Composite Dilation Wavelets with High Degrees

Tian-Xiao He, Illinois Wesleyan University
Composite Dilation Wavelets with High Degrees

Tian-Xiao He
Dept. Math, Illinois Wesleyan University
Bloomington, IL, USA

International Congress of Mathematicians
Analysis and Its Applications, SC08-13-02
Coex, Seoul, Korea
August 18, 2014
Introduction

Construction of multivariate wavelets on arbitrary triangulation

- Haar-type non-separable constant wavelets: “twin dragon,” Belogay and Wang [99], Flaherty and Wang [99], and Gröchenig and Madych [92]; wavelet with composite dilations, Guo, Kutyniok, and Labate [04], Krishtal, Robinson, Weiss, and Wilson [08], Krommweh [09], Krommwek and Plonka [09], Blanchard [09], MacArthur and Taylor [11], Blanchard and Krishtal [12], Grohs [13].

- Continuous piecewise linear wavelet: Yserenntant [86], Vassilevski and Wang [97], Stevenson [97, 98], Liu [06], Floater and Quak [99, 00], Hardin and Hong [03].

- $C^1$ quadratic splines and spline wavelets: Powell [73], Powell-Sabin [77], Chui and He[90], Chui, Chui, and He[93], Chui and Jiang [04]; Oswald [92], Davydov and Petrushev [03,05], Maes and Bultheel [07, 08, 09, 10], Windmolders, Vanraes, Dierckx, and Bultheel [03], Maes, Vanraes, Dierckx, and Bultheel [04], Speleers, Dierckx, and Vandewalle [06, 07, 08, 09].
Splines and elements, spline wavelets, wavelets with composite dilations

- Splines and their BB-expressions: Farin [88,90,93], Chui [87], etc.
- Characterization of compactly supported refinable splines: Lawton, Lee, and Shen [95], Sun [96], Goodman [98], Guan and He [09], etc.
- Spline wavelets: Chui and Wang [92,93], Chui, Stöckler, and Ward [92], Jia and Micchelli [91], Riemenschneider and Shen [92], Lorentz and Oswald [00, Sobolev spaces], Jia, Wang, and Zhou [03], Jia and Liu [08], etc.
- Wavelets with composite dilations: Guo, Labate, Lim, Weiss, and Wilson [04, 06, 06], Guo and Labate [07, 08,10, 11,12,13], Guo and Labate and Lim [09], etc.
A function $f$ supported in $[0, 1]^2$ wish a discontinuity across a nice curve $\Gamma$ and otherwise smoothness.

$\tilde{f}_F^m$ standard Fourier approximation built from the best $m$ nonzero Fourier terms.

$$\|f - \tilde{f}_F^m\|_2^2 \leq cm^{-1/2}, \quad m \to \infty.$$ 

$\tilde{f}_W^m$ wavelet non-adaptive approximation built from the best $m$ nonzero wavelet terms.

$$\|f - \tilde{f}_W^m\|_2^2 \leq cm^{-1}, \quad m \to \infty.$$ 

$\tilde{f}_A^m$ wavelet adaptive approximation built from the best $m$ nonzero wavelet terms.

$$\|f - \tilde{f}_A^m\|_2^2 \leq cm^{-2}, \quad m \to \infty.$$ 

$\tilde{f}_C^m$ (non-adaptive) curvelet $m$-term approximation-summing the $m$ biggest terms in the curvelet frame expansion.

$$\|f - \tilde{f}_C^m\|_2^2 \leq Cm^{-2}(\log m)^3, \quad m \to \infty.$$
Composite dilations: Curvelets (Candés and Donoho [03])

\[ \psi_{a,b,\theta}(x) = a^{-\frac{3}{4}} \psi(D_a R_\theta(x - b)), \]

\[ D_a = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad R_\theta^{-1} = R_\theta^T = R_{-\theta} \]

\[ c(a, b, \theta) = \langle f, \psi_{a,b,\theta} \rangle = \int_{\mathbb{R}^2} f(x) \overline{\psi_{a,b,\theta}(x)} \, dx \]

Define \( \psi_j \) by \( \hat{\psi}_j(\xi) = \bigcup_j(\xi) := 2^{-\frac{3j}{4}} W(2^{-j} \gamma) V(\frac{2^{j/2}}{2\pi} \theta) \) with \( \sum_{j=\infty} W^2(2^j \gamma) = 1 \) and \( \sum_{l=-\infty}^{\infty} V^2(t - l) = 1. \) When \( r \in (\frac{3}{4}, \frac{3}{2}) \) and \( t \in (-\frac{1}{2}, \frac{1}{2}) \)

\[ \psi_{j,l,k} = \psi_j(R_{\theta_l}(x - x_j^{l,k})) \quad f = \sum_{j,l,k} \langle f, \psi_{j,l,k} \rangle \psi_{j,l,k} \]

\[ \|f\|^2 = \sum_{j,l,k} |\langle f, \psi_{j,l,k} \rangle|^2 \]
Composite dilations: Shearlets (Krishtal, Robinson, Weiss, and Wilson [08])

Let \( A_{ast}(\psi) := \{ \psi_{ast}(x) = a^{-\frac{3}{4}} \psi(M_{as}^{-1}(x - t)) : a \in R^+, s \in R, \ t \in R^2 \} \), where \( M_{as} := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} = \begin{pmatrix} a & \sqrt{as} \\ 0 & \sqrt{a} \end{pmatrix} \). We have

\[
\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right),
\]

where \( \hat{\psi}_1 \in C^\infty(\mathbb{R}) \) with \( \text{supp} \ \hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] \) and \( \hat{\psi}_2 \in C^\infty(\mathbb{R}) \) with \( \text{supp} \ \hat{\psi}_2 \subset [-1, 1], \hat{\psi}_2 > 0 \) on \((-1, 1)\), and \( \| \psi_2 \| = 1 \). The family \( \{ \psi_{ast}(x) \} \) is a reproducing system for \( L^2(\mathbb{R}^2) \), i.e., it satisfies the Calderson’s formula

\[
\| f \|^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{0}^{\infty} |\langle f, \psi_{ast} \rangle|^2 \frac{da}{a^3} ds dt
\]

for all \( f \in L^2(\mathbb{R}^2) \).
Wavelets with composite dilations—Shearlets-2

Continuous Shearlet transform

\[ S_f(a, s, t) = \langle f, \psi_{ast} \rangle, \quad a \in \mathbb{R}^+, \; s \in \mathbb{R}, \; t \in \mathbb{R}^2 \]

The continuous Shearlet are not only able to locate a discontinuity curve, but also to identify its orientation. That is, for \( a \to 0 \), the Shearlet transforms \( S_f(a, s, t) \) tends to 0 rapidly unless \( t \) is at the singularity and \( s \) describe the direction that is perpendicular to the discontinuity curve. Thus

\[ \hat{\psi}_{ast} = a^{-\frac{3}{4}} e^{-2\pi i \xi t} \hat{\psi}_1(a\xi_1) \hat{\psi}_2(a^{-\frac{1}{2}}(s + \frac{\xi_2}{\xi_1})) \]

where \( \text{supp } \hat{\psi}_{ast} \subset \{ (\xi_1, \xi_2) : \xi_1 \in [-\frac{2}{a}, -\frac{1}{2a}] \cup [\frac{1}{2a}, \frac{2}{a}], |s + \frac{\xi_2}{\xi_1}| \leq \sqrt{a} \} \).

Tian-Xiao He
Dept. Math, Illinois Wesleyan University Bloomington, IL, USA

Composite Dilation Wavelets with High Degrees
Wavelets with composite dilations-Shearlets-3

**Ex.** Let $f = X_D$, $D$ is the unit disc in $\mathbb{R}^2$ for $a \to 0$ if $t \in \partial D$ and $s$ describes the direction normal to $\partial D$, then $|S_f(a, s, t)| \leq ca^\frac{3}{4}$. Otherwise, for each $N = 1, 2, \ldots$ we have $|S_f(a, s, t)| \leq ca^N$. Let $a = 2$, denote $M_{2s}$ by $M_{ij} = \begin{pmatrix} 2^i & 0 \\ 0 & 2^{\frac{i}{2}} \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$. Then $A_{ijt}(\psi)$ gives a discrete system. Define $\psi \in L^2(\mathbb{R}^2)$ by $\hat{\psi} = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_1}{\xi_2})$ where $\hat{\psi}_1 \in L^2(\mathbb{R})$ with $\text{supp} \ \hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and $\hat{\psi}_2 \in L^2(\mathbb{R})$ with $\text{supp} \ \hat{\psi}_2 \subset [-1, 1]$ satisfying

$$\sum_{j \in \mathbb{Z}} |\psi_2(\xi + j)|^2 = 1 \quad \text{and} \quad \sum_{j \in \mathbb{Z}} |\psi_1(2^j \xi)|^2 = 1 \quad a.e \xi \in \mathbb{R}.$$ 

Thus $\hat{\psi} \in C^\infty(\mathbb{R}^2)$, $|\psi(x)| \leq k_N(1 + |x|)^{-N}$ where $k_N > 0$ for any $N \in \mathbb{N}$. In addition, $\sum_{i,j,k} |\langle f, \psi_{i,j,k} \rangle|^2 = \|f\|^2$ for all $f \in L^2(\mathbb{R}^2)$.
Let $A \in GL_2(\mathbb{R})$ and 
$\{B_j : j \in \mathbb{Z}\} \subset \tilde{S}L_2(\mathbb{Z}) = \{b \in GL_n(\mathbb{R}) : |\det b| = 1\}$.

(i) $D_{B_j} T_k V_0 = V_0$ for any $j \in \mathbb{Z}, \ k \in \mathbb{Z}^2$.

(ii) For each $i \in \mathbb{Z}, \ V_i \subset V_{i+1}$, where $V_i = D_{A^{-i}}V_0$.

(iii) $\bigcap_i V_i = \{0\}$ and $\bigcup_i V_i = L^2(\mathbb{R}^2)$.

(iv) $\exists \phi \in L^2(\mathbb{R}^2)$ such that 
$\Phi_B = \{D_{B_j} T_k \phi : j \in \mathbb{Z}, \ k \in \mathbb{Z}^2\}$ is a tight frame (resp. ON basis) for $V_0$.

$V_0$: AB scaling space, $\phi$: AB scaling function (resp. ON AB scaling function) for $V_0$. 
$AB$-MRA shearlet basis: $A_{AB}(\tilde{\Psi}) = \{D_{A^i} D_{B_j} T_k \Psi : k \in \mathbb{Z}^2, \ i, j \in \mathbb{Z}\}$, where $\tilde{\Psi} = (\psi^1, \psi^2, ..., \psi^L)$ and $\{B_j : j \in \mathbb{Z}, |\det B_j| = 1\}$.
Wavelets with composite dilations-AB-MRA-2

Let \( \bar{\psi} = (\psi^1, \psi^2, \ldots, \psi^L) \in L^2(\mathbb{R}^2) \) be such that
\[ \{ D_{B_j} T_k \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, \ldots, L \} \]
is an ON basis (resp. tight frame) for \( W_0 \), the o.r. complement of \( V_0 \) in \( V_1 \). Then \( \bar{\Psi} \) is an ON (resp. tight frame) AB-multiwavelet.

**EX.** Let \( A := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \ B_j := B^j, j \in \mathbb{Z}, B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \)

\( S_0 = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| < \frac{1}{4} \} \) is the vertical strip of width \( \frac{1}{2} \) bounded by the lines \( \xi_1 = \pm \frac{1}{4} \).

\[ S_i = A^i S_0, i \in \mathbb{Z}, \{ \xi = (\xi_1, \xi_2) : |\xi_1| < 2^{2i-2} \} \]

\[ (B^T)^j \xi = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ j\xi_1 + \xi_2 \end{pmatrix} \Rightarrow (B^T)^j S_0 \subseteq S_0, j \in \mathbb{Z} \]

We have \( (i) S_i \subseteq S_{i+1}, (ii) \bigcup_{i \in \mathbb{Z}} S_i = \mathbb{R}^2 \), \( (iii) \bigcap_{i \in \mathbb{Z}} S_i = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \xi_1 = 0 \} \).
Wavelets with composite dilations-AB-MRA-3

Define \( L^2(S) = \{ f \in L^2(R^2) : \text{supp } \hat{f} \subseteq S \} \).

(i) \( D^j_B T_k L^2(S_0) = L^2(S_0) \), for any \( j \in \mathbb{Z}, k \in \mathbb{Z}^2 \).

(ii) \( L^2(S_i) \subseteq L^2(S_{i+1}) \)

(iii) \( \cap_{i \in \mathbb{Z}} L^2(S_i) = 0 \) and \( \cup_{i \in \mathbb{Z}} L^2(S_i) = L^2(R^2) \).

Let \( \hat{\phi} = X_U, U = U^+ \cup U^-, U^+ \) is the triangle with vertices \((0,0), (\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4})\) and \( U^- = -U^+ \). Thus \( S_0 = \cup_{i \in \mathbb{Z}} (B^T)^i U \), where the union is disjoint. Hence, \( \Phi_B \) is a tight frame for \( V_0 \).

Therefore, \( \{L^2(S_i) = V_i : i \in \mathbb{Z}\} \) is a AB-MRA. AB-MRA shearlet: \( R_0 := S_1/S_0 \). \( W_0 = L^2(R_0) \) is the complement of \( V_0 \) in \( V_1 \). Set \( I = I^+ \cup I^- \) in \( R_0 \), which is defined by \( I^+ \) is the trapezoid with vertices \((\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}), (1, 0) & (1, 1) \) & \( I^- = -I^+ \). We have \( \hat{\psi} = X_I \{D^j_B T^k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^2\} \) is a tight frame for \( W_0 \) and \( \psi \) is a tight frame AB-wavelet.
Spline wavelets with composite dilations-1

Haar-type wavelets

\[ D_c : (D_c f)(x) = |\det c|^{-\frac{1}{2}} f(c^{-1} x) \]

\[ \Gamma : a \text{ lattice in } \mathbb{R}^d \text{ with } \Gamma = M \mathbb{Z}^d. \]

\[ T_\gamma : \gamma \in \Gamma \quad (T_\gamma f)(x) = f(x - \gamma) \]

\[ \Psi = \{\psi^1, \psi^2, ..., \psi^L\}, \psi^l \in L^2(\mathbb{R}^d), l = 1, ..., L \]

\( \{D_c T_\gamma \psi^l : c \in C, \gamma \in \Gamma, l = 1, 2, ..., L\} \) is an orthonormal wavelet system or multiwavelet associated with dilation C and lattice \( \Gamma \).

Consider \( C = AB, A, B \in GL_d(\mathbb{R}) : |\det b| = 1 \text{ for all } b \in B \text{ and } |\det a| \leq 1 \text{ for all } a \in A. \) B is finite, \( A(\Gamma) \subset \Gamma \) and \( B(\Gamma) = \Gamma \).
Spline wavelets with composite dilations

MRA associated with a sequence of dilations \( \{a^j\}_{j \in \mathbb{Z}} = A \), \( a \) is an expanding matrix, in an increasing sequence \( \{V_j\}_{j \in \mathbb{Z}} \) of closed subspaces of \( L^2(\mathbb{R}^d) \) such that

(i) \( V_j = D_{a^{-j}} V_0 \)

(ii) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \)

(iii) \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^d) \)

(iv) \( \exists \varphi \in V_0 \) such that \( \{T_\gamma \varphi\}, \gamma \in \Gamma \), is an orthonormal basis of \( V_0 \).

Since \( B(\Gamma) = \Gamma \), the operators generated by the dilations \( D_b, b \in B \), and the translations \( T_\gamma, \gamma \in \Gamma \), form a group with operation \((c, \tau), (b, \gamma) = (cb, b^{-1} \tau + \gamma)\), i.e. \((D_c T_c)(D_b T_\gamma)f = D_{cb} T_{b^{-1} \tau + \gamma}\). \( B\Gamma \) is a semi-direct product of \( B\&\Gamma \). \( B\Gamma \)-invariant space \( V \subset L^2(\mathbb{R}^d) : D_b T_\gamma f \in V \) for every \( f \in V, b \in B \), and \( \gamma \in \Gamma \). Thus, \((iv)' \) \( \varphi \in V_0 \) such that \( \{D_b T_\gamma \varphi : b \in B, \gamma \in \Gamma\} \) is an orthonormal basis for \( V_0 \). \( AB\)-MRA: (i)-(iii) and (iv)’
Spline wavelets with composite dilations-3

An example of compactly supported multiwavelet: 
\( \Psi = \{ \psi^1, \psi^2, \ldots, \psi^L \} \) such that the system 
\( \{ D_{a^j} D_b T_\gamma \psi^l : j \in \mathbb{Z}, b \in B, \gamma \in \Gamma, l = 1, \ldots, L \} \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \): Let 
\( a = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} =: q \) be the expanding matrix, 
and let \( B = \{ b_i : i = 0, 1, \ldots, 7 \} \) be the group of matrices

\[
\begin{align*}
  b_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
  b_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
  b_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
  b_3 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\\
  b_i = -b_{i-4}, i = 4, 5, 6, 7
\]

\( R_i = b_i R_0 \) for \( i = 0, 1, \ldots, 7 \).
\( \varphi := 2\sqrt{2} X_{R_0}, V_0 := \{ D_b T_k \varphi : b \in B, k \in \mathbb{Z}^2 \} \) is an orthonormal basis for its closed linear span \( V_j := D_{q^{-j}} V_0, j \in \mathbb{Z} \).
Spline wavelets with composite dilations-4

Hence, $V_0$ is the subspace of $L^2(R^2)$ consisting of all square integrable functions that are constant on each $Z^2$-translate of the triangles $R_i, i = 0, 1, 2, .., 7$. $V_1$ consists of all functions in $L^2(R^2)$ that are constant on each $q^{-1}Z^2$-translate of the triangles $q^{-1}R_i, i = 0, 1, .., 7. V_0 \subset V_1$ and consequently, $V_j \subset V_{j+1}$ for all $j \in Z$. $V_j, j \in Z$ form an AB-MRA with $\varphi$ as a scaling function.

As we mentioned above, another point of view is to consider the column vector

$$
\Phi = \begin{pmatrix}
D_{b_0} \varphi \\
\vdots \\
D_{b_7} \varphi
\end{pmatrix} = \begin{pmatrix}
\varphi_0 \\
\vdots \\
\varphi_7
\end{pmatrix}
$$

to be scaling function vector for this MRA.

$$
R_0 = q^{-1}R_1 \cup \left[ q^{-1}R_6 + \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right) \right] = q^{-1}R_1 \cup q^{-1} \left[ R_6 + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right].
$$
Spline wavelets with composite dilations-5

Hence, \( X_{R_0}(x) = X_{q^{-1}R_1}(x) + X_{q^{-1}(R_6 + \begin{pmatrix} 0 \\ 1 \end{pmatrix})} \) or equivalently

\[ \phi^0(x) = \phi^1(qx) + \phi^6(qx - \begin{pmatrix} 0 \\ 1 \end{pmatrix}). \]

Applying \( D_{b_i} \) to the above

\[
\begin{align*}
\phi^0(x) &= \phi^1(qx) + \phi^6(qx - \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \\
\phi^1(x) &= \phi^2(qx) + \phi^5(qx - \begin{pmatrix} 1 \\ 0 \end{pmatrix}), \\
\phi^2(x) &= \phi^3(qx) + \phi^0(qx + \begin{pmatrix} 1 \\ 0 \end{pmatrix}), \\
\phi^3(x) &= \phi^4(qx) + \phi^7(qx + \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \\
\phi^4(x) &= \phi^5(qx) + \phi^2(qx + \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \\
\phi^5(x) &= \phi^6(qx) + \phi^1(qx + \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \\
\phi^6(x) &= \phi^7(qx) + \phi^4(qx - \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \\
\phi^7(x) &= \phi^0(qx) + \phi^3(qx - \begin{pmatrix} 1 \\ 0 \end{pmatrix}).
\end{align*}
\]
Spline wavelets with composite dilations-6

\( \psi^0(x) = \varphi^1(qx) - \varphi^6(qx - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \) is a Haar-type spline wavelet.

Moreover, the system \( \{D_{q_i} D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^2\} \) is an orthonormal basis for \( L^2(\mathbb{R}^2) \). Since the matrices in \( B \) have the integer entries and \( |\text{det} \ B| = 1, b \in B, \)

\[
\{D_{q_i} D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^2\} = \{D_{q_i} T_k (D_b \psi) : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^2\},
\]

where \( \psi^i := D_{b_i} \psi, i = 0, 1, .., 7 \). Thus, \( (\psi^0, .., \psi^7)^T \) forms a Haar-type spline multiwavelet.

\[
\hat{\Phi} := \begin{pmatrix} \hat{\varphi}^0 \\ \vdots \\ \hat{\varphi}^7 \end{pmatrix}, \quad \hat{\Psi} := \begin{pmatrix} \hat{\psi}^0 \\ \vdots \\ \hat{\psi}^7 \end{pmatrix}
\]
Spline wavelets with composite dilations

We have \( \xi q = (\xi_1 + \xi_2, \xi_2 - \xi_1) \), and \( \hat{\Phi}(\xi q) = M_0(\xi)\hat{\Phi}(\xi) \), where low-pass filter is

\[
M_0(\xi) = \frac{1}{2}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & e(-\xi_2) & 0 \\
0 & 0 & 1 & 0 & 0 & e(-\xi_2) & 0 & 0 \\
e(\xi_1) & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & e(\xi_2) & 0 & 0 & 1 & 0 & 0 \\
0 & e(\xi_2) & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & e(-\xi_1) & 0 & 0 & 1 \\
1 & 0 & 0 & e(-\xi_1) & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

where \( e(\alpha) = e^{2\pi i \alpha} \) for \( \alpha \in \mathbb{R} \). The high-pass filter \( M_1 \) is \( M_1(\xi) = M_0(\xi + \beta) \), \( \beta = (\frac{1}{2}, \frac{1}{2}) \) satisfying \( M_0(\xi)M_0^*(\xi) + M_1(\xi)M_1^*(\xi) = I \), \( M^* \) is the conjugate transpose of \( M \). Hence, \( \hat{\psi}(\xi q) = M_1(\xi)\hat{\phi}(\xi) \).
BB-expressions of polynomials and splines-1

Let $x^0, \ldots, x^d \in \mathbb{R}^d$, $d \geq 1$, $x^i = (x^i_1, \ldots, x^i_d)$ and consider the convex hull

$$T_d := \langle x^0, \ldots, x^d \rangle = \left\{ \sum_{i=0}^{d} \alpha_i x^i : \sum_{i=0}^{d} \alpha_i = 1, \alpha_i \geq 0 \right\}.$$ 

This convex hull is called an $d$-simplex if its signed volume $Vol_d \langle x^0, \ldots, x^d \rangle$ is nonzero. Suppose that $\langle x^0, \ldots, x^d \rangle$ is an $d$-simplex. Then any $x \in \mathbb{R}^d$ can be identified by an $(d + 1)$-tuple $\lambda = (\lambda_0, \ldots, \lambda_d)$, the barycentric coordinates of $x$ relative to the $d$-simplex $\langle x^0, \ldots, x^d \rangle$, where

$$\lambda_i = \lambda_i(x) = \frac{Vol_d \langle x^0, \ldots, x^{i-1}, x, x^{i+1}, \ldots, x^d \rangle}{Vol_d \langle x^0, \ldots, x^d \rangle}.$$
Thus, each $\lambda_i = \lambda_i(x)$ is a linear polynomial in $x$ with $\sum_{i=0}^{d} \lambda_i = 1$, and if $x \in \langle x^0, \ldots, x^d \rangle$, then $\lambda_i \geq 0$. For any $b = (\beta_0, \ldots, \beta_d) \in \mathbb{Z}_{d+1}^{d+1}$, and $n \in \mathbb{Z}_+$, we will use the usual multivariate notation $\lambda^b = \lambda_0^{\beta_0} \cdots \lambda_d^{\beta_d}$, $b! = \beta_0! \cdots \beta_d!$, and $|b| = \beta_0 + \cdots + \beta_d$. Hence,

$$\phi^n_b(\lambda) := \frac{n!}{b!} \lambda^b$$

is a polynomial in $\pi^d_{|\beta|}$, the space of all polynomials in $d$ variables of order $|\beta| + 1$, or degree at most $|\beta|$.
BB-expressions of polynomials and splines-3

With any set \( \{a^n_\beta\} = \{a^n_\beta\}_{\beta \in \mathbb{Z}^+_{d+1}, |\beta|=n} \subset \mathbb{R} \) one may associate the polynomial

\[
p_n(x) = B_n[\{a^n_\beta\}; \lambda] = \sum_{|\beta|=n} a^n_\beta \phi^n_\beta(\lambda),
\]

which is called a Bernstein-Bézier polynomial (BB polynomial) of total degree \( n \) relative to the \( d \)-simplex \( \langle x^0, \ldots, x^d \rangle \). In addition, \( \{a^n_\beta : |\beta| = n\} \) shown as in (2) is called the set of Bézier coefficients of the polynomial \( p_n \). The piecewise linear interpolant to the points \( (\beta/n, a^n_\beta) \) is said to be the Bézier net or control net and is displayed schematically in Figure 1 for the case of \( n = 2 \) and \( d = 2 \).
BB-expressions of polynomials and splines-4

Denote $D_y = \sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i}$, where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. For $y = x^i - x^j$, we denote

$$D_{ij} = D_y = D_{x^i - x^j}, \quad i \neq j.$$ 

By using the barycentric coordinates $\{\lambda_\ell\}_{\ell=0}^{d}$ of $x \in \mathbb{R}^d$ relative to an $d$-simplex $T_d = \langle x^0, \ldots, x^d \rangle$, we can write $x = \sum_{\ell=0}^{d} \lambda_\ell x^\ell$. If we define

$$E_ia_\alpha := a_\alpha + e^i$$

and

$$\triangle_{ij}a^n_\alpha = E_ia^n_\alpha - E_ja^n_\alpha,$$

where $e^i = (\delta_{ij})_{j=0}^{d}$ denotes the $i^{th}$ coordinate vector in $\mathbb{R}^{d+1}$, then

$$D_{ij}p_n = n \sum_{|\alpha|=n-1} (E_i - E_j)a^n_\alpha \phi^{n-1}_\alpha(\lambda) = n \sum_{|\alpha|=n-1} \triangle_{ij}a^n_\alpha \phi^{n-1}_\alpha(\lambda).$$
Continuous Wavelets with Composite Dilation-1

Let $B$ be the group of order 3 generated by the matrix
\[ \rho = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \]
which is the counter-clockwise rotation by $2\pi/3$. Consider the hexagon $H$ centered at the origin consisting of the diamonds $R_i = (v_{i0}, v_{i1}, v_{i2}, v_{i3})$, ($i = 0, 1, 2$), where $v_{i0}$, $v_{i1}$, $v_{i2}$, and $v_{i3}$ are vertices of $R_i$, and $R_0$ has vertices $v_{00} = (0, 0), v_{01} = (\sqrt{3}/4, -3/4), v_{02} = (\sqrt{3}/2, 0), v_{03} = (\sqrt{3}/4, 3/4)$. 
The elements of $B$ map $R_0$ onto other diamonds $R_i = \rho^i R_0$ $(i = 1, 2)$. Let $C = \frac{1}{4} \begin{pmatrix} 0 & 3\sqrt{3} \\ 6 & 3 \end{pmatrix}$ and $\Gamma_0 = C\mathbb{Z}^2$. The translates of the hexagon by $\gamma \in \Gamma_0$ form a partition of $\mathbb{R}^2$ with the centers of the hexagons in the partition being the lattice points $\gamma$. Let $q = \begin{pmatrix} 1 \\ \sqrt{3} \\ -\sqrt{3} \\ 1 \end{pmatrix}$. The MRA is now generated by the composite dilation system $\{D_{q^j}D_{\rho^i}T_\gamma: j \in \mathbb{Z}, i = 0, 1, 2, \gamma \in \Gamma_0\}$ applied to the linear scaling function $\phi(x)$ with $\phi(v_{00}) = 1$, $\phi(v_{01}) = \phi(v_{02}) = \phi(v_{03}) = 0$ (i.e., the Bézier coefficient vector of $\phi$ is $(1, 0, 0, 0)$). Here, the vertex $v_{00}$ at which $\phi$ has value 1 is the initial vertex of diamond boundary. The space $V_j$ are $q^{-j}$ dilates of $V_0$, i.e., $V_j = D_{q^{-j}}V_0$ $(j \in \mathbb{Z})$. 
The space \( V_0 \subset L^2(\mathbb{R}^2) \) consists of the linear functions \( \phi_i(x) \) defined on \( R_i \) \( (i = 0, 1, 2) \), with values at vertices of \( R_i \) as \( \phi(v_{i0}) = 1 \) and \( \phi(v_{i1}) = \phi(v_{i2}) = \phi(v_{i3}) = 0 \), and their translations defined on \( \Gamma_0 \)-translates of the diamonds \( R_i \) \( (i = 0, 1, 2) \). In order to describe the space \( V_1 \) we consider the original hexagon \( H \) and, within \( H \), the smaller hexagon \( q^{-1}H \), which is the disjoint union of the diamonds \( R_i = \rho^i R_0 \) \( (i = 0, 1, 2) \) and their translations. \( \Phi = [\phi_0, \phi_1, \phi_2]^T \) is refinable. The corresponding multiwavelet \( \Psi \) and the duals of the \( \Phi \) and \( \Psi \) are constructed. (More details available upon request.)
\( \phi_0(x) \) is refinable:

\[
\phi_0(x) = \phi_2 \left( q^{-1} \rho^2 x + \left( \frac{\sqrt{3}}{4} \right) \right) + \frac{1}{2} \left[ \phi_2 \left( q^{-1} \rho^2 x + \left( \frac{3\sqrt{3}}{8} \right) \right) \\
+ \phi_2 \left( q^{-1} \rho^2 x + \left( \frac{3\sqrt{3}}{8} \right) \right) + \phi_2 \left( q^{-1} \rho^2 x + \left( \frac{\sqrt{3}}{2} \right) \right) \right]
\]  

(4)
Similarly, we have

\[ \phi_1(x) = \phi_0 \left( q^{-1} \rho^2 x + \left( \frac{-\sqrt{3}}{8} \right) \right) + \frac{1}{2} \left[ \phi_0 \left( q^{-1} \rho^2 x + \left( \frac{0}{3} \right) \right) + \phi_0 \left( q^{-1} \rho^2 x + \left( \frac{-3\sqrt{3}}{8} \right) \right) \right] \] (5)

\[ \phi_2(x) = \phi_1 \left( q^{-1} \rho^2 x - \left( \frac{\sqrt{3}}{8} \right) \right) + \frac{1}{2} \left[ \phi_1 \left( q^{-1} \rho^2 x + \left( \frac{-3\sqrt{3}}{8} \right) \right) + \phi_1 \left( q^{-1} \rho^2 x - \left( \frac{0}{3} \right) \right) \right] \] (6)
In the above three expressions, the initial vertices of the supports of functions \( \phi_2 \left( q^{-1} \rho^2 x + \left( \frac{\sqrt{3}}{4} \right) \right) \), \( \phi_0 \left( q^{-1} \rho^2 x + \left( \frac{-\sqrt{3}}{8} \right) \right) \), and \( \phi_1 \left( q^{-1} \rho^2 x - \left( \frac{\sqrt{3}}{8} \right) \right) \) are the original, which is an element in \( \Gamma_0 \). It can be seen that all coefficients of those functions in expressions (4)-(6) are 1. Furthermore, the initial vertices of the boundaries of the supports of all other functions \( \phi' \)'s with coefficients 1/2 in the expressions are not in \( \Gamma_0 \). Those functions \( \phi_i(qx + \cdot) \), which supports have no initial vertices in \( \Gamma_1 \), will be defined as our wavelet functions \( \psi_i \cdot(x) \) \( (i = 0, 1, 2) \) with a certain translations. Hence, any element \( \phi_i(qx + \cdot) \) in \( V_1 \) is either \( \psi_i(x) \) or the difference of \( \phi_j(x) \) and \( \psi_j(x + \cdot)'s \), where \( j = 2, 0, 1 \) when \( i = 0, 1, 2 \), respectively.
We define the dual scaling functions $\tilde{\phi}_i(x - \gamma) = \delta_{\gamma}$ ($\gamma \in \Gamma_1$), where $\delta_{\gamma}$ is the Dirac distribution at the node $\gamma$. We define dual wavelet of $\psi_{i,\sigma}$ by $	ilde{\psi}_{i,\sigma} = \delta_{\sigma} - (1/2)(\delta_{\sigma_1} + \delta_{\sigma_2})$, where $\sigma_1, \sigma_2 \in \Gamma_1$ and $\sigma = (\sigma_1 + \sigma_2)/2$ (we can do it because each $\sigma$ is a middle point of two points in $\Gamma_1$). It can be seen that $\{\phi, \psi\}$ and $\{\tilde{\phi}, \tilde{\psi}\}$ are generalized biorthogonal in the Radon measurement. And for a function $f \in V_1$, it can be decomposed in terms of the dual bases. In order to obtain a more stable decomposition, we shall use the lifting-scheme (Sweldens [96]) to modify the bases.
Consider the hexagonal lattice $\Delta$ in $\mathbb{R}^2$ defined by $C\mathbb{Z}^2$ with
\[ C = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}. \]
Let $\Delta^3$ be the type-3 refinement of $\Delta$. We call $\phi \in S^1_2(\Delta^3)$ a Powell-Sabin (PS) spline or macroelement. For any $k \in \Delta$, the Hermite interpolation problem
\[
\begin{pmatrix} \phi_k,0(\ell) & D_1\phi_k,0(\ell) & D_2\phi_k,0(\ell) \\ \phi_k,1(\ell) & D_1\phi_k,1(\ell) & D_2\phi_k,1(\ell) \\ \phi_k,2(\ell) & D_1\phi_k,2(\ell) & D_2\phi_k,2(\ell) \end{pmatrix} = \delta_{k,\ell} I
\]
has a unique solution $\Phi_k = (\phi_k,0, \phi_k,1, \phi_k,2)^T$. And
\[
\{ \Phi_{0,k} \equiv \Phi(x - \Gamma k) : k \in \mathbb{Z}^2 \}
\]
is a basis of $S^1_2(\Delta^3)$. The BB-expressions of $\Phi_{0,i}$ ($i = 0, 1, 2$) are given.
$C^1$ Quadratic Prewavelets with Composite Dilations-2

$\Phi$ is refinable with respect to the dilation matrix $D = 2I$:

$$\Phi(x) = \sum_{k \in \mathbb{Z}^2} C_k \Phi(Dx - \Gamma k), \quad x \in \mathbb{R}^2.$$ 

The refinement $\Delta_j := D^{-j} \Delta$ is the mid-edge subdivision that generates PS partition $\Delta^3_j := D^{-j} \Delta^3$. The corresponding nested subspaces $V_j = S_2^1(\Delta^3_j) \subset L^2(\mathbb{R}^2), \ j \in \mathbb{Z}$, form a MRA of multiplicity 3.
$C^1$ Quadratic Prewavelets with Composite Dilation-3

$C_{1,0} = \frac{1}{12} \begin{pmatrix} 0 & 0 & 0 \\ 3 & -6 & -2\sqrt{3} \\ 3 & -6 & 2\sqrt{3} \end{pmatrix}$

$C_{1,1} = \frac{1}{12} \begin{pmatrix} 1 & -4 & 0 \\ 1 & 2 & -2\sqrt{3} \\ 4 & -4 & -4\sqrt{3} \end{pmatrix}$

$C_{0,0} = \frac{1}{3} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -\sqrt{3} \\ 1 & 1 & \sqrt{3} \end{pmatrix}$

$C_{0,-1} = \frac{1}{12} \begin{pmatrix} 1 & -4 & 0 \\ 4 & -4 & 4\sqrt{3} \\ 1 & 2 & 2\sqrt{3} \end{pmatrix}$

$C_{0,1} = \frac{1}{12} \begin{pmatrix} 3 & 0 & -4\sqrt{3} \\ 0 & 0 & 0 \\ 3 & 6 & -2\sqrt{3} \end{pmatrix}$

$C_{-1,0} = \frac{1}{12} \begin{pmatrix} 4 & 8 & 0 \\ 1 & 2 & -2\sqrt{3} \\ 1 & 2 & 2\sqrt{3} \end{pmatrix}$

$C_{-1,-1} = \frac{1}{12} \begin{pmatrix} 3 & 4 & 4\sqrt{3} \\ 3 & 6 & 2\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}$
Let $H$ be the Hermit interpolation operator, and let $H^r_2(\Omega)$ be the Sobolev space with norm $\|f\|_{r,\Omega} := \left(\sum_{|\alpha| \leq r} \|D^\alpha f\|_2^2\right)^{1/2}$, where $\|\|_2$ is the $L^2$ norm on $\Omega$ and $D^\alpha f = (\partial^{|\alpha|}/\partial x^{\alpha_1} \partial x^{\alpha_2})f$, $\alpha = (\alpha_1, \alpha_2)$. Thus, there exists an apopositive constant $C$ such that

$$\|f - Hf\|_{r,\Omega} \leq C\delta^{3-r}\|f\|_{3,r}$$

for all $f \in H^3_2(\Omega)$ and $r = 0, \ldots, 3$, where $\delta = |\Delta|$ the maximum of the diameters of the triangles in the triangulation $\Delta$. Particularly, for $r = 0$, there holds $\|f - Hf\|_\infty \leq C\delta^3\|f\|_{3,\Omega}$ for all $f \in H^3_2(\Omega)$.
$C^1$ Quadratic Prewavelets with Composite Dilation-5

Find the dual basis $\{\tilde{\Phi}_{j,k} : k \in \Delta_j\} \subset V_{j+1}$ using the matrix system

$$\langle \tilde{\Phi}_{j,k}, \Phi^T_{j,k_i} \rangle = \delta_{i,0} \| \Phi_{j,k} \|^2,$$

where $\| \Phi_{j,k} \|^2 = \text{diag} \left( \langle \Phi_{j,k}, \Phi^T_{j,k} \rangle \right)$, $k_0$ is the center of the Hexagon and $k_i$ ($i = 1, 2, \ldots, 6$) are its boundary vertices. In addition, $\tilde{\Phi}_{j,k}$ has the compact expression

$$\tilde{\Phi}_{j,k} = \sum_{i=0}^{6} P_i \Phi_{j+1, \ell_i},$$

where $\ell_0 = k_0$ and $\ell_i$ is the middle point of $k_0k_i$ ($i = 1, 2, \ldots, 6$).
$C^1$ Quadratic Prewavelets with Composite Dilation-6

$\{2^{j+1}\tilde{\Phi}_{j,k} : k \in \Delta_j\} \cup \{2^{j+1}\Phi_{j+1,\ell} : \ell \in \Delta_{j+1} \setminus \Delta_j\}$ forms a Riesz basis of $V_{j+1}$. Since $\langle \tilde{\Phi}_{j,k}, \Phi_{j,\ell}^T \rangle = \delta_{k,\ell} \|\Phi_{j,k}\|^2$, we know the

$$\psi_{j,\ell} := \Phi_{j+1,\ell} - \sum_{k \in \Delta_j} \langle \Phi_{j+1,\ell}, \Phi_{j,k}^T \rangle \|\Phi_{j,k}\|^{-2} \tilde{\Phi}_{j,k}$$

is in $V_{j+1} \cap V_{j+1}^T$, which yields the Riesz basis, $\{2^{j+1}\psi_{j,\ell} : \ell \in \Delta_{j+1} \setminus \Delta_j\}$, of $W_j$, and $L^2(\mathbb{R}^2) = \bigoplus_{j \in \mathbb{Z}} W_j$. 
Acknowledgement

Thanks for the comments and suggestions by Guido Weiss and Edward Wilson, Washington University, St. Louis, USA.