirling numbers of the first kind is

$$[s(n,k)]_{0\le k\le n} = \left[\frac{k!}{n!}s(n,k)\right]_{0\le k\le n} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & -1 & 1 & & \\ 0 & 2 & -3 & 1 & \\ 0 & -6 & 11 & -6 & 1 \end{bmatrix},$$

which gives all $s(n,k) = S(n,k,1,0,0)$ for $0 \le k \le 4$. For instance, $s(4,0) = 0$, $s(4,1) = -6$, $s(4,2) = 11$, $s(4,3) = -6$, and $s(4,4) = 1$. Of course, the Stirling numbers of the first kind can be evaluated more easily by using formula (2.5) in Theorem 2.2, namely,

$$s(n,k) \equiv S(n,k,1,0,0) = \frac{1}{k!}\, \frac{d^k}{dz^2}[z]_n\bigg|_{z=0},$$

which are simply the coefficients of the powers of z in the expansion of $[z]_n$.

If $c = \{c_k = k!\}_{k\ge 0}$, the corresponding (c)-Riordan array $[d_{n,k}]_{n\ge k\ge 0}$ shown in (3.3) is called an exponential Riordan array in [13]), where $d_{n,k}$ is presented in (3.2). [13] gives an interesting algorithm in computation of $d_{n,k}$ by using two different sequences, c-sequence and r-sequence. More precisely, we cite Proposition 4.1 of [13] as follows.

Proposition 3.4 *([13], Proposition 4.1) Let* $[d_{n,k}]_{n\ge k\ge k\ge 0} = (d(z), h(z))$ *be an exponential Riordan array and let*

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots, \quad r(x) = r_0 + r_1 x + r_2 x^2 + \cdots \tag{3.14}$$

be two formal power seeies such that

$$c(h(z)) = d'(z)/d(z), \qquad r(h(z)) = h'(z). \tag{3.15}$$

Then

$$d_{n+1,0} = \sum_{i\geq 0} i! c_i d_{n,i}, \tag{3.16}$$

$$d_{n+1,k} = r_0 d_{n,k-1} + \frac{1}{k!}\sum_{i\geq k} i!(c_{i-k} + kr_{i-k+1})d_{n,i}, \tag{3.17}$$

or, defining $c_{-1} = 0$,

$$d_{n+1,k} = \frac{1}{k!}\sum_{i\geq k-1} i!(c_{i-k} + kr_{i-k+1})d_{n,i} \tag{3.18}$$

for all $k \geq 0$.

Conversely, starting from the sequence defined by (3.14), the infinite array $[d_{n,k}]_{n\geq k\geq 0}$ *defined by (3.18) is an exponential Riordan array.*

The exponential Riordan array $[S(n,k)] \equiv [S(n,k,\alpha,\beta,r)]$ of the generalized Stirling numbers have the generating functions shown in (3.7). Thus $[S(n,k)] = (d(z), h(z))$, where

$$d(z) = (1+\alpha z)^{r/\alpha}, \quad h(z) = \frac{(1+\alpha z)^{\beta/\alpha} - 1}{\beta}. \tag{3.19}$$

It is obvious that the compositional inverse of $h(t)$ is

$$\bar{h}(z) = \frac{(1+\beta z)^{\alpha/\beta} - 1}{\alpha}. \tag{3.20}$$

From (3.15) we obtain the generating functions of c-sequence and r-sequence

$$c(x) = \left.\frac{d'(x)}{d(x)}\right|_{x=\bar{h}(x)} = \left.\frac{r}{1+\alpha x}\right|_{x=\bar{h}(x)} = r(1+\beta x)^{-\alpha/\beta} \tag{3.21}$$

$$r(x) = h'(x)|_{x=\bar{h}(x)} = \left.(1+\alpha z)^{(\beta/\alpha)-1}\right|_{x=\bar{h}(x)} = (1-\beta x)^{1-\alpha/\beta}. \tag{3.22}$$

In particular, if $\alpha = \beta \neq 0$, then

$$\begin{aligned} c(x) &= r(1+\beta x)^{-1} = r - r\beta x + r\beta^2 x^2 - r\beta^3 x^3 + \cdots \\ r(x) &= 1. \end{aligned} \tag{3.23}$$

Thus, we obtain a recursive formula for the computation of $S(n,k) = S(n,k,\beta,\beta,r)$:

$$S(n+1,0) = \sum_{i=0}^{n} i! c_i S(n,i), \tag{3.24}$$

$$S(n+1,k) = S(n,k-1) + \frac{1}{k!}\sum_{i=k}^{n} i! c_{i-k} S(n,i), \ \ k>0, \tag{3.25}$$

where $c_0 = r$, $c_1 = -r\beta$, $c_2 = r\beta^2$, $\ldots$, shown in (3.23).

References

[1] G. E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications Vol. 2. Addison-Wesley Publishing Com., London 1976.

[2] C. deBoor, Divided difference, Surveys in Approximation Theory, 1 (2005), 46-67.

[3] A. Z. Broder, The r-Stirling numbers, Discrete Math., 49 (1984), 241-259.

[4] P. Butzer and M. Hauss, Eulerian numbers with fractional order parameters, Aequationes Mathematicae, 46 (1993), 119-142.

[5] L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, Utilitas Mathematica 15 (1979), 1-88.

[6] L. Carlitz, Weighted Stirling numbers of the first and second kind-I, II, Fibonacci Quart. 18 (1980), 147-162, 242-257.

[7] C. A. Charalambides, On weighted Stirling and other related numbers and some combinatorial applications, Fibonacci Quart., 22 (1984), 296-309.

[8] C. A. Charalambides and M. Koutras, On the differences of the generalized factorials at an arbitrary point and their combinatorial applications, Discrete Math., 47 (1983), 183-201.

[9] G.-S. Cheon, H. Kim, L. W. Shapiro, Riordan group involutions. *Linear Algebra Appl.*, 428 (2008), 941–952.

[10] G.-S. Cheon, H. Kim, L. W. Shapiro, An algebraic structure for Faber polynomials, *Linear Algebra Appl.*, 433 (2010), 1170 -1179.

[11] L. Comtet, Advanced Combinatorics-The Art of Finite and Infinite expansions, Dordrecht: Reidel, 1974.

[12] H. B. Curry and I. J. Schoenberg, On Pólya frequency functions IV: the fundamental spline functions and their limits, J. Analyse Math. 17 (1966), 71-107.

[13] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays. Ann. Comb. 13 (2009), no. 1, 6585.

[14] R. Díaz and E. Pariguan, On hypergeometric functions and k-Pochhammer symbol, arXiv: math/0405596v2.

[15] P. Doubilet, G.-C. Rota, and R. Stanley, On the foundations of combinatorial theory (VI): The idea of generating function, in "Probability Theory," Vol. 2, Sixth Berkeley Symposium on Mathematical Statistics and Probability, 267-318, Univ. of California, 1972.

[16] A. O. Gel'fond, Calculus of Finite Differences, Delhi, Hindustan Publ. Corp., 1971.

[17] H.W. Gould and A. T. Hopper, Operational formulas connected with two generalizations of Hermit polynomials, Duke Math. J., 29 (1962), 51-63.

[18] T. X. He, A Unified Approach to Generalized Stirling Functions, J. Math. Research and Appl., to appear, 2012.

[19] T. X. He, L. C. Hsu, P. J.-S. Shiue. The Sheffer Group and the Riordan Group. *Discrete Applied Mathematics*, 155 (2007) 1895–1909.

[20] T. X. He and P. J.-S. Shiue, A note of Horner's method, manuscript, 2010.

[21] T. X. He and R. Sprugnoli. Sequence characterization of Riordan arrays, *Discrete Mathematics*, 309 (2009), 3962-3974.

[22] Power-type generating functions, in "Colloquia Mahematica Societatis Janos Bolyai," Vol. 58, Approximation Theory, Kesckemet, Hungary, 1990, 405-412.

[23] F. T. Howard, Degenerated weighted Stirling numbers, Discrete Math. 57 (1985), 45-58.

[24] L. C. Hsu and P. J.-S. Shiue, A unified approach to generalized Stirling numbers, Advances Appl. Math. 20 (1998), 366-384.

[25] S. A. Joni, G.-C. Rota, and B. Sagan, From sets to functions: Three elementary examples, Discrete Math., 37 (1981), 193-202.

[26] Ch. Jordan, Calculus of Finite Differences, Chelsea, New York, 1965.

[27] M. Mansour, Determining the k-generalized Gamma function $\Gamma_k(x)$ by functional equations, Int. J. Contemp. Math. Sciences, 4 (2009), No. 21, 1037-1042.

[28] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. On some alternative characterizations of Riordan arrays. *Canadian J. Mathematics*, 49 (1997) 301–320.

[29] D. S. Mitrinović and J. D. Keckić, The Cauchy method of residues, English ed., D. Reidel Publishing Company, Dordrecht, Holland, 1984.

[30] L. Moser and M. Wyman, Asymptotic development of Stirling numbers of the first kind, J. London Math, Soc., 33 (1958), 133-146.

[31] S. B. Nandi and S. K. Dutta, On associated and generalized Lah numbers and applications to discrete distribution, Fibonacci Quart., 25 (1987), 128-136.

[32] A. Nkwanta. A Riordan matrix approach to unifying a selected class of combinatorial arrays. *Congressus Numerantium*, 160 (2003) 33–45.

[33] P. Peart, and W.-J. Woan. A divisibility property for a subgroup of Riordan matrices. *Discrete Applied Mathematics*, 98 (2000) 255–263.

[34] J. Riordan, Moment recurrence relations for binomial POisson and hypergeometric frequency distribution, Ann. Math. Statist., 8 (1937), 103-111.

[35] J. Riordan, Combinatorial Identities, R. E. Krieger Publishing Com., Huntington, New York 1979.

[36] D. G. Rogers. Pascal triangles, Catalan numbers and renewal arrays. *Discrete Mathematics*, 22 (1978) 301–310.

[37] G.-C. Rota, The number of partitions of a set, Amer. Math. Monthly, 71 (1964), 498-504.

[38] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integral and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.

[39] L. W. Shapiro. A survey of the Riordan Group. Talk at a meeting of the American Mathematical Society, Richmond, Virginia, 1994.

[40] L. W. Shapiro. Some open questions about random walks, involutions, limiting distributions and generating functions. *Advances in Applied Mathematics*, 27 (2001) 585–596.

[41] L. W. Shapiro. Bijections and the Riordan group. *Theoretical Computer Science*, 307 (2003) 403–413.

[42] L. W. Shapiro, S. Getu, W.-J. Woan, and L. Woodson. The Riordan group. *Discrete Applied Mathematics*, 34 (1991) 229–239.

[43] R. Sprugnoli, Riordan arrays and combinatorial sums. *Discrete Mathematics*, 132 (1994) 267–290.

[44] R. Sprugnoli, Riordan arrays and the Abel-Gould identity. *Discrete Mathematics*, 142 (1995) 213–233.

[45] P. G. Todorov, Taylor expansions of analytic functions related to $(1+z)^x - 1$, J. Math. Anal. Appl., 132 (1988), 264-280.

[46] E. G. Tsylova, The asymptotic behavior of generalized Stirling numbers, in "Combinatorial-Algebraic Methds in Applied Mathematics," 143-154, 158, Gorkov. Gos. Univ. Gorkin, 1985 (in Russian).

[47] C. G. Wagner, Surjections, differences and binomial lattices, Stud. Appl. Math. 93 (199), 15-27.