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Asymptotic Expansions and Computation of Generalized Stirling Numbers and Generalized Stirling Functions

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Abstract

Here presented is a unified approach to Stirling numbers and their generalizations as well as generalized Stirling functions by using generalized factorial functions, k-Gamma functions, and generalized divided difference. Previous well-known extensions of Stirling numbers due to Riordan, Carlitz, Howard, Charalambides-Koutras, Gould-Hopper, Hsu-Shiue, Tsylova Todorov, Ahuja-Enneking, and Stirling functions introduced by Butzer and Hauss, Butzer, Kilbas, and Trujilloet and others are included as particular cases of our generalization. Some basic properties related to our general pattern such as their recursive relations and generating functions are discussed. Some asymptotic expansions for the generalized Stirling functions and generalized Stirling numbers are established. In addition, four algorithms for calculating the Stirling numbers based on our generalization are also given.

Generalized Stirling Numbers in a Unified Form

- Classical Stirling numbers
- Generalized Stirling numbers
- Generalized divided difference
- Unified expression of generalized Stirling numbers
- Generalized fractional difference operators
- Definition of generalized Stirling functions
- Existence and recurrence relationship of generalized Stirling functions
- Generating functions of generalized Stirling functions
- Asymptotic properties of generalized Stirling functions
- Algorithms of generalized Stirling numbers

Classical Stirling Numbers

$$[z]_n = \sum_{k=0}^{n} s(n,k) z^k, \quad z^n = \sum_{k=0}^{n} S(n,k)[z]_k, \tag{1}$$

with the convention $s(n,0) = S(n,0) = \delta_{n,0}$, the Kronecker symbol, where $z \in \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the falling factorial polynomials $[z]_n = z(z-1)\cdots(z-n+1)$.

$$\frac{(\log(1+x))^k}{k!} = \sum_{n \geq k} s(n,k) \frac{x^n}{n!}, \quad \frac{(e^x - 1)^k}{k!} = \sum_{n \geq k} S(n,k) \frac{x^n}{n!}, \tag{2}$$

where $|x| < 1$ and $k \in \mathbb{N}_0$.

"Stirling's numbers are of the greatest utility. This however has not been fully recognized." "Stirling's numbers are as important or even more so than Bernoulli's numbers." ——-Ch. Jordan

Unified Form of Classical Stirling numbers

$$
\begin{aligned}
S(n,k) &:= \frac{1}{k!}\,\Delta^k z^n\Big|_{z=0} = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n \\
&= \frac{1}{k!}\sum_{j=1}^{k}(-1)^{j}\binom{k}{j}(k-j)^n. \qquad (3)
\end{aligned}
$$

Denote $\langle z\rangle_{n,\alpha} := z(z+\alpha)\cdots(z+(n-1)\alpha)$ for $n=1,2,\ldots,$ and $\langle z\rangle_{0,\alpha}=1$, where $\langle z\rangle_{n,\alpha}$ is called the generalized factorial of z with increment α. Thus, $\langle z\rangle_{n,-1}=[z]_n$ with $[z]_0=1$, and $\langle z\rangle_{n,0}=z^n$.

$$
\langle z\rangle_{n,-\alpha} = \sum_{k=0}^{n} S(n,k,\alpha,\beta)\langle z\rangle_{n,-\beta}, \qquad (4)
$$

$S(n,k,1,0)=s(n,k)$, the Stirling numbers of the first kind and $S(n,k,0,1)=S(n,k)$. the Stirling numbers of the second kind.

Generalized Stirling Numbers

Definition

Let $n \in \mathbb{N}$ and $\alpha, \beta, r \in \mathbb{R}$. A generalized Stirling number denoted by $S(n, k, \alpha, \beta, r)$ is defined by

$$\langle z \rangle_{n,-\alpha} = \sum_{k=0}^{n} S(n, k, \alpha, \beta, r) \langle z - r \rangle_{k,-\beta}. \tag{5}$$

In particular, if $(\alpha, \beta, r) = (1, 0, 0)$, $S(n, k, 1, 0, 0)$ is reduced to the unified form of Classical Stirling numbers defined by (4).

This definition is originally given by Hsu and Shiue in 1998 associated with $-r$. We prefer this form for the convenience of expression and computation.

Generalized Stirling Numbers (Cont.)

Each $\langle z\rangle_{n,-\alpha}$ does have exactly one such expansion (5) for any given z. Since *deg* $\langle z-r\rangle_{k,-\beta}=k$ for all k, which generates a graded basis for $\Pi\subset\mathbb{F}\to\mathbb{F}$, the linear spaces of polynomials in one real (when $\mathbb{F}=\mathbb{R}$) or complex (when $\mathbb{F}=\mathbb{C}$), in the sense that, for each n, $\{\langle z-r\rangle_{n,-\beta}\}$ is a basis for $\Pi_n\subset\Pi$, the subspace of all polynomials of degree $<n$. In other wards, the column map

$$W_z:\mathbb{F}_0^N\to\Pi:s\mapsto\sum_{k\geq 0}S(n,k,\alpha,\beta,r)\langle z\rangle_{k,-\beta},$$

from the space $\mathbb{F}_0^N$ of scalar sequences with finitely many nonzero entries to the space Π is one-to-one and onto, hence invertible. In particular, for each $n\in\mathbb{N}$, the coefficient $c(n)$ in the Newton form (5) for $\langle z\rangle_{n,-\alpha}$ depends linearly on $\langle z\rangle_{n,-\alpha}$, i.e., $\langle z\rangle_{n,-\alpha}\mapsto s(n)=(W_z^{-1}\langle z\rangle_{n,-\alpha})(n)$, the set of $S(n,k,\alpha,\beta,r)$, is a well-defined linear functional on Π, and vanishes on $\Pi_{<n-1}$.

Generalized Stirling Numbers (Cont.)

Stirling-type pair $\{S^1, S^2\} = \{S^1(n,k), S^2(n,k)\} \equiv \{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}$ can be defined by the inverse relations

$$\langle z\rangle_{n,-\alpha} = \sum_{k=0}^{n} S^1(n,k)\langle z-r\rangle_{k,-\beta} \quad \langle z\rangle_{n,-\beta} = \sum_{k=0}^{n} S^2(n,k)\langle z+r\rangle_{k,-\alpha}, \quad (6)$$

where $n \in \mathbb{N}$ and the parameter triple $(\alpha,\beta,r) \neq (0,0,0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$.

$$S(n,k;0,0,1) = \binom{n}{k}$$

$$s(n,k) = S^1(n,k;1,0,0) \quad S(n,k) = S^2(n,k;1,0,0).$$

$$S(0,0) = 1, \quad S(n,n) = 1, \quad S(1,0) = r, \quad \textit{and} \quad S(n,0) = \langle r\rangle_{n,-\alpha}. \quad (7)$$

(α, β, r)	dual	$S(n,k)$ pairs	Name of Stirling
$(-1,1,0)$	$(1,-1,0)$	$n!\binom{n-1}{k-1}/k!$ $(-1)^{n-k}n!\binom{n-1}{k-1}/k!$	*Lah number pair*
$(-1,0,0)$	$(0,-1,0)$	$\lvert s(n,k)\rvert$ $(-1)^{n-k}S(n,k)$	*signless Stirling nur*
$(1,\theta,0)(\theta\neq 0)$	$(\theta,1,0)$	$S(n,k,1,\theta,0)$ $S(n,k,\theta,1,0)$	*Carlitz's degenera number pair*
$(1,0,-\lambda)$	$(0,1,\lambda)$	$S(n,k,1,0,-\lambda)$ $S(n,k,0,1,\lambda)$	*Carlitz's weighted number pair*
$(1,\theta,-\lambda)$	$(\theta,1,\lambda)$	$S(n,k,1,\theta,-\lambda)$ $S(n,k,\theta,1,\lambda)$	*Howard's weightec Stirling number pa*
$(0,1,-a+b)$	$(1,0,-b+a)$	$S(n,k,0,1,-a+b)$ $S(n,k,1,0,-b+a)$	*Gould − Hopper's number pair*
$(1/s,1,-a+b)$	$(1,1/s,-b+a)$	$S(n,k,1/s,1,-a+b)$ $S(n,k,1,1/s,-b+a)$	*Charalambides− Koutras's number*

(α, β, r)	dual	$S(n,k)$ *pairs*	*Name of Stirling numbers*
$(1, 0, b-a)$	$(0, 1, a-b)$	$S(n,k,1,0,b-a)$ $S(n,k,0,1,a-b)$	$Riordan's\ non-central\ Sti$ $number\ pair$
$(\alpha, \beta, 0)$	$(\beta, \alpha, 0)$	$A_{\alpha\beta}(r,m)$ $B_{\alpha\beta}(r,m)$	$Tsylova's\ Stirling\ number\ pa$
(α, β, r)	$(\beta, \alpha, -r)$	$S(n,k,\alpha,\beta,r)$ $S(n,k,\beta,\alpha,-r)$	$Hsu-Shiue's\ Stirling$ $number\ pair$
$(1, x, 0)$	–	$a_{nk}(x)$	$Todorov's\ Stirling\ numbers$
$(-1/r, 1, 0)$	–	$B(n,r,k)$	$Ahuja-Enneking's\ associa$ $Lah\ numbers$
$(-1, 0, r)$	–	$S(n-r, k-r, -1, 0, r)$	$Broder's\ r-Stirling\ number$

Table 1. Some generalized Stirling Numbers and Stirling Number pairs

Generalized factorial polynomials

The classical falling factorial polynomials $[z]_n = z(z-1)\cdots(z-n+1)$ and classical rising factorial polynomials $[z]^n = z(z+1)\cdots(z+n-1)$, $z \in \mathbb{C}$ and $n \in \mathbb{N}$, can be unified to the expression $\langle z\rangle_{n,\pm 1} := z(z\pm 1)\cdots(z\pm(n-1))$, using the *generalized factorial polynomial* expression

$$\langle z\rangle_{n,k} := z(z+k)\cdots(z+(n-1)k) = \langle z+(n-1)k\rangle_{n,-k} \quad (z \in \mathbb{C}, n \in \mathbb{N}). \tag{8}$$

Thus $\langle z\rangle_{n,1} = [z]^n$ and $\langle z\rangle_{n,-1} = [z]_n$.

$$\langle z\rangle_{n,k} = k^n[z/k]^n \quad (z \in \mathbb{C}, n \in \mathbb{N}, k > 0). \tag{9}$$

$$\langle z\rangle_{n,-k} = z(z-k)\cdots(z-(n-1)k) = k^n[z/k]_n \quad (z \in \mathbb{C}, n \in \mathbb{N}, k > 0). \tag{10}$$

The history as well as some important basic results of the generalized factorials can be found in Chapter II of Ch. Jordan's book, and an application is shown on Page 31 of Gel'fond's book.

Generalized difference operator

Difference operator in terms of β ($\beta \neq 0$) is defined by

$$\Delta_\beta^k f = \Delta_\beta(\Delta_\beta^{k-1} f) \quad (k \geq 2) \quad and \quad \Delta_\beta f(t) := f(t+\beta) - f(t). \tag{11}$$

Hence,

$$\Delta_\beta^k f(t) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(t + (k-j)\beta).$$

It can be seen that $\Delta_\beta^k \langle z \rangle_{j,-\beta}\Big|_{z=0} = \beta^k k! \delta_{k,j}$, where $\delta_{k,j}$ is the Kronecker delta symbol; i.e., $\delta_{k,j} = 1$ when $k = j$ and 0 otherwise. Evidently, from (10) there holds

$$\Delta_\beta^k \langle z \rangle_{j,-\beta}\Big|_{z=0} = \Delta_\beta^k \beta^j \left[\frac{t}{\beta}\right]_j \Bigg|_{z=0} = \beta^j \Delta^k [t]_j \Big|_{z=0} = \beta^k k! \delta_{k,j}. \tag{12}$$

Divided difference

Denote the divided difference of $f(t)$ at $t+i$, $i=0,1,\ldots,k$, by $f[t,t+1,\ldots,t+k]$, or $[t,t+1,\ldots,t+k]f(t)$, here

$$f[t,t+1]:=\frac{f(t+1)-f(t)}{(t+1)-t}=\Delta f(t),$$

$$f[t,t+1,t+2]:=\frac{f[t+1,t+2]-f[t,t+1]}{(t+2)-t}=\frac{\Delta^2 f(t)}{2!}, etc.$$

Hence,

$$\frac{1}{k!}\Delta^k f(t)=f[t,t+1,\ldots,t+k]=[t,t+1,\ldots,t+k]f(t)$$

and

$$\frac{1}{\beta^k k!}\Delta_\beta^k f(t)=f[t,t+\beta,t+2\beta,\ldots,t+k\beta]=[t,t+\beta,\ldots,t+k\beta]f(t).$$

Generalized divided difference

We now give the following definition of the *generalized divided differences.*

Definition

We define $\underline{\triangle}_{\beta}^{k} f(t)$ by

$$\underline{\triangle}_{\beta}^{k} f(t) = \begin{cases} \frac{1}{\beta^k k!}\Delta_{\beta}^{k} f(t) = f[t, t+\beta, \ldots, t+k\beta] & \textit{if } \beta \neq 0 \\ \frac{1}{k!} D^k f(t) & \textit{if } \beta = 0 \end{cases}, \tag{13}$$

where $\Delta_{\beta}^{k} f(t)$ is shown in (11), $f[t, t+\beta, \ldots, t+k\beta] \equiv [t, t+\beta, \ldots, t+k\beta]f$ is the kth divided difference of f in terms of $\{t, t+\beta, \ldots, t+k\beta\}$, and $D^k f(t)$ is the kth derivative of $f(t)$.

Unified expression of generalized Stirling numbers

We now give a unified expression of the generalized Stirling numbers in terms of the the generalized divided differences.

Theorem

Let $n, k \in \mathbb{N}_0$ and the parameter triple $(\alpha, \beta, r) \neq (0, 0, 0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. For the generalized Stirling numbers defined by (5), there holds

$$S(n,k,\alpha,\beta,r) = \underline{\triangle}_{\beta}^{k}\langle z\rangle_{n,-\alpha}\Big|_{z=r}$$
$$= \begin{cases} \frac{1}{\beta^k k!}\,\Delta_{\beta}^{k}\langle z\rangle_{n,-\alpha}\Big|_{z=r} = [r, r+\beta, \ldots, r+k\beta]\langle z\rangle_{n,-\alpha} & \text{if } \beta \neq 0 \\ \frac{1}{k!}\,D^k\langle z\rangle_{n,-\alpha}\big|_{z=r} & \text{if } \beta = 0. \end{cases} \tag{14}$$

Unified expression of generalized Stirling numbers (Cont.)

In particular, for the generalized Stirling number pair defined by (6), we have the expressions

$$S^1(n,k) \equiv S^1(n,k,\alpha,\beta,r) = \triangle_{\beta}^{k}\langle z\rangle_{n,-\alpha}\Big|_{z=r}$$
$$S^2(n,k) \equiv S^2(n,k,\beta,\alpha,-r) = \triangle_{\alpha}^{k}\langle z\rangle_{n,-\beta}\Big|_{z=-r} \tag{15}$$

Furthermore, if $(\alpha,\beta,r)=(1,0,0)$, then (14) is reduced to the classical Stirling numbers of the first kind defined by (1) with the expression $s(n,k)=S(n,k,1,0,0)=\frac{1}{k!}D^k[z]_n\Big|_{z=0}$. If (α,β,r) $=(0,1,0)$, then (14) is reduced to the classical Stirling numbers of the second kind shown in (3) with the following divided difference expression form:

$$S(n,k)=S(n,k,0,1,0)=[0,1,2,\ldots,k]z^n|_{z=0}. \tag{16}$$

Unified expression of generalized Stirling numbers (Cont.)

The following corollary is obvious due to the expansion formula of the divided differences generated from their definition.

Corollary

Let $n, k \in \mathbb{N}_0$ and the parameter triple $(\alpha, \beta, r) \neq (0,0,0)$ is in $\mathbb{R}^3$ or $\mathbb{C}^3$. If $\beta \neq 0$, for the generalized Stirling numbers defined by (5), there holds

$$S(n,k) \equiv S(n,k,\alpha,\beta,r) = \frac{1}{\beta^k k!} \sum_{j=0}^{n} (-1)^j \binom{k}{j} \langle r+(k-j)\beta \rangle_{n,-\alpha} \quad (n \neq 0) \tag{17}$$

and $S(0,k) = \delta_{0k}$ *and* $S(n,0) = \langle r \rangle_{n,-\alpha}$.

Newton algorithm of generalized Stirling numbers

If $\beta \neq 0$, we denote

$$\underline{\triangle}_{\beta}^{j} f(t+\ell\beta) := f[t, t+\ell\beta, t+(\ell+1)\beta, \ldots, t+j\beta] \qquad (18)$$

Thus, from (14) in Theorem 3,

$$\underline{\triangle}_{\beta}^{j} f(t+\ell\beta) = \frac{1}{j\beta}(\underline{\triangle}_{\beta}^{j-1} f(t+(\ell+1)\beta) - \underline{\triangle}_{\beta}^{j-1} f(t+\ell\beta)), \quad (19)$$

Algorithm

This Newton algorithm of evaluating the generalized Stirling numbers is based on the construction of the following lower triangle array by using (18) and (19).

$$\begin{array}{llll}
\langle z\rangle_{n,-\alpha}\big|_{z=r} & & & \\
\langle z+\beta\rangle_{n,-\alpha}\big|_{z=r} & \underline{\triangle}_{\beta}\langle z\rangle_{n,-\alpha}\Big|_{z=r} & & \\
\langle z+2\beta\rangle_{n,-\alpha}\big|_{z=r} & \underline{\triangle}_{\beta}\langle z+\beta\rangle_{n,-\alpha}\Big|_{z=r} & \underline{\triangle}_{\beta}^{2}\langle z\rangle_{n,-\alpha}\Big|_{z=r} & \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Table 2. Newton algorithm for the The generalized Stirling numbers

Thus, the diagonal of the above lower triangle array gives $S(n,i,\alpha,\beta,r)=\underline{\triangle}_{\beta}^{i}\langle z\rangle_{n,-\alpha}\Big|_{z=r}$ *for* $i=0,1,\ldots,k$.

Example

We now use Algorithm 5 shown in Table 2 to evaluate the classical Stirling numbers of the second kind $S(4,k) = S(4,k,0,1,0)$ $(k = 1,2,3,4)$, which are re-expressed by (16). Thus,

$$\begin{array}{lllll}
0 & & & & \\
1 & 1 & & & \\
2^4 = 16 & 15 & 7 & & \\
3^4 = 81 & 65 & 25 & 6 & \\
4^4 = 256 & 175 & 55 & 10 & 1
\end{array}$$

From the diagonal of the above lower triangular matrix, we may read $S(4,0) = 0$, $S(4,1) = 1$, $S(4,2) = 7$, $S(4,3) = 6$, and $S(4,4) = 1$. Meanwhile, the subdiagonal gives $S(5,1) = 1$, $S(5,2) = 15$, $S(5,3) = 25$, and $S(5,4) = 10$.

Example

For the Howard's weighted degenerate Stirling numbers $S(4,k) = S(4,k,1,2,-1)$, using Algorithm 5, we obtain $S(4,0) = 24$, $S(4,1) = -12$, $S(4,2) = 3$, $S(4,3) = 2$, and $S(4,4) = 1$ reading from the following table.

$$\begin{array}{llllll}
\langle z\rangle_{4,-1}|_{z=-1} = 24 & & & & \\
\langle z+2\rangle_{4,-1}|_{z=-1} = 0 & -12 & & & \\
\langle z+4\rangle_{4,-1}|_{z=-1} = 0 & 0 & 3 & & \\
\langle z+6\rangle_{4,-1}|_{z=-1} = 120 & 60 & 15 & 2 & \\
\langle z+8\rangle_{4,-1}|_{z=-1} = 840 & 360 & 75 & 10 & 1
\end{array}$$

Generalized fractional difference operators

First, in order to cover as large a function class as possible, we recall that the *generalized fractional difference operator* $\Delta_\beta^{\eta,\epsilon}$ with an exponential factor, which is introduced in [**?**]. More precisely, for $\eta \in \mathbb{C}$, $\beta \in \mathbb{R}_+$, $\epsilon \geq 0$, the generalized fractional difference operator $\Delta_\beta^{\eta,\epsilon}$ is defined for "sufficient good" functions f by

$$\Delta_\beta^{\eta,\epsilon} f(z) := \sum_{j\geq 0} (-1)^j \binom{\eta}{j} e^{(\eta-j)\epsilon} f(z+(\eta-j)\beta) \quad (z \in \mathbb{C}), \quad (20)$$

where $\binom{\eta}{j}$ are the general binomial coefficients given by

$$\binom{\eta}{j} = \frac{[\eta]_j}{j!} := \frac{\eta(\eta-1)\cdots(\eta-j+1)}{j!} \quad (j \in \mathbb{N}), \qquad (21)$$

with $[\beta]_0 = 1$.

Definition of generalized Stirling functions

Definition

The generalized degenerated Stirling functions, $S(\gamma, \eta, \alpha, \beta, r; \epsilon)$ for any complex numbers γ and η are given by

$$S(\gamma, \eta; \epsilon) \equiv S(\gamma, \eta, \alpha, \beta, r; \epsilon) := \frac{1}{\beta^\eta \Gamma(\eta+1)} \lim_{z \to r} \Delta_\beta^{\eta,\epsilon}(\langle z \rangle_{\gamma,-\alpha}) \quad (\epsilon \geq 0), \tag{22}$$

provided the limit exists; or equivalently, by

$$S(\gamma, \eta; \epsilon) = \frac{1}{\beta^\eta \Gamma(\eta+1)} \sum_{j \geq 0} (-1)^j \binom{\eta}{j} e^{(\eta-j)\epsilon} \langle r+(\eta-j)\beta \rangle_{\gamma,-\alpha} \quad (\gamma \neq 0), \tag{23}$$

provided the series converges absolutely, and $S(0, \eta) = \frac{(e^\epsilon - 1)^\eta}{\beta^\eta \Gamma(\eta+1)}$.

Existence of generalized Stirling functions

Theorem

If $\gamma \in \mathbb{C}$ and either of the conditions $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}$), $\epsilon > 0$, or $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}$, $Re(\eta) > Re(\gamma)$), $\epsilon = 0$ hold, then the generalized Stirling functions $S(\gamma, \eta; \epsilon)$ can be represented in the form (23) and $S(0, \eta; \epsilon) = \delta_{\eta,0}$. In particular, if $n \in \mathbb{N}_0$, $\eta = k \in \mathbb{N}$, and $\epsilon \geq 0$, then the corresponding generalized Stirling functions $S(n, k; \epsilon)$ has the representation (23).

Recurrence relationship of generalized Stirling functions

There hold the following three results. (a) For $\gamma \in \mathbb{C}$, $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}$), and $\epsilon > 0$, the generalized Stirling functions $S(\gamma, \eta; \epsilon)$ defined by (23) satisfy

$$S(\gamma, \eta; \epsilon) = (r+\eta\beta-(\gamma-1)\alpha)S(\gamma-1, \eta; \epsilon)+S(\gamma-1, \eta-1; \epsilon). \quad (24)$$

(b) Let $\gamma \in \mathbb{C}$, $\eta \in \mathbb{C}$ ($\eta \notin \mathbb{Z}$), and $Re(\eta) > Re(\gamma)$). The generalized Stirling functions $S(\gamma, \eta) \equiv S(\gamma, \eta; 0)$ satisfy

$$S(\gamma, \eta) = (r + \eta\beta - (\gamma - 1)\alpha)S(\gamma - 1, \eta) + S(\gamma - 1, \eta - 1). \quad (25)$$

(c) For $\gamma \in \mathbb{C}$, $k \in \mathbb{N}$, and $\epsilon \geq 0$, the generalized Stirling functions $S(\gamma, k; \epsilon; h)$ defined by (23) satisfy

$$S(\gamma, k; \epsilon) = (r+k\beta-(\gamma-1)\alpha)S(\gamma-1, k; \epsilon)+S(\gamma-1, k-1; \epsilon). \quad (26)$$

In particular,
$S(\gamma, k) = (r + k\beta - (\gamma - 1)\alpha)S(\gamma - 1, k) + S(\gamma - 1, k - 1)$.

Generating functions of generalized Stirling functions

Theorem

Let $z \in \mathbb{C}$, $\eta \in \mathbb{C}$, and $\epsilon \geq 0$. The generating function for the generalized Stirling functions $S(\gamma, \eta; \epsilon)$ defined by (23) with $\gamma = n$ and $\alpha\beta \neq 0$ is

$$\frac{1}{\Gamma(\eta+1)}(1+\alpha z)^{r/\alpha}\left(\frac{e^{\epsilon}(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{\eta} = \sum_{n\geq 0} S(n, \eta; \epsilon)\frac{z^n}{n!} \tag{27}$$

for $\eta \notin \mathbb{Z}$ and $\epsilon > 0$, and

$$\frac{1}{k!}(1+\alpha z)^{r/\alpha}\left(\frac{e^{\epsilon}(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{k} = \sum_{n\geq 0} S(n, k; \epsilon)\frac{z^n}{n!} \tag{28}$$

for $\eta = k \in \mathbb{N}_0$ and $\epsilon \geq 0$.

Asymptotics of generalized Stirling functions

Asymptotic expansion of classical Stirling numbers can be found in a numerous papers. For instance, Hsu[1948], Moser and Wyman [1958, 1958], Bleick and Wang [1974], Knessl and Keller [1991], Themme [1993], Flajolet and Prodinger [1999], Chelluri, Richmond, and Temme [2000], Louchard [2010], etc. The asymptotic property in the case of $m \sim n$ (both large) for the three parameter generalized Stirling numbers is still unknown.

Let $\phi(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series over the complex field $\mathbb{C}$ in $\mathcal{F}_0$, with $a_0 = g(0) = 1$. For every j $(0 \leq j < n)$ define

$$W(n,j) = \sum_{\sigma(n,n-j)} \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!}, \tag{29}$$

where the summation is taken over all such partition $1k_1 + 2k_2 + \cdots + nk_n = n$ that have $k_1 + k_2 + \cdots + k_n = n - j$ parts.

Asymptotics of generalized Stirling functions (Cont.)

For a fixed $m \in \mathbb{N}$ and for large μ and n such that $n = o(\mu^{1/2})$ $(\mu \to \infty)$, we have the asymptotic expansion

$$\frac{1}{[\mu]_n}[z^n](\phi(z))^\mu = \sum_{j=0}^{m} \frac{W(n,j)}{[\mu - n + j]_j} + o\left(\frac{W(n,m)}{[\mu - n + m]_m}\right), \quad (30)$$

where $W(n,j)$ are given by (29).
Let $g(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series over the complex field $\mathbb{C}$ in $\mathcal{F}_0$, with $a_0 = g(0) \neq 0$. We may write

$$g(z) = a_0 \sum_{n\geq 0} \frac{a_n}{a_0} z^n.$$

Asymptotics of generalized Stirling functions (Cont.)

For a fixed $m \in \mathbb{N}$ and for large μ and n such that $n = o(\mu^{1/2})$ $(\mu \to \infty)$, From formulas (29) and (30) we have the asymptotic expansion

$$\frac{1}{[\mu]_n}[z^n](g(z))^\mu = \sum_{j=0}^{m} \frac{W(n,j)}{a_0^{n-\mu-j}[\mu-n+j]_j} + o\left(\frac{W(n,m)}{a_0^{m-\mu}[\mu-n+m]_m}\right), \tag{31}$$

where $W(n,j)$ are given by (29).

$$W(n,j) = a_0 \sum_{\sigma(n,n-j)} \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!}. \tag{32}$$

In particular, when n is fixed, the remainder estimate becomes $O(\mu^{-m-1})$.

Asymptotics of generalized Stirling functions (Cont.)

Denote

$$g(z)=(1+\alpha z)^{r/\alpha}\frac{e^{\epsilon}(1+\alpha z)^{\beta/\alpha}-1}{\beta}=\sum_{n\geq 0}\frac{S(n,1;\epsilon)}{n!}z^n \qquad (33)$$

when $\epsilon\neq 0$, and

$$\bar{g}(z)=(1+\alpha z)^{r/\alpha}\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta z}=\sum_{n\geq 0}\frac{S(n+1,1)}{(n+1)!}z^n \qquad (34)$$

when $\epsilon=0$, we have

$$\begin{aligned}(g(z))^{\mu} &= (1+\alpha z)^{\mu r/\alpha}\left(\frac{e^{\epsilon}(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^{\mu}\\ &= \mu!\sum_{n\geq 0}\frac{S(n,\mu,\alpha,\beta,\mu r;\epsilon)}{n!}z^n \qquad (35)\end{aligned}$$

for $\epsilon \neq 0$, and

$$(\bar{g}(z))^{\mu} = (1+\alpha z)^{\mu r/\alpha}\left(\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta z}\right)^{\mu} = \mu! \sum_{n\geq 0} \frac{S(n+\mu,\mu,\alpha,\beta,\mu r)}{(n+\mu)!} z^n \tag{36}$$

for $\epsilon = 0$. Therefore, making use of (31) yields

$$\frac{S(n,\mu,\alpha,\beta,\mu r;\epsilon)}{[\mu]_n[n]_\mu} = \left(\frac{\beta}{e^{\epsilon}-1}\right)^{n-\mu} \times \sum_{j=0}^{m}\left(\frac{e^{\epsilon}-1}{\beta}\right)^{j}\frac{W(n,j)}{[\mu-n+j]_j} + o\left(\left(\frac{\beta}{e^{\epsilon}-1}\right)^{n-\mu}\frac{W(n,m)}{[\mu-n+m]_m}\right) \tag{37}$$

for $\epsilon \neq 0$,

and

$$\frac{S(n+\mu,\mu,\alpha,\beta,\mu r)}{[\mu]_n[n+\mu]_\mu}=\sum_{j=0}^{m}\frac{W(n,j)}{[\mu-n+j]_j}+o\left(\frac{W(n,m)}{[\mu-n+m]_m}\right) \tag{38}$$

for $\epsilon=0$, where $n=o(\mu^{1/2})$ as $\mu\to\infty$ and $W(n,j)$ $(j=0,1,2,\ldots)$ are given by (29).
For $\epsilon\neq 0$, there holds the asymptotic expansion (37) of $S(n,\mu,\mu r;\epsilon)\equiv S(n,\mu,\alpha,\beta,\mu r;\epsilon)$ for n with $n=o(\mu^{1/2})$ $(\mu\to\infty)$, where $W(n,j)$ is defined by (29) with $a_0=(e^\epsilon-1)/\beta$ and $a_j=\frac{1}{j!\beta}\left[\langle r+\beta\rangle_{j,-\alpha}+(e^\epsilon-2)\langle r\rangle_{j,-\alpha}\right]\quad(j=1,2,\ldots)$. For $\epsilon=0$, there holds the asymptotic expansion (38) of $S(n+\mu,\mu,\mu r)\equiv S(n+\mu,\mu,\alpha,\beta,\mu r)$ for n with $n=o(\mu^{1/2})$ $(\mu\to\infty)$, where $W(n,j)$ is defined by (29) with

$$a_j=\frac{1}{(j+1)!\beta}\left[\langle r+\beta\rangle_{j+1,-\alpha}-\langle r\rangle_{j+1,-\alpha}\right]\quad j=0,1,\ldots.$$

Corollary

For $\epsilon \neq 0$, by replacing the quantity r by r/μ, the asymptotic expansion (37) is also applicable to $S(n,\mu,r;\epsilon) \equiv S(n,\mu,\alpha,\beta,r;\epsilon)$ for n with $n = o(\mu^{1/2})$ $(\mu \to \infty)$, where $W(n,j)$ is defined by (29) with $a_0 = (e^\epsilon - 1)/\beta$ and

$$a_j = \frac{1}{j!\beta}\left[\left\langle \frac{r}{\mu} + \beta \right\rangle_{j,-\alpha} + (e^\epsilon - 2)\left\langle \frac{r}{\mu} \right\rangle_{j,-\alpha}\right] \quad (j = 1, 2, \ldots).$$

For $\epsilon = 0$, by replacing the quantity r by r/μ, the asymptotic expansion (38) is also applicable to $S(n+\mu,\mu,r) \equiv S(n+\mu,\mu,\alpha,\beta,r)$ for n with $n = o(\mu^{1/2})$ $(\mu \to \infty)$, where $W(n,j)$ is defined by (29) with

$$a_j = \frac{1}{(j+1)!\beta}\left[\left\langle \frac{r}{\mu} + \beta \right\rangle_{j+1,-\alpha} - \left\langle \frac{r}{\mu} \right\rangle_{j+1,-\alpha}\right] \quad j = 0, 1, \ldots.$$

Riordan algorithm of generalized Stirling numbers

Definition

Let $n \in \mathbb{N}$ and $\alpha, \beta, r \in \mathbb{R}$. A generalized Stirling number denoted by $S(n, k, \alpha, \beta, r)$ is defined by

$$\langle z \rangle_{n,-\alpha} = \sum_{k=0}^{n} S(n, k, \alpha, \beta, r) \langle z - r \rangle_{k,-\beta}. \tag{39}$$

In particular, if $(\alpha, \beta, r) = (1, 0, 0)$, $S(n, k, 1, 0, 0)$ is reduced to the unified form of Classical Stirling numbers.

$$\frac{1}{k!}(1 + \alpha z)^{r/\alpha} \left(\frac{(1 + \alpha z)^{\beta/\alpha} - 1}{\beta} \right)^k = \sum_{n \geq 0} S(n, k) \frac{z^n}{n!}. \tag{40}$$

Let $\alpha\beta \neq 0$. The $A-$ sequence $(a_n)_{n\in\mathbb{N}_0}$ of the Riordan array of the generalized Stirling number array $[d_{n,k} = k!S(n,k)/n!]_{0\le k\le n}$ satisfies

$$a_0 = 1, \quad a_n = -\frac{1}{\alpha}\sum_{k=1}^{n} a_{n-k}\frac{\langle\alpha\rangle_{k+1,-\beta}}{(k+1)!} \tag{41}$$

for all $n \geq 1$. To find the fist column of the array $[d_{n,k}]_{0\le k\le n}$, we consider (40) for $k = 0$ and have $(1+\alpha z)^{r/\alpha} = \sum_{n\ge 0}\frac{S(n,0)}{n!}z^n$. On the other hand, $(1+\alpha z)^{r/\alpha} = \sum_{n\ge 0}\binom{r/\alpha}{n}(\alpha z)^n$. Comparing the right-hand sides of the last two equations, we obtain

$$S(n,0) \equiv S(n,0,\alpha,\beta,r) = n!\binom{r/\alpha}{n}\alpha^a = \langle r\rangle_{n,-\alpha}. \tag{42}$$

$$[d_{n,k}]_{0\le k\le n} = \left[\frac{k!}{n!}S(n,k)\right]_{0\le k\le n}, S(n,k) \equiv S(n,k,\alpha,\beta,r)(\alpha\beta \neq 0). \tag{43}$$

Riordan algorithm of generalized Stirling numbers

Denote $d(t) = (1+\alpha z)^{r/\alpha}$ and $h(z) = ((1+\alpha z)^{\beta/\alpha} - 1)/\beta$ $(\alpha\beta \neq 0)$. Let $n, k \in \mathbb{N}_0$ and $\alpha\beta \neq 0$. Then we may find A-sequence $(a_n)_{n\in\mathbb{N}_0}$ shown in (41) and establish the array (43) except its first column:

$$\frac{k!}{n!}S(n,k) = \sum_{j\geq 0} a_j \frac{(k+j-1)!}{(n-1)!} S(n-1, k+j-1) \qquad (44)$$

for all $1 \leq k \leq n$. The first column of array (43) can be constructed by using (42).

$$\frac{1}{n!}S(n,0) = \frac{\langle r\rangle_{n,-\alpha}}{n!}. \qquad (45)$$

Finally, all $S(n,k) \equiv S(n,k,\alpha,\beta,r)$ $(0 \le k \le n)$ can be read from a modification of array (43); namely from

$$\left[\frac{n!}{k!} d_{n,k}\right]_{0\le k\le n} = [S(n,k)]_{0\le k\le n},$$

where $S(n,k) = n\sum_{j\ge 0} a_j [k+j-1]_{j-1} S(n-1,k+j-1)$ when $1 \le k \le n$, and $S(n,0)$ can be obtained from (45) or (42).

For the Howard's weighted degenerated Stirling numbers $S(n,k) \equiv S(n,k,1,1,-1)$. From Algorithm 3, we have $A(z) = 1$. Then

$$\left[\frac{k!}{n!}S(n,k)\right]_{0\le k\le n} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -1 & 1 & & \\ -1 & 1 & -1 & 1 & \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

$$[S(n,k)]_{0\le k\le n} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 2 & -2 & 1 & & \\ -6 & 6 & -3 & 1 & \\ 24 & -24 & 12 & -4 & 1 \end{bmatrix},$$

which gives $S(0,0) = 1$; $S(1,0) = -1$, $S(1,1) = 1$; $S(2,0) = 2$, $S(2,1) = -2$, $S(2,2) = 1$; $S(3,0) = -6$, $S(3,1) = 6$, $S(3,2) = -3$, $S(3,3) = 1$; and $S(4,0) = 24$, $S(4,1) = -24$, $S(4,2) = 12$, $S(4,3) = -4$, and $S(4,4) = 1$ row by row.

Thank You !