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On an Extension of Riordan Array and Its Application in the Construction of Convolution-type and Abel-type Identities

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Abstract

Using the basic fact that any formal power series over the real or complex number field can always be expressed in terms of given polynomials $\{p_n(t)\}$, where $p_n(t)$ is of degree n, we extend the ordinary Riordan array (resp. Riordan group) to a generalized Riordan array (resp. generalized Riordan group) associated with $\{p_n(t)\}$. As new application of the latter, a rather general Vandermonde-type convolution formula and certain of its particular forms are presented. The construction of the Abel type identities using the generalized Riordan arrays is also discussed.

Key words formal power series, expansion formula, Riordan group, matrix multiplication, convolution formula.

Mathematics subject Classification (2000) 05A15, 11B68, 11B83, 13F25, 41A58

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1 Introduction

In the recent literature, special emphasis has been given to the concept of *Riordan arrays* associated with power series, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function (GF) of their columns. They form a group, called *the Riordan group* (cf. Shapiro, Getu, Woan, and Woodson [31]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli[32, 33], on subgroups of the Riordan group in Peart and Woan [23] and Shapiro [28], on some characterizations of Riordan matrices in Rogers [24], Merlini, Rogers, Sprugnoli, and Verri [20], and He and Sprugnoli [16], and on many interesting related results in Cheon, Kim, and Shapiro [2, 3], He [9], He, Hsu, and Shiue [13], Nkwanta [22], Shapiro [29, 30], Wang and Wang [34], Yang, Zheng, Yuan, and He [36] , and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $f^*(t)$ exists such that $f(f^*(t)) = f^*(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the GF whose coefficients make-up the entries of column k.

It is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t)) \cdot (d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{2}$$

The Riordan array $I = (1,\, t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it is easily seen that I acts as an identity for this product, that is, $(1,\, t) \cdot (d(t),\, h(t)) = (d(t),\, h(t)) \cdot (1, t) = (d(t),\, h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(h^*(t))},\, h^*(t)\right) \tag{3}$$

where $h^*(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays is a group.

Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$. Then the polynomials $u_n(x)$ ($n = 0, 1, 2, \cdots$) defined by the GF

$$d(t)e^{xh(t)} = \sum_{n\geq 0} u_n(x)t^n \tag{4}$$

are called Sheffer-type polynomials with $u_0(x) = 1$. The set of all Sheffer-type polynomial sequences $\{u_n(x) = [t^n]d(t)e^{xh(t)}\}$ with an operation, "umbral composition" (*cf.* [25] and [27]), forms a group called the Sheffer group. [13] presents the isomorphism between the Riordan group and Sheffer group.

Rogers [24] introduced the concept of the A-*sequence* for Riordan arrays; Merlini, Rogers, Sprugnoli, and Verri [20] introduced the related concept of the Z-*sequence* and showed that these two concepts, together with the element $d_{0,0}$, completely characterize a proper Riordan array. He and Sprugnoli [16] presented the characterization of Riordan arrays by means of the A- and Z-sequences for some subgroups of $\mathcal{R}$ and the products and the inverses of Riordan arrays.

In [8], one of the authors defined a generalized Sheffer-type polynomial sequences as follows.

Definition 1.1 *[8] Let $d(t)$, $U(t)$, and $h(t)$ be any formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $d(0) = 1$, $U(0) = 1$, $h(0) = 0$, and $h'(0) \neq 0$. Then the polynomials $u_n(x)$ ($n = 0, 1, 2, \cdots$) defined by the GF*

$$d(t)U(xh(t)) = \sum_{n\geq 0} u_n(x)t^n \tag{5}$$

are called the generalized Sheffer-type polynomials associated with $(d(t), h(t))_{U(t)}$. Accordingly, $u_n(D)$ with $D \equiv d/dt$ is called Sheffer-type differential operator of degree n associated with $(d(t), h(t))_{U(t)}$. Particularly, $u_0(D) \equiv I$ is the identity operator due to $u_0(x) = 1$.

One of the authors [9] shows that for every $U(t)$ there exists a one-to-one correspondence between $(d(t), h(t))$ and $\{u_n(x)\}$, and the collection, P_U, of all polynomial sequences $\{u_n(x)\}$ with respect to $V(t) = \sum_{n\geq 0} a_n t^n$, defined by (5), forms a group $(P_U, \tilde{\#})$ under the operation $\tilde{\#}$, defined by

$$\{p_n(x)\}\tilde{\#}\{q_n(x)\} = \{r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k : r_{n,k} = \sum_{\ell=k}^{n} p_{n,\ell}q_{\ell,k}/a_\ell, n \geq k\},$$

which is isomorphic to the Riordan group. Hence, for different power series $U(t)$ and $V(t)$, groups $(P_U, \tilde{\#})$ and $(P_V, \tilde{\#})$, defined by (5) associated with $U(t)$ and $V(t)$, respectively, are isomorphic.

Let $c = (c_0, c_1, c_2, \ldots)$ be a sequence satisfying $c_0 = 1$, $c_k > 0$ for all $k = 1, 2, \ldots$. We call the element $A \in \mathcal{F}$ with the form $A(x) = \sum_{k\geq 0} \frac{x^k}{c_k}$ a generalized power series (GPS) associated with $\{c_n\}$ or, simply, a (c)-GPS, and $\mathcal{F}$ the GPS set associated with $\{c_n\}$. In [7], a (c)-Riordan array generated by $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$ with respect to $A(x)$ and $\{c_k\}_{k\geq 0}$ is an infinite complex matrix $[d_{n,k}]_{0\leq k\leq n}$, whose bivariate GF has the form

$$\sum_{n,k\geq 0} d_{n,k} \frac{t^n}{c_n} x^k = d(t) g(xh(t)). \tag{6}$$

Hence, we denote $[d_{n,k}] = (d(t), h(t))$. Particularly, if $h'(0) \neq 0$, the corresponding Riordan array is called a proper Riordan array. Otherwise, it is called an improper Riordan array.

Furthermore, if $c_k = 1$ $(k = 0, 1, 2, \ldots)$, i.e., the corresponding series $d(t)$ and $h(t)$ are ordinary power series, then expression (6) is written as

$$\sum_{n,k\geq 0} d_{n,k} t^n x^k = \frac{d(t)}{1 - xh(t)}, \tag{7}$$

which defines the classical Riordan array, called (1)-Riordan array. If $c_k = k!$ $(k = 0, 1, 2, \ldots)$, i.e., the corresponding series $d(t)$ and $h(t)$ are exponential power series, then expression (6) is written as

$$\sum_{n,k\geq 0} d_{n,k} \frac{t^n}{n!} x^k = d(t) e^{xh(t)}, \tag{8}$$

which defines the Sheffer type Riordan array. If $c_0 = 1$ and $c_k = k$ $(k = 1, 2, \ldots)$, i.e., the corresponding series $d(t)$ and $h(t)$ are Dirichlet series, then expression (6) is written as

$$d_{0,0} + \sum_{1\leq k\leq n} d_{n,k} \frac{t^n}{n} x^k = d(t) \left(1 - \ln(1 - xh(t))\right), \tag{9}$$

which is called the Dirichlet Riordan series. The improper Riordan arrays derived from the bivariate GF $d(t) \ln(1/(1 - xh(t))$ is worth being investigated.

In the above definitions, the (n, k) entry of (c)-Riordan array $[d_{n,k}]$ is

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] d(t) \frac{h(t)^k}{c_k} = [p_n(t)] d(t) p_k(h(t)), \quad p_j(t) = \frac{t^j}{c_j}, \tag{10}$$

for all $0 \leq k \leq n$ and $d_{n,k} = 0$ otherwise. Obviously, we have $d_{n,k} = [t^n] d(t)(h(t))^k$ and $d_{n,k} = [t^n] d(t)(h(t))^k / n!$, $0 \leq k \leq n$, for the classical $(1)-$Riordan arrays and the Sheffer Riordan arrays, respectively, and $d_{n,k} =$

$[t^n/n]d(t)(h(t))^k/n$ $(1 \le k \le n)$ for the Dirichlet Riordan arrays. Notation $[t^n/c_n]f(t)$ was introduced by Knuth [18] in 1996 and is not popularly used due to $[t^n/c_n]f(t) = c_n[t^n]f(t)$.

Gould and He [7] considered the characterization of (c)-Riordan arrays by means of the A- and Z-sequences. They also showed a one-to-one correspondence between Gegenbauer-Humbert-type polynomial sequences and the set of (c)-Riordan arrays, which generates the sequence characterization of Gegenbauer-Humbert-type polynomial sequences.

In this paper, we will not only consider the (c) extension of Riordan arrays, but also the change of the basic sets, on which the algebraic structure of the Riordan group is built. More precisely, we will consider a generalized case when the set of polynomial $\{p_n(t) = t^n/c_n\}$ is extended to a basic set defined below. From the definition, we will see $\{x^n/c_n\}$ is a special case of basic sets. As an application of the Riordan arrays, we give a new method to construct convolution-type identities using the extended Riordan arrays.

Given a polynomial $p(t)$ in t of degree n, we may denote $deg\, p(t) = n$. If $f(t)$ is a formal power series in $\mathcal{F}_m$, then its lowest order is m, which is denoted by $ord\, f(t) = m$. Particularly, the case $ord\, f(t) = 1$ means that $f(t) \in \mathcal{F}_1$ is an compositionally invertible series. We need the following definitions (cf.[1]).

Definition 1.2 *A sequence of polynomials $\{p_n(t)\}$ is called a normal sequence, if $p_0(t) \equiv 0$ and $deg\, p_n(t) = n$ $(n \ge 1)$.*

Definition 1.3 *$\{p_n(t)\}$ is said to be a basic set of polynomials if it is a normal sequences and every formal power series $f(t)$ can be written uniquely as*

$$f(t) = \sum_{n=0}^{\infty} a_n p_n(t),$$

in terms of $\{p_n(t)\}$ and real or complex coefficients $\{a_n\}_{n\ge 0}$, in the sense that $f(t) \equiv 0$ implies all coefficients $a_n = 0$, $n = 0, 1, \ldots$.

Note that a normal sequences may be not a basic set. The simple example is given by

$$q_0(t) \equiv 1, q_1(t) = -t, q_n(t) = t^{n-1}/(n-1)! - t^n/n! \ (n \ge 2).$$

Clearly $\{q_n(t)\}$ is a normal sequence, but it is not a basic set since $f(t) = 1$ has two representations:

$$f(t) = q_0(t), f(t) = q_0(t) + \sum_{n=1}^{\infty} q_n(t).$$

Also, it is known that $\{t^n\}$ and $\{(t)_n\}$ are the simplest basic sets of polynomials, where $(t)_n$ are the falling factorial polynomials:

$$(t)_0 = 1, (t)_n = t(t-1)\cdots(t-n+1) \text{ for } n \geq 1.$$

Similarly, we have basic sets $\{t^n/c_n\}$ and $\{(t)_n/c_n\}$, where $c_n \neq 0$. Particularly, if $c_n = 1$, $n!$, and n, the corresponding basic sets are called the classical, Sheffer-type, and Dirichlet-type basic sets. For example, the $\{p_n(t) = t^n/c_n\}$ defined by (7)-(9) with $x = 1$ are classical, Sheffer-type, and Dirichlet-type basic sets, respectively.

We shall show that a basic set is an infinite linearly independent normal sequence based on the following definition of the infinite linearly independent sequence.

Definition 1.4 *Let $\{f_n(t)\}_{n\geq 0}$ be a function sequence defined on a region Ω of $\mathbb{R}$ or $\mathbb{C}$ with its every finite subsequence being linearly independent on Ω, i.e., for every finite subset $N \subseteq \mathbb{N}_0$, $\{f_n(t)\}_{n\in N}$ is linearly independent on Ω. We say $\{f_n(t)\}_{n\geq 0}$ is infinite linearly independent on Ω, if every series $\sum_{n\geq 0}\gamma_n f_n(t)$ vanishing on Ω has zero partial sum sequence, or equivalently, $\sum_{n\geq 0}\gamma_n f_n(t) \equiv 0$ implies its partial sum sequence $\{s_n(t) = \sum_{k=0}^{n}\gamma_k f_k(t)\}_{n\geq 0}$ is a zero sequence at every $t \in \Omega$.*

Roughly speaking, an infinite function sequence is said to be linearly independent if its zero linear combination implies its every partial sum is identically zero. Or equivalently, an infinite function sequence is linearly dependent if there exists a vanishing linear combination of the sequence that has a non-vanishing partial sum. Thus, the sequence $\{1, t, t^2 - t, t^3 - t^2, \ldots\}$ is linearly dependent, while both sequences $\{1, t/c_1, t^2/c_2, t^3/c_3, \ldots\}$, $c_n = 1$, $n!$ and n, respectively, are linearly independent. If $c_n = 1$, then the corresponding basic set is called the standard basic set, which is also a standard normal polynomial sequence defined in Definition 1.2.

Proposition 1.5 *A normal polynomial sequence defined by Definition 1.2 is a basic set if and only if it is linearly independent.*

Proof. Let $\{p_n(t)\}_{n\geq 0}$ be a normal polynomial sequence. Then any formal power series can be written as a linear combination of $\{p_n(t) = \sum_{j=1}^{n}\alpha_{n,j}t^j\}$, where $\alpha_{n,n} \neq 0$. More precisely, in a formal power series $f(t) = \sum_{n\geq 0} f_n t^n$, we may take a transform of the standard normal polynomial sequence $\{t^n\}$ to an arbitrary normal polynomial sequence $\{p_n(t)\}$. First, $t = p_1(t)/\alpha_{1,1}$. Assume that $t^i = \sum_{j=1}^{i}\beta_{i,j}p_j(t)$ for all $1 \leq i \leq k-1$. Then,

$$
\begin{aligned}
t^k &= \frac{1}{\alpha_{k,k}} p_k(t) - \frac{1}{\alpha_{k,k}} \sum_{i=1}^{k-1} \alpha_{k,i} t^i \\
&= \frac{1}{\alpha_{k,k}} p_k(t) - \frac{1}{\alpha_{k,k}} \sum_{i=1}^{k-1} \alpha_{k,i} \sum_{j=1}^{i} \beta_{i,j} p_j(t) \\
&= \frac{1}{\alpha_{k,k}} p_k(t) - \frac{1}{\alpha_{k,k}} \sum_{j=1}^{k-1} \left(\sum_{i=j}^{k-1} \alpha_{k,i} \beta_{i,j} \right) p_j(t).
\end{aligned}
$$

Hence we have proved that the transform $t^n \mapsto \{p_k(t)\}_{1 \le k \le n}$ holds by using the mathematical induction. Therefore, any formal power series can be written as a linear expression in terms of $\{p_n(t) = \sum_{j=1}^n \alpha_{n,j} t^j\}$. We should note that the expression may not be unique.

If $\{p_n(t)\}$ is not a basic set, i.e., there exists a formal power series $f(t)$ defined on $\Omega \subset \mathbb{R}$ (or $\mathbb{C}$) has two different linear sums in terms of $\{p_n(t)\}$, say

$$
f(t) = \sum_{n \ge 0} a_n p_n(t) \quad f(t) = \sum_{n \ge 0} b_n p_n(t),
$$

and assume n_0 is the first subindex such that $a_{n_0} \ne b_{n_0}$, then the vanishing series $\sum_{n \ge 0} (a_n - b_n) p_n(t)$ has a partial sum $s_{n_0} = \sum_{k=0}^{n_0} (a_k - b_k) p_k(t)$ not being identically zero on Ω, i.e., $s_{n_0} \not\equiv 0$ on Ω, due to $a_{n_0} \ne b_{n_0}$. Thus, $\{p_n(t)\}_{n \ge 0}$ is not linearly independent from the definition of the infinite linear independence.

Conversely, if $\{p_n(t)\}$ defined on $\Omega \subset \mathbb{R}$ is not infinite linearly independent, then there exists a vanishing series in terms of $\{p_n(t)\}$, denoted by $\sum_{n \ge 0} a_n p_n(t) \equiv 0$, such that it has a partial sum $s_{n_0} = \sum_{k=0}^{n_0} a_k p_k(t)$ not being zero at some point $t_0 \in \Omega$. Hence, there exists at least one nonzero coefficient a_{n_0} in the partial sum s_{n_0}. If $f(t)$, a formal power series, is a linear sum in terms of $\{p_n(t)\}_{n \ge 0}$ shown as $f(t) = \sum_{n \ge 0} b_n p_n(t)$, then $\sum_{n \ge 0} (a_n + b_n) p_n(t)$ is also a linear expression of $f(t)$ in terms of $\{p_n(t)\}$. Therefore, we have

$$
f(t) = \sum_{n \ge 0} b_n p_n(t) = \sum_{n \ge 0} (a_n + b_n) p_n(t),
$$

which implies that $f(t)$ has two different linear expressions in terms of $\{p_n(t)\}$ because of $b_{n_0} \ne a_{n_0} + b_{n_0}$.

$\blacksquare$

From Proposition 1.5, we know that $\{1, t, t^2 - t, t^3 - t^2, \ldots\}$ is not a basic set while $\{1, t/c_1, t^2/c_2, t^3/c_3, \ldots\}$ with $c_n = n!$ and n are basic sets.

Remark 1.1 Definition 1.3 states that a normal polynomial sequence is a basic set if every formal power series $f(t) = \sum_{n\geq 0} f_n t^n$ can be written as a unique linear sum in terms of $\{p_n(t)\}$:

$$f(t) = \sum_{n\geq 0} a_n p_n(t), \tag{11}$$

or equivalently, the above coefficient set $\{a_n\}$ is the unique solution set of the system of the linear equations

$$a_0 = f_0, \quad \sum_{i\geq n} a_i \alpha_{i,n} = f_n, \quad n \geq 1, \tag{12}$$

which is called the *characterization system for $f(t)$ associated with $\{p_n(t)\}$*. In the proof of Proposition 1.5, we have shown that every formal power series can be written as a linear sum shown in (11). Thus, system (12) is always solvable provided that $\{p_n(t)\}$ is a normal polynomial sequence. The following example shows the solution may not be unique: For $\{p_n(t) = t^n - t^{n-1}\}_{n\geq 1}$ and $p_0(t) = 1$, system (12) becomes

$$a_0 = f_0, \quad a_n - a_{n+1} = f_n, \quad n \geq 1,$$

which has infinitely many solutions. Thus a normal polynomial sequence is a basic set can be characterized as follows.

Proposition 1.6 *A normal polynomial sequence $\{p_n(t)\}$ is a basic set if for every formal power series, the characterization system associated with $\{p_n(t)\}$ has a unique solution set.*

Next are three useful type of basic sets of polynomials frequently used in the literature.

(i) $\{p_n(t) = t^n/c_n : c_0 = 1, c_n \neq 0, n \geq 1\}$ (cf. also [9]).

(ii) $\{p_n(t) = (t)_n/c_n : c_0 = 1, c_n \neq 0, n \geq 1\}$ including a special case $\{p_n(t) = (t)_n/n! = \binom{t}{n}\}_{n\geq 0}$.

(iii) Note that both $t^n/n!$ and $\binom{t}{n}$ are the simplest Sheffer-type polynomials. Certainly every special kind of Sheffer polynomials $\{p_n(t)\}$ could be used as a basic set.

In Section 3, we will present a method (or an algorithm) to find more basic sets based on Proposition 1.6.

In next section, we define the generalized Riordan arrays with respect to basic sets and prove the set of those arrays with a basic set forms a group, called the Riordan group with respect to the basic set. We will also show some subgroups of the Riordan group. Different Riordan groups with respect to several different basic sets are given. And the isomorphism between the Riordan group and the Sheffer group with the same basic set and the isomorphism between two Riordan groups with different basic sets are also shown. In Section 3, we construct a general class of convolution identities using the formal expressions, shown in [13], of entire functions in terms of the Sheffer-type polynomials and the generalized Riordan arrays. Three different classes identities with respect to three different type basic sets as well as the corresponding algorithms are given. A general method to identify basic sets will also be given in Section 3. Finally, we present Abel type identities using the generalized Riordan arrays in Section 4.

2 Generalized Riordan arrays

Given a normal basic set of polynomials $\{p_n(t)\}$ with $p_0(t) = 1$ and $deg\, p(t) = n$ $(n \geq 1)$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$. Consequently, $d(t)$ has reciprocal $d(t)^{-1}$ and $h(t)$ has compositional inverse $h^*(t)$. Especially noteworthy is that $d(t)p_k(h(t))$ is a formal power series so that it can be expressed uniquely as linear sums in terms of $p_n(t)'s$ with coefficients $d_{n,k}$, viz.,

$$d(t)p_k(h(t)) = \sum_{n=0}^{\infty} d_{n,k}p_n(t).$$

Thus, based on this relation we may write, upon using of the extracting-coefficient operator $[p_n(t)]$, by

$$d_{n,k} = [p_n(t)]d(t)p_k(h(t)), \tag{13}$$

where $n, k \in \mathbb{N}$, the set of non-negative integers. A combination of these facts allows us to introduce

Definition 2.1 *The matrix $(d_{n,k})$ defined above is said to be a generalized Riordan array with respect to the basic set $\{p_n(t)\}$. Following Shapiro–Getu–Woan–Woodson [31], we write it by*

$$(d_{n,k}) = (d(t), h(t)).$$

Although we use the same notation for our generalized Riordan arrays $(d_{n,k})$ and the classical Riordan arrays $(\bar{d}_{n,k})$, the entries are usually different, where $d_{n,k}$ is defined by (13), and $\bar{d}_{n,k} = [t^n]d(t)(h(t))^k$. Let $\{p_n(t)\}_{n\geq 0}$ be a basic set, where $p_0(t) = 1$ and $p_k(t) = \sum_{j=1}^{k} \alpha_{k,j} t^j$ $(\alpha_{k,j} \neq 0,\ k \geq 1)$. From (13) we obtain a relationship between $d_{n,k}$ and $\bar{d}_{n,k}$:

$$d_{n,k} = \sum_{\ell=1}^{n} \frac{1}{\alpha_{n,\ell}}[t^\ell]d(t)\sum_{j=1}^{k}\alpha_{k,j}(h(t))^j = \sum_{\ell=1}^{n}\sum_{j=1}^{k}\frac{\alpha_{k,j}}{\alpha_{n,\ell}}\bar{d}_{\ell,j}. \tag{14}$$

Remark 2.1 Some coefficients $\alpha_{k,j}$, $1 \leq j \leq k-1$, may be vanishing in $p_k(t) = \sum_{j=1}^{k}\alpha_{k,j}t^j$. If it happens, then the corresponding terms in the expression (14) are also missing. However, we emphasize that $\alpha_{k,k}$ $(k \geq 0)$ never be zero.

As may be expected, two arbitrary Riordan matrices, $(d_{n,k}) = (d(t), h(t))$ and $(c_{n,k}) = (f(t), g(t))$, $d(t), f(t) \in \mathcal{F}_0$ and $h(t), g(t) \in \mathcal{F}_1$, of such sort can also carry out the usual matrix multiplications. To make this effect, let us denote

$$\xi_{n,k} = \sum_{\lambda=0}^{\infty} d_{n,\lambda}c_{\lambda,k}, \quad (n,k) \in \mathbb{N}\times\mathbb{N},$$

where $\xi_{n,k}$ may be real or complex numbers. Particularly, we denote $\xi_{n,k} = \infty$ in case $\left|\sum_{\lambda=0}^{\infty} d_{n,\lambda}c_{\lambda,k}\right|$ diverges to $+\infty$.

Theorem 2.2 *Let $(d_{n,k}) = (d(t), h(t))$ and $(c_{n,k}) = (f(t), g(t))$ be two generalized Riordan arrays with respect to the basic set $\{p_n(t)\}$, where $d(t), f(t) \in \mathcal{F}_0$ and $h(t), g(t) \in \mathcal{F}_1$. There holds the matrix multiplication*

$$(d_{n,k}) \cdot (c_{n,k}) =: (\xi_{n,k}) = (d(t)f(h(t)), g(h(t))), \tag{15}$$

or equivalently,

$$(d(t), h(t)) \cdot (f(t), g(t)) = (\xi_{n,k}) = (d(t)f(h(t)), g(h(t))). \tag{16}$$

*Proof.*The formal proof is entirely similar to that for the ordinary Riordan matrices. In order to justify the relation (15) formally, we make the replacement $t \to h(t)$ in $c_{n,k} = [p_n(t)]f(t)p_k(g(t))$, so that we have

$$f(h(t))p_k(g(h(t)) = \sum_{\lambda=0}^{\infty} c_{\lambda,k}p_\lambda(h(t)). \tag{17}$$

It is worthy of note that both the left-hand side of (17) and $p_\lambda(g(t))$ are formal power series, and $d(t)f(h(t))|_{t=0} = d(0)f(0) = 1, g(h(t))|_{t=0} = g(0) =$

0 and $\big(g(h(t))\big)^{'}|_{t=0} = \big(\frac{d}{dt}\big)g(h(t))|_{t=0} = g^{'}(0)h^{'}(0) = 1$. Thus, in view of (17) we may compute

$$\begin{aligned}
& [p_n(t)]d(t)f(h(t))p_k(g(h(t))) \\
= \; & [p_n(t)]d(t)\sum_{\lambda=0}^{\infty} c_{\lambda,k}p_\lambda(h(t)) \\
= \; & [p_n(t)]\sum_{\lambda=0}^{\infty} c_{\lambda,k}\sum_{m=0}^{\infty} d_{m,\lambda}p_m(t) \\
= \; & [p_n(t)]\sum_{m=0}^{\infty} p_m(t)\left\{\sum_{\lambda=0}^{\infty} d_{m,\lambda}c_{\lambda,k}\right\} \\
= \; & \sum_{\lambda=0}^{\infty} d_{n,\lambda}c_{\lambda,k} = \xi_{n,k}.
\end{aligned}$$

This shows that

$$(\xi_{n,k}) = (d(t)f(h(t)), g(h(t))).$$

Hence the theorem is verified.

■

As usual, we have the inverse for the matrix $(d(t), h(t))$:

$$(d(t), h(t))^{-1} = \left(\frac{1}{d(h^*(t))}, h^*(t)\right). \tag{18}$$

Indeed

$$(d(t), h(t)) \cdot \left(\frac{1}{d(h^*(t))}, h^*(t)\right) = \left(\frac{d(t)}{d(h(h^*(t)))}, h(h^*(t))\right) = (1, t).$$

Here $(1, t)$ denotes the unit matrix $(\delta_{n,k}) = \text{diag}(1, 1, 1, \cdots)$, since for the case $d(t) = 1, h(t) = t$, we have

$$d_{n,k} = [p_n(t)]p_k(t) = \delta_{n,k}, \quad (d_{n,k}) = (\delta_{n,k}).$$

Moreover, the associative law for the matrix multiplication rule as given by (16) can also be confirmed without any difficulty. Thus for given basic set $\{p_n(t)\}$ of polynomials, all matrices $(d(t), h(t))$ formed by formal power series $d(t)$ and $h(t)$ satisfying those conditions mentioned previously just yields a group with the multiplication $\cdot$, in which the inverse and the unit are given by (18) and $(1, t) = (\delta_{n,k})$, respectively.

All these can be summarized by

Proposition 2.3 *All the matrices $(d_{n,k}) = (d(t), h(t))$ associated with a given basic set $\{p_n(t)\}$ and with $d_{n,k}$ being defined by (13) form a generalized Riordan group, denoted by $\mathcal{R}$, associated with $\{p_n(t)\}$ with respect to the multiplication $\cdot$ as given by (16), and with the inverse element $(d_{n,k})^{-1}$ and the unit element being given by (18) and $(1, t) = (\delta_{n,k})$, respectively.*

Particular two subgroups of $\mathcal{R}$ are important and have been considered in the literature:

- the set $\mathcal{A}$ of *Appell arrays*, that is the (c)-Riordan arrays $R = (d(t),\, h(t))$ for which $h(t) = t$; it is an invariant subgroup and is isomorphic to the group of f.p.s.'s of order 0, with the usual product as group operation;
- the set $\mathcal{L}$ of *Lagrange arrays*, that is the (c)-Riordan arrays $R = (d(t),\, h(t))$ for which $d(t) = 1$; it is also called the *associated subgroup*; it is isomorphic with the group of f.p.s.'s of order 1, with composition as group operation;

Next are three useful cases associated with some basic sets of polynomials will be investigated later.

(i) If the basic set $\{p_n(t)\}$ is taken to be $\{t^n/c_n\}$, where $c_0 = 1$ and $c_n \neq 0$ for all $n \geq 1$, then we get the (c)- Riordan matrices and Riordan group (cf. [9]).

(ii) If the basic set $\{p_n(t)\}$ is taken to be $\{(t)_n/c_n\}$, where $c_0 = 1$ and $c_n \neq 0$ for all $n \geq 1$, which includes the special case $\{(t)_n/n! = \binom{t}{n}\}$, then we get a new type of Riordan group, which will be discussed later.

(iii) Note that both $t^n/n!$ and $\binom{t}{n}$ are the simplest Sheffer-type polynomials. Certainly every special kind of Sheffer polynomials $\{p_n(t)\}$ could be used as a basic set, thereby producing a kind of Riordan group.

Remark 2.2 Analogue to the isomorphic property of the generalized Sheffer-type polynomial groups shown in [9], we have an isomorphic property for our generalized Riordan groups. Let $P = \{p_n(t)\}$ and $Q = \{q_n(t)\}$ be two distinct basic sets of polynomials, and let G_P and G_Q be the generalized Riordan groups associated with P and Q, respectively. Then it can be shown that G_P and G_Q are isomorphic in the sense that a one-to-one correspondence can be established between the elements of G_P and G_Q.

More precisely, let $(d_{n,k})_P = (d(t), h(t))_P$ and $(c_{n,k})_P = (f(t), g(t))_P$ be two generalized Riordan arrays with respect to the basic set $P = \{p_n(t)\}$, where $d(t), f(t) \in \mathcal{F}_0$ and $h(t), g(t) \in \mathcal{F}_1$. From Theorem 2.2, we have their product

$$(\xi_{n,k})_P = (d(t), h(t))_P \cdot (f(t), g(t))_P = (d(t)f(h(t)), g(h(t)))_P.$$

Similarly, $(d(t), h(t))_Q$, $(f(t), g(t))_P$, and $(d(t)f(h(t)), g(h(t)))_Q$ denote the corresponding elements of G_Q associated with $Q = \{q_n\}$, where $(d(t), h(t))_Q = (d'_{n,k})_Q$ with $d'_{n,k} = [q_n(t)]d(t)q_k(h(t))$, etc. Then G_P and G_Q are isomorphic under the one-to-one correspondence relations $(d(t), h(t))_P \leftrightarrow (d(t), h(t))_Q$, $(f(t), g(t))_P \leftrightarrow (f(t), g(t))_Q$, which imply

$$(d(t), h(t))_P \cdot (f(t), g(t))_P \leftrightarrow (d(t), h(t))_Q \cdot (f(t), g(t))_Q$$

and

$$(d(t), h(t))_P^{-1} = \left(\frac{1}{d(h^*(t))}, h^*(t)\right)_P \leftrightarrow (d(t), h(t))_Q^{-1} = \left(\frac{1}{d(h^*(t))}, h^*(t)\right)_Q.$$

Note the unit element $(1, t) = (\delta_{n,k})$ is common to both G_P and G_Q. Consequently, all the generalized Riordan groups are isomorphic to the ordinary Riordan group with $\{t^n/n!\}$. However, concrete structures and analytic and combinatorial implications are different between different groups. More details and applications demonstrating the differences will be presented in another paper.

3 A general class of convolution identities

We are not going to details of the convergence of series expansions presented in this paper. In order to secure convergence in power series expansions, throughout this section, $\alpha(t), \beta(t), \phi(t)$ and $\psi(t)$ might be assumed to be real entire functions having power series expansions in $\mathbb{R}$. We may also assume $\phi(t), \psi(t) \in \mathcal{F}_1$, so that both $\phi(t)$ and $\psi(t)$ are compositionally invertible functions. Hence $\psi(\phi(t))$ is also an entire function in $\mathcal{F}_1$. In addition, some weak conditions for the convergence can be found by using the similar arguments shown in He, Hsu, and Shiue [12, 14] and in He, Hsu, and Yin [15]. However, our results in this paper are not limited to the above assumption but hold for formal power series.

Let $\{p_n(t)\}$ be a basic set of polynomials with real coefficients, and let $\alpha(0) = \beta(0) = 1$. Then parallel to (13), we define

$$\begin{aligned} d_{n,k} &= [p_n(t)]\alpha(t)p_k(\phi(t)) \quad (19)\\ c_{n,k} &= [p_n(t)]\beta(t)p_k(\psi(t)) \quad (20)\end{aligned}$$

By Theorem 2.2 we have

$$(\alpha(t),\phi(t))\cdot(\beta(t),\psi(t)) = (\alpha(t)\beta(\phi(t)),\psi(\phi(t)). \quad (21)$$

This implies

Theorem 3.1 (Vandermonde-type convolution formula) *With all the same assumptions as above, we have*

$$\begin{aligned}&\sum_{\lambda\geq 0}\left([p_n(t)]\alpha(t)p_\lambda(\phi(t))\right)\left([p_\lambda(t)]\beta(t)p_k(\psi(t))\right)\\ &\qquad = [p_n(t)]\alpha(t)\beta(\phi(t))p_k(\psi(\phi(t))). \quad (22)\end{aligned}$$

Remark 3.1 Let $(\alpha(t),\phi(t))$ and $(\beta(t),\psi(t))$ be the generalized Riordan arrays with respect to a basic set $\{p_n(t)\}_{n\geq 0}$. In (21), if $\psi(\phi(t)) = \phi(\psi(t)) = t$ and $\alpha(t) = 1/\beta(\phi(t))$, then a pair of inverse matrices, $(\alpha(t),\phi(t))$ and $(\beta(t),\psi(t))$, with respect to the basic set $\{p_n(t)\}_{n\geq 0}$ is defined. Following Definition 2.1, we denote $(\alpha(t),\phi(t))$ and its inverse by $(d_{n,k})$ and $(e_{n,k})$, respectively. Then a pair of inverse matrices $(d_{n,k})$ and $(e_{n,k})$ can be used to generalize the combinatorial inversion

$$f_n = \sum_{k=0}^{n} d_{n,k}g_k \Leftrightarrow g_n = \sum_{k=0}^{n} e_{n,k}f_k, \quad (23)$$

or equivalently,

$$F(t) = \alpha(t)G(\phi(t)) \Leftrightarrow G(t) = \beta(t)F(\psi(t)),$$

where $F(t)$ and $G(t)$ are the GFs of the sequences $\{f_n\}$ and $\{g_n\}$, respectively. The problem of inverting combinatorial sums by means of Riordan arrays was studied in Merlini, Sprugnoli, and Verri in [21]. It is easy to see

$$(d_{n,k})^{-1} \equiv (d(t),h(t))^{-1} \equiv (\tilde{d}(t),h^*(t)) \equiv (e_{n,k}), \quad (24)$$

where $h^*(t)$ is the compositional inverse of $h(t)$ as shown in (3) and

$$\tilde{d}(t) = \frac{1}{d(h^*(t))}. \quad (25)$$

As an example, for the Pascal triangle $(d(t),h(t)) = (1/(1-t), t/(1-t)) = (\binom{n}{k})$ from (25) there holds

$$(\tilde{d}(t), h^*(t)) = \left(\frac{1}{1+t}, \frac{t}{1+t}\right) = \left((-1)^{n-k}\binom{n}{k}\right),$$

which yields the well known binomial inversion

$$f_n = \sum_{k=0}^{n}\binom{n}{k}g_k \Leftrightarrow g_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f_k.$$

Another example can be found in Catalan matrix $(C(t), tC(t)) = (d^c_{n,k})$, where

$$C(t) = \frac{1-\sqrt{1-4t}}{2t}$$

and the entries of the Catalan matrix are (cf., for example, [4, 10, 19, 28])

$$d^c_{n,k} = [t^n]t^k C(t)^{k+1} = \frac{k+1}{n+1}\binom{2n-k}{n}, \quad 0 \le k \le n.$$

It is easy to find

$$(C(t), tC(t))^{-1} = (1-t, t(1-t)) = (e^c_{n,k}),$$

where the matrix entries are

$$e^c_{n,k} = [t^n]t^k(1-t)^{k+1} = (-1)^{n-k}\binom{k+1}{n-k}, \quad 0 \le k \le n.$$

Thus we have the sum inversion

$$f_n = \sum_{k=0}^{n}\frac{k+1}{n+1}\binom{2n-k}{n}g_k \Leftrightarrow g_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{k+1}{n-k}f_k.$$

Remark 3.2 It may be shown that (22) is a valid finite convolution identity for the case where $\alpha(t)$, $\beta(t)$, $\phi(t)$, and $\psi(t)$ are arbitrary polynomials. This is not a particular case of Theorem 3.1, but can be proved similarly. Let $\alpha(t)$ and $\phi(t)$ be arbitrary polynomials. Especially noteworthy is that $\alpha(t)p_k(\phi(t))$ is a polynomial so that it can be expressed uniquely as linear sums in terms of $p_n(t)'s$ with coefficients $d_{n,k}$, viz.,

$$\alpha(t)p_k(\phi(t)) = \sum_{n=0}^{\infty} d_{n,k}p_n(t).$$

Thus, based on this relation we may write, upon using of the extracting-coefficient operator $[p_n(t)]$, by

$$d_{n,k} = [p_n(t)]\alpha(t)p_k(\phi(t)),$$

where $n, k \in J$, $J \subset \mathbb{N}$. We denote the matrix $(d_{n,k})$ by $(\alpha(t), \phi(t))$. For arbitrary polynomials $\beta(t)$ and $\psi(t)$, we may define

$$c_{n,k} = [p_n(t)]\beta(t)p_k(\psi(t))$$

and denote $(c_{n,k}) = (\beta(t), \psi(t))$, where $n, k \in J$. Thus there holds matrix multiplication

$$(\alpha(t), \phi(t)) \cdot (\beta(t), \psi(t)) = (\alpha(t)\beta(\phi(t)), \psi(\phi(t)),$$

which implies a finite convolution identity (22), in which there is no infinite summation involved.

Before proceeding to applications of (22), we need

Lemma 3.2 (cf.: He-Hsu-Shiue [11, Theorem 3.7]) *Let $f(t)$ be an entire function. Then we have a formal expansion of $f(t)$ in terms of a sequence of Sheffer-type polynomials $\{p_n(t)\}$, namely*

$$f(t) = \sum_{k\geq 0} \alpha_k \, p_k(t), \tag{26}$$

where

$$\alpha_k = \Lambda_k(D)f(0), D = \frac{d}{dt}, \tag{27}$$

$$\Lambda_k(D) = \sum_{n\geq k} \frac{k!}{n!}\sigma^*(n,k)D^n, \tag{28}$$

$$\sigma^*(n,k) = \left[\frac{t^n}{n!}\right]\frac{1}{k!}\frac{(h^*(t))^k}{d(h^*(t))} \tag{29}$$

and

$$p_k(t) = [\tau^k]d(\tau)\exp(th(\tau)) \tag{30}$$

being resulted from the definition $d(\tau)\exp(th(\tau)) = \sum_{k\geq 0} p_k(t)\tau^k$.

Applying Lemma 3.2 to (22) leads us to the following

Corollary 3.3 *Let the differential operator $\Lambda_k(D)$ be defined by (28) with (29) and (30), where $A(t)$ and $g(t)$ satisfies the conditions $A(0) = 1, ord\, g(t) = 1$. Then there holds the general convolution formula of the form*

$$\begin{aligned} &\sum_{\lambda\geq 0} \Lambda_n(D)\big(\alpha(t)p_\lambda(\phi(t))\big)_{t=0}\Lambda_\lambda(D)\big(\beta(t)p_k(\psi(t))\big)_{t=0} \\ = \;& \Lambda_n(D)\big(\alpha(t)\beta(\phi(t))p_k(\psi(\phi(t))\big)_{t=0}. \end{aligned} \tag{31}$$

In fact, all the functions appearing in (31) are entire functions so that differential operators could apply. However, a practical application of formula (31) would seem quite complicated.

The key to construct convolution formulas using (31) is to find the basic sets with respect to corresponding operators Λ_n similar to those defined by Lemma 3.2. More precisely, we call $\Lambda \equiv \{\Lambda_n\}_{n=0}^{\infty}$ a sequence of the *function-to-sequence operators* if there exist $\Lambda_n : f \mapsto a_n$ $(n \geq 0)$ to map every real entire function $f(t)$ to a real number sequence $\{a_n\}_{n=0}^{\infty}$ with the property that the particular case $f(t) \equiv 0$ leads to the zero-sequence $\{a_n\} = \{0\}$. Given a simple normal polynomial sequence $\{p_n(t)\}_{t=0}^{\infty}$, suppose that there is a function-to-sequence operator sequence $\{\Lambda_n\}$ that makes every real entire function $f(t)$ expressible formally or analytically in the form

$$f(t) = \sum_{n\geq 0} a_n p_n(t),$$

where $\{a_n\}$ is determined from $f(t)$ via Λ_n. Then from Proposition 1.6 $\{p_n(t)\}$ is a basic set, called a basic set for entire functions with respect to the operator sequence $\Lambda \equiv \{\Lambda_n\}$.

Here are some examples of basic sets with respect to some function-to-sequence operators.

Example 3.1 Generally, every Sheffer-type polynomial sequence $\{p_n(t)\}$ is a basic set with respect to $\Lambda(D) \equiv \{\Lambda_n(D)\}$ defined by (28). Let the Sheffer-type sequence $\{p_n(t)\}$ be given by the GFs $(d(z), h(z))$ as

$$d(z)exp(th(z)) = \sum_{n\geq 0} p_n(t) z^n.$$

Let $d(z)$ and $h(z)$ have convergence radii ρ_1 and ρ_2, respectively, and let $\rho = \min\{\rho_1, \rho_2\}$. Then the absolute convergence of $\sum_{n\geq 0} p_n(t)z^n$ with $|z| < \rho$ implies the convergence of $\sum_{n\geq 0} a_n p_n(t)$ with $\overline{\lim}_{n\to\infty} |a_n|^{1/n} < \rho$. Accordingly, for the basic set $\{p_n(t)\}$ with respect to the operator $\Lambda(D)$ we see that the expression

$$f(t) = \sum_{n\geq 0} (\Lambda_n(D) f(0)) p_n(t)$$

is an absolutely convergent expansion for every entire function $f(t)$ satisfying the condition

$$\overline{\lim}_{n\to\infty} |\Lambda_n(D) f(0)|^{1/n} < \rho.$$

Example 3.2 Bernoulli polynomial sequence $\{B_n(t)/n!\}$ and Euler polynomial sequence $\{E_n/n!\}$ are basic sets for entire functions with respect to the operators $\Lambda(D) = \{\Delta D^{n-1}\}$ and $\Lambda(D) = \{MD^n\}$ acting to $f(t)$ at $t = 0$, respectively, so that there hold formal expressions

$$f(t) = \sum_{n\geq 0} \left(\Delta D^{n-1} f(0)\right) \frac{B_n(t)}{n!}, \quad f(t) = \sum_{n\geq 0} \left(MD^n f(0)\right) \frac{E_n(t)}{n!}.$$

Particularly, the above expressions are convergent analytic expressions in case $f(t)$ are taken to be entire functions satisfying the following conditions, respectively:

$$\overline{\lim}_{n\to\infty} \left|\Delta D^{n-1} f(0)\right|^{1/n} < 2\pi, \quad \overline{\lim}_{n\to\infty} \left|MD^n f(0)\right|^{1/n} < \pi.$$

These follow from from the following GFs, respectively,

$$\begin{aligned} \left(\frac{z}{e^z - 1}\right) e^{tz} &= \sum_{n\geq 0} \frac{B_n(t)}{n!} z^n \ (|z| < 2\pi), \\ \left(\frac{2}{e^z + 1}\right) e^{tz} &= \sum_{n\geq 0} \frac{E_n(t)}{n!} z^n \ (|z| < \pi). \end{aligned}$$

In the end of this section, we consider in detail the convolution formulas constructed by using the following basic sets.

(i) $p_n(t) = t^n/n!$;

(ii) $p_n(t) = (t)_n/n!$;

(iii) $p_n(t) = t^{[n]}/n!$, where $t^{[n]} = t\left(t + \frac{n}{2} - 1\right)_{n-1}$;

(iv) $p_n(t) = B_n(t)/n!$, where $B_n(t)$ is the nth order Bernoulli polynomial of the first kind;

(v) $p_n(t) = E_n(t)$, where $E_n(t)$ is the nth order Euler polynomial;

(vi) $p_n(t) = \xi_n(t)$, where $\xi_n(t)$ is the nth order Boole's polynomial;

(vii) $p_n(t) \equiv p_n(t,x) := t(t - nx)^{n-1}/n!$ $(n \geq 1)$ and $p_0(t) \equiv p_0(t,x) = 1$ for fixed x, where $p_n(t)$ is the nth order Appel polynomial.

For (i), the Maclaurin expansion

$$f(t) = \sum_{n\geq 0} f^{(n)}(0) \frac{t^n}{n!} = \sum_{n\geq 0} D^n f(0) p_n(t)$$

shows that $\Lambda_n(D) = D^n$. Consequently we get the convolution formula

$$\sum_{\lambda\geq 0} D^n\left(\alpha(t)\frac{\phi^\lambda(t)}{\lambda!}\right)_0 D^\lambda\left(\beta(t)\frac{\psi^k(t)}{k!}\right)_0 = D^n\left(\alpha(t)\beta(\phi(t))\frac{\psi(\phi(t))^k}{k!}\right)_0, \quad (32)$$

where the index 0 in above expression means the values of the corresponding functions at $t = 0$.

For Lagrange arrays $(\alpha(t), \phi(t))$ and $(\beta(t), \psi(t))$, i.e., $\alpha(t) = \beta(t) \equiv 1$, we have the following lemma to evaluate the high order derivatives in (32).

Proposition 3.4 *Let $h(t) \in \mathcal{F}_1$ with compositional inverse $h^*(t)$. Then*

$$D^n\left(h^k(t)\right)\Big|_{t=0} = (n-1)!k[u^{n-k}]\left(\frac{u}{h^*(u)}\right)^n \quad (33)$$

for all $n \geq k \geq 0$.

Proof. First, for $p_n(t) = t^n/n!$ and $f(t) \in \mathcal{F}$ we observe the nth coefficient of Maclaurin expansion of

$$f(t) = \sum_{k\geq 0} D^k f(0) p_k(t)$$

can be written as

$$D^n f(0) = [p_n(t)]f(t).$$

Let $f(u) = u^k$ and $u = h(t)$. Then

$$D^n f(h(t))|_{t=0} = [p_n(t)]f(h(t)) = n![t^n]f(h(t)).$$

Using the Lagrange inverse formula (cf., for example, Theorem 5.1 in [35]) and noting the compositional inverse of $u = h(t)$ satisfies $u = ut/h^*(u)$ with $u/h^*(u) \in \mathcal{F}_0$, we can write the last equation as

$$\begin{aligned} D^n f(h(t))|_{t=0} &= D^n h^k(t)\Big|_{t=0} = n![t^n]f(h(t)) \\ &= n!\frac{1}{n}[u^{n-1}]Df(u)\left(\frac{u}{h^*(u)}\right)^n \\ &= (n-1)!k[u^{n-1}]u^{k-1}\left(\frac{u}{h^*(u)}\right)^n, \end{aligned}$$

which implies (33).

∎

Corollary 3.5 *Let $h(t) \in \mathcal{F}_1$, and let $d(t) \in \mathcal{F}_0$ be the GF of sequence $\{d_n\}_{n\geq 0}$. Then*

$$D^n \left(d(t)h^k(t) \right)\Big|_{t=0} = kn! \sum_{\ell=\max\{k,1\}}^{n} \frac{d_{n-\ell}}{\ell} [u^{\ell-k}] \left(\frac{u}{h^*(u)} \right)^{\ell} \tag{34}$$

for all $n \geq k \geq 0$.

Proof. Noting $h(t) \in \mathcal{F}_1$, from the Leibnitz formula

$$D^n(f(t)g(t)) = \sum_{\ell=0}^{n} \binom{n}{\ell} D^{\ell} f(t) D^{n-\ell} g(t)$$

and formula (33) we have

$$D^n \left(d(t)h^k(t) \right)\Big|_{t=0} = \sum_{\ell=\max\{k,1\}}^{n} \binom{n}{\ell} \; D^{n-\ell} d(t)\Big|_{t=0} \; D^{\ell} h^k(t)\Big|_{t=0}$$

$$= \sum_{\ell=\max\{k,1\}}^{n} \binom{n}{\ell} (n-\ell)! d_{n-\ell} (\ell-1)! k [u^{\ell-k}] \left(\frac{u}{h^*(u)} \right)^{\ell},$$

which implies (34).

■

Denote the compositional inverses of $\phi(t)$, $\psi(t)$, and $(\psi \circ \phi)(t)$ by $\phi^*(t)$, $\psi^*(t)$, and $(\psi \circ \phi)^*(t)$, respectively. Using Proposition 3.4, we can immediately write Identity (32) as

$$\sum_{\lambda=k}^{n} \left([u^{n-\lambda}] \left(\frac{u}{\phi^*(u)} \right)^{n} \right) \left([u^{\lambda-k}] \left(\frac{u}{\psi^*(u)} \right)^{\lambda} \right) = [u^{n-k}] \left(\frac{u}{(\psi \circ \phi)^*(u)} \right)^{n} \tag{35}$$

for $\alpha(t) = \beta(t) \equiv 1$.

Let $\alpha(t), \beta(t) \in \mathcal{F}_0$ with $\alpha(t)\beta(t) \not\equiv 1$ and $\phi(t), \psi(t) \in \mathcal{F}_1$. Denote the compositional inverses of $\phi(t)$, $\psi(t)$, and $(\psi \circ \phi)(t)$ by $\phi^*(t)$, $\psi^*(t)$, and $(\psi \circ \phi)^*(t)$, respectively. Using Corollary 3.5, we can modify Identity (32) as

$$
\begin{aligned}
&\sum_{\lambda=0}^{n}\lambda\left(\sum_{\ell=\max\{\lambda,1\}}^{n}\frac{1}{\ell}\left([t^{n-\ell}]\alpha(t)\right)\left([u^{\ell-\lambda}]\left(\frac{u}{\phi^*(u)}\right)^{\ell}\right)\right)\\
&\times\left(\sum_{j=\max\{k,1\}}^{\lambda}\frac{1}{j}\left([t^{\lambda-j}]\beta(t)\right)\left([u^{j-k}]\left(\frac{u}{\psi^*(u)}\right)^{j}\right)\right)\\
=\;&\sum_{\ell=\max\{k,1\}}^{n}\frac{1}{\ell}\left([t^{n-\ell}]\alpha(t)\beta(\phi(t))\right)\left([u^{\ell-k}]\left(\frac{u}{(\psi\circ\phi)^*(u)}\right)^{\ell}\right)
\end{aligned}\tag{36}
$$

for all $n \geq k \geq 0$.

As an application of identity (36), if $\alpha(t) = \beta(t) = 1/(1-t)$, and $\phi(t) = \psi(t) = t/(1-t)$, then $\alpha(t)\beta(\phi(t)) = 1/(1-2t)$. Similarly, there holds identity

$$
\begin{aligned}
&\sum_{\lambda=k}^{n}\lambda\left(\sum_{\ell=\max\{\lambda,1\}}^{n}\frac{1}{\ell}\binom{\ell}{\ell-\lambda}\right)\left(\sum_{j=\max\{k,1\}}^{\lambda}\frac{1}{j}\binom{j}{j-k}\right)\\
=\;&2^{n-k}\sum_{\ell=\max\{k,1\}}^{n}\frac{1}{\ell}\binom{\ell}{\ell-k}.
\end{aligned}\tag{37}
$$

For (ii), the Newton series

$$
f(t) = \sum_{n\geq 0}\Delta^n f(0)\frac{(t)_n}{n!} = \sum_{n\geq 0}\Delta^n f(0)p_n(t)
$$

implies that $\Lambda_n(D) = \Delta^n$. Consequently we obtain the convolution formula

$$
\sum_{\lambda\geq 0}\Delta^n\left(\alpha(t)\binom{\phi(t)}{\lambda}\right)_0\Delta^\lambda\left(\beta(t)\binom{\psi(t)}{k}\right)_0 \tag{38}
$$

$$
= \quad \Delta^n\left(\alpha(t)\beta(\phi(t))\binom{\psi(\phi(t))}{k}\right)_0, \tag{39}
$$

among which the particular case with $\alpha(t) = \beta(t) = 1$ seems more interesting, since

$$
\sum_{\lambda\geq 0}\Delta^n\binom{\phi(t)}{\lambda}_0\Delta^\lambda\binom{\psi(t)}{k}_0 = \Delta^n\binom{\psi(\phi(t))}{k}_0. \tag{40}
$$

This is actually a "deep generalization" of the Vandermonde convolution formula. Indeed, let $k > n$ and take $\phi(t) = t + a$ and $\psi(t) = t + b$. It follows from Remark 3.2 that

$$\sum_{\lambda\geq 0}\Delta^n\binom{t+a}{\lambda}_0\Delta^\lambda\binom{t+b}{k}_0 = \sum_{\lambda\geq 0}\binom{a}{\lambda-n}\binom{b}{k-\lambda}$$
$$= \Delta^n\binom{t+a+b}{k}_0. \tag{41}$$

The last equality gives

$$\sum_{\lambda\geq 0}\binom{a}{\lambda-n}\binom{b}{k-\lambda}=\binom{a+b}{k-n}. \tag{42}$$

For (iii), the Newton series in terms of central difference

$$f(t)=\sum_{n\geq 0}\delta^{(n)}f(0)\frac{t^{[n]}}{n!}=\sum_{n\geq 0}\delta^n f(0)p_n(t)$$

implies that $\Lambda_n(D)=\delta^n$. Consequently, noting

$$p_n(t)=t^{[n]}/n!=\frac{t}{t-\frac{n}{2}}\binom{t+\frac{n}{2}-1}{n},$$

we obtain the convolution formula

$$\sum_{\lambda\geq 0}\delta^n\left(\frac{\alpha(t)\phi(t)}{\phi(t)-\frac{\lambda}{2}}\binom{\phi(t)+\frac{\lambda}{2}-1}{\lambda}\right)_0\delta^\lambda\left(\frac{\beta(t)\psi(t)}{\psi(t)-\frac{k}{2}}\binom{\psi(t)+\frac{k}{2}-1}{k}\right)_0$$
$$= \delta^n\left(\frac{\alpha(t)\beta(\phi(t))\psi(\phi(t))}{\psi(\phi(t))-\frac{k}{2}}\binom{\psi(\phi(t))+\frac{k}{2}-1}{k}\right)_0, \tag{43}$$

where $2\phi(0), 2\psi(0), 2\psi(\phi(0))\notin\mathbb{N}\cup\{0\}$, among which the particular case with $\alpha(t)=\beta(t)=1$ yields

$$\sum_{\lambda\geq 0}\delta^n\left(\frac{\phi(t)}{\phi(t)-\frac{\lambda}{2}}\binom{\phi(t)+\frac{\lambda}{2}-1}{\lambda}\right)_0\delta^\lambda\left(\frac{\psi(t)}{\psi(t)-\frac{k}{2}}\binom{\psi(t)+\frac{k}{2}-1}{k}\right)_0$$
$$= \delta^n\left(\frac{\psi(\phi(t))}{\psi(\phi(t)-\frac{k}{2}}\binom{\psi(\phi(t))+\frac{k}{2}-1}{k}\right)_0. \tag{44}$$

Taking $\phi(t)=t+a$ and $\psi(t)=t+b$, $2a, 2b, 2(a+b)\notin\mathbb{N}\cup\{0\}$, we obtain the convolution formula

$$\sum_{\lambda\ge 0}\delta^n\left(\frac{t+a}{t+a-\frac{\lambda}{2}}\binom{t+a+\frac{\lambda}{2}-1}{\lambda}\right)_0\delta^\lambda\left(\frac{t+b}{t+b-\frac{k}{2}}\binom{t+b+\frac{k}{2}-1}{k}\right)_0$$
$$= \delta^n\left(\frac{t+a+b}{t+a+b-\frac{k}{2}}\binom{t+a+b+\frac{k}{2}-1}{k}\right)_0 \tag{45}$$

for $k>n$, or equivalently,

$$\sum_{\lambda\ge 0}\frac{ab}{(a-\frac{\lambda-n}{2})(b-\frac{k-\lambda}{2})}\binom{a+\frac{\lambda-n}{2}-1}{\lambda-n}\binom{b+\frac{k-\lambda}{2}-1}{k-\lambda}$$
$$= \frac{a+b}{a+b-\frac{k-n}{2}}\binom{a+b+\frac{k-n}{2}-1}{k-n} \tag{46}$$

for all $k\ge n$ due to $\delta^k p_n(t)=p_{n-k}(t)$.

For (iv), using the property of Bernoulli polynomials (cf.[5]) we have the following series expansion

$$f(t)=\sum_{n\ge 0}(f^{(n-1)}(1)-f^{(n-1)}(0))\frac{B_n(t)}{n!}=\sum_{n\ge 0}[D^{n-1}f(t)]_0^1\frac{B_n(t)}{n!}$$

where

$$[D^{-1}f(t)]_0^1=\int_0^1 f(t)dt.$$

Consequently we obtain the convolution formula

$$\sum_{\lambda\ge 0}\left[D^{n-1}\frac{B_\lambda(\phi(t))}{\lambda!}\right]_0^1\left[D^{\lambda-1}\frac{B_k(\phi(t))}{k!}\right]_0^1=\left[D^{n-1}\frac{B_k(\psi(\phi(t)))}{k!}\right]_0^1, \tag{47}$$

where we may assume $n\ge 2$.

For (v), we apply the property of Euler polynomials (cf.[5]) to have the following series expansion

$$f(t)=\sum_{n\ge 0}\frac{1}{2}(f^{(n)}(1)+f^{(n)}(0))E_n(t)=\sum_{n\ge 0}[MD^nf(0)]E_n(t),$$

where $Mf(t)=(f(t)+f(t+1))/2$. Consequently we obtain the convolution formula

$$\sum_{\lambda\ge 0}\Big[MD^nE_\lambda(\phi(t))\Big]_0\Big[MD^\lambda E_k(\phi(t))\Big]_0=\Big[MD^nE_k(\psi(\phi(t)))\Big]_0, \tag{48}$$

where we may assume $n\ge 2$.

For (vi), similar to (iv) and (v), from the property of Boole's polynomials (cf.[5]) we have the series expansion

$$f(t) = \sum_{n\geq 0} \frac{1}{2}(\Delta^n f(1) + \Delta^n f(0))\xi_n(t) = \sum_{n\geq 0}[M\Delta^n f(0)]\xi_n(t),$$

where $Mf(t) = (f(t)+f(t+1))/2$. Thus there holds the convolution formula

$$\sum_{\lambda\geq 0} \left[M\Delta^n \xi_\lambda(\phi(t))\right]_0 \left[M\Delta^\lambda \xi_k(\phi(t))\right]_0 = \left[M\Delta^n \xi_k(\psi(\phi(t)))\right]_0, \tag{49}$$

where we may assume $n \geq 2$.

Finally, for (vii), from Theorem C on page 130 of [5] there holds

$$f(t) = \sum_{n\geq 0} f^{(n)}(nx)p_n(t,x)$$

for all $f \in \mathcal{F}$, where $f^{(k)}$ is the kth derivative of f and x is fixed. If $x = 0$, then the above expansion is the ordinary (formal) Taylor formula. Noting

$$[p_n(t)]\, f(t) \equiv [p_n(t,x)]\, f(t) = f^{(n)}(nx),$$

we find out from Vandermonde-type convolution formula (22) in Theorem 3.1

$$\begin{aligned}&\sum_{\lambda\geq 0} D_t^n \left.(\alpha(t)p_\lambda(\phi(t)))\right|_{t=nx} D_t^\lambda \left.(\beta(t)p_k(\psi(t)))\right|_{t=nx} \\ &\quad = D_t^n \left.(\alpha(t)\beta(\phi(t))p_k(\psi(\phi(t))))\right|_{t=nx}.\end{aligned} \tag{50}$$

Remark 3.3 Clearly (21) subject to (27) reduces to finite summations whenever $\psi(t)$ is polynomial. The final result is:

$$\sum_{\lambda\geq 0} \Delta^n \binom{\phi(t)}{\lambda}_0 \Delta^\lambda \binom{\psi(t)}{k}_0 = \Delta^n \binom{\psi(\phi(t))}{k}_0. \tag{51}$$

Remark 3.4 The Lagrange inverse formula in [5, Theorem 3.7] guarantees the relation

$$\left[\frac{t^n}{n!}\right] \frac{h^*(t)^k}{k!} = D_0^n \frac{h^*(t)^k}{k!} = \binom{n-1}{k-1} D_0^{n-k} \left(\frac{t}{h(t)}\right)^k,$$

suggesting that for the case $d(t) \equiv 1$, (29) may be reformulated in the form

$$\begin{aligned} \sigma^*(n,k) &= \left[\frac{t^n}{n!}\right]\frac{h^*(t)^k}{k!} \\ &= \binom{n-1}{k-1}\left[\frac{t^{n-k}}{(n-k)!}\right]\left(\frac{t}{h(t)}\right)^n \\ &= \binom{n-1}{k-1}D^{n-k}\left(\frac{t}{h(t)}\right)^n_{t=0}. \end{aligned} \tag{52}$$

This applies to the case when $h^*(t)$, the compositional inverse of $h(t)$, is not easily computed. For instance, [10] consider $(d(t), h(t)) = \left(\frac{1-et}{1+(c-e)t}, \frac{t(1-et)}{1+(c-e)t}\right)$, which has a complicated inverse $(d^*(t), h^*(t)) = (d_{c,e}(t), td_{c,e}(t))$, where

$$d_{c,e}(t) = \frac{1-(c-e)t-\sqrt{1-2(c+e)t+(c-e)^2t^2}}{2et}, \quad e \neq 0.$$

4 A general class of Abel identities

Inspired by [7, 33], we will use the fundamental property of the generalized Riordan arrays and its alternating form with respect to basic sets to give two ways for the construction of a general class of Abel identities. First, we present the fundamental property of the generalized Riordan arrays with respect to a basic set using a similar argument of [28]. Let $(d(t), h(t))$, $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$, be a generalized Riordan array with respect to the basic set $\{p_n(t)\}$, and let $f(t)$ be the GF of any sequence $\{f_n\}_{n\geq 0}$ in terms of $\{p_n(t)\}$, i.e., $f(t) = \sum_{n\geq 0} f_n p_n(t)$. We have

$$\sum_{k\geq 0} d_{n,k} f_k = [p_n(t)]d(t)f(h(t)), \tag{53}$$

which comes from the following observation

$$\begin{aligned} &[p_n(t)]d(t)f(h(t)) = [p_n(t)]d(t)\sum_{k\geq 0} f_k p_k(h(t)) \\ = &\sum_{k\geq 0} f_k [p_n(t)]d(t)p_k(h(t)) = \sum_{k\geq 0} d_{n,k} f_k. \end{aligned} \tag{54}$$

Proposition 4.1 *Let $(d(t),\ h(t)) = (d_{n,k})_{n\geq k\geq 0}$ be a generalized Riordan array with respect to the basic set $\{p_n(t)\}_{n\geq 0}$ with $p_0(t) = 1$ and $p_n(t) = \sum_{j=1}^n \alpha_{n,j} t^j$ ($n \geq 1$, $\alpha_{n,j} \neq 0$), where $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$, and let $f(t) = \sum_{n\geq 0} f_n p_n(t)$. Then there holds*

$$\sum_{k=1}^{n} d_{n,k} f_k = \sum_{j=1}^{n} \sum_{\ell=1}^{j} \frac{1}{\alpha_{n,j}} [t^\ell] f(h(t)) [t^{j-\ell}] d(t). \tag{55}$$

Furthermore, if $h^*(t)$ is the compositional inverse of $h(t)$, then (55) can be written as

$$\sum_{k=1}^{n} d_{n,k} f_k = \sum_{j=1}^{n} \sum_{\ell=1}^{j} \frac{1}{\ell \alpha_{n,j}} [u^{\ell-1}] f'(u) \left(\frac{u}{h^*(u)} \right)^\ell [t^{j-\ell}] d(t). \tag{56}$$

Particularly, if $p_n(t) = t^n/c_n$, $c_0 = 1$, $c_n \neq 0$ for all $n \geq 1$, then (56) is specified to

$$\sum_{k=1}^{n} d_{n,k} f_k = c_n \sum_{\ell=1}^{n} \frac{1}{\ell} [u^{\ell-1}] f'(u) \left(\frac{u}{h^*(u)} \right)^\ell [t^{n-\ell}] d(t). \tag{57}$$

Proof. For $n \geq 1$, substituting $p_n(t) = \sum_{j=1}^{n} \alpha_{n,j} t^j$ ($\alpha_{n,j} \neq 0$ or see the following Remark 4.1) into (53), we can write it as

$$\begin{aligned}
& \sum_{k=0}^{n} d_{n,k} f_k = \sum_{j=1}^{n} \frac{1}{\alpha_{n,j}} [t^j] d(t) f(h(t)) \\
= & \sum_{j=1}^{n} \sum_{\ell=0}^{j} \frac{1}{\alpha_{n,j}} [t^\ell] f(h(t)) [t^{j-\ell}] d(t) \\
= & \sum_{j=1}^{n} \frac{1}{\alpha_{n,j}} [1] f(h(t)) [t^j] d(t) + \sum_{\ell=1}^{n} \sum_{j=\ell}^{n} \frac{1}{\alpha_{n,j}} [t^\ell] f(h(t)) [t^{j-\ell}] d(t) \\
= & f_0 [p_n(t)] d(t) + \sum_{\ell=1}^{n} \sum_{j=\ell}^{n} \frac{1}{\alpha_{n,j}} [t^\ell] f(h(t)) [t^{j-\ell}] d(t),
\end{aligned}$$

which implies (55). Noting the compositional inverse of $h(t)$, which exists due to $h(t) \in \mathcal{F}_1$, satisfies $u = t(u/h^*(u))$, we may use the Lagrange inverse formula to write

$$[t^\ell] f(h(t)) = \frac{1}{\ell} [u^{\ell-1}] f'(u) \left(\frac{u}{h^*(u)} \right)^\ell, \quad \ell \geq 1.$$

Thus (56) is derived by (55).

∎

Remark 4.1 From the proof of (55), it is easy to see that the initial terms of both sides of (55) can be extended to $k = 0$ and $\ell = 0$, respectively, namely,

$$\sum_{k=0}^{n} d_{n,k} f_k = \sum_{j=1}^{n} \sum_{\ell=0}^{j} \frac{1}{\alpha_{n,j}} [t^\ell] f(h(t)) [t^{j-\ell}] d(t).$$

The fundamental property (53) of the generalized Riordan arrays has an alternating form shown below.

Theorem 4.2 *Let $(d(t), h(t))$, $d(t) \in \mathcal{F}_0$, $h(t) \in \mathcal{F}_1$, be a generalized Riordan array with respect to $\{p_n(t)\}$, where $p_0(t) = 1$ and $p_n(t) = \sum_{k=1}^{n} \alpha_{n,j} t^j$, $\alpha_{n,j} \neq 0$, for $n \geq 1$, and let $f(t)$ be a formal power series. Denote $f(h(t))$ by $g(t)$, and $[p_n(t)]g(h^*(t))$ by g_n. Then there holds*

$$\sum_{k \geq 0} d_{n,k} g_k = [p_n(t)] d(t) g(t) \tag{58}$$

where $g_0 = f_0$ and

$$g_k = [p_k(t)] g(h^*(t)) = \sum_{j=1}^{k} \frac{1}{j \alpha_{k,j}} [u^{j-1}] g'(u) \left(\frac{u}{h(u)} \right)^j \tag{59}$$

for all $k \geq 1$. Particularly, if $p_n(t) = \alpha_{n,n} t^n$, $\alpha_{n,n} \neq 0$, then

$$g_k = \frac{1}{k \alpha_{k,k}} [u^{k-1}] g'(u) \left(\frac{u}{h(u)} \right)^k \tag{60}$$

for all $k \geq 1$.

Proof. Since $g(t) = f(h(t))$, then $g(h^*(t)) = f(t)$, where $h^*(t)$ is the compositional inverse of $h(t)$. Thus, (54) becomes

$$[p_n(t)] d(t) g(t) = \sum_{k \geq 0} d_{n,k} f_k,$$

where $f_k = [p_k(t)] f(t) = [p_k(t)] g(h^*(t)) = g_k$. Denote $u = h^*(t)$. We have $t = h(u)$, which satisfies $u = t(u/h(u))$. Using the Lagrange inverse formula shown in Theorem 5.1 of [35] in which $\phi(u) = u/h(u)$, we may further obtain

$$\begin{aligned} g_k &= [p_k(t)] g(h^*(t)) = \sum_{j=1}^{k} \frac{1}{\alpha_{k,j}} [t^j] g(h^*(t)) \\ &= \sum_{j=1}^{k} \frac{1}{j \alpha_{k,j}} [u^{j-1}] g'(u) \left(\frac{u}{h(u)} \right)^j \end{aligned}$$

for all $k \geq 1$. As for $k=0$, $g_0 = [1]g(h^*(t)) = g(h^*(0)) = g(0) = f(0) = f_0$, completing the proof.

■

Substituting (58) into (59), we obtain

Corollary 4.3 *If $d(t)$, $h(t)$, $f(t)$, and $g(t)$ are defined as the same as in Theorem 4.2, then we have identity.*

$$\sum_{k\geq 0} d_{n,k} g_k = d_{n,0} g_0 + \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{d_{n,k}}{j\alpha_{k,j}} [u^{j-1}] g'(u) \left(\frac{u}{h(u)} \right)^j = [p_n(t)] d(t) g(t). \tag{61}$$

Particularly, if $p_n(t) = t^n / c_n$, then (61) becomes

$$\sum_{k\geq 0} d_{n,k} g_k = d_{n,0} g_0 + \sum_{k=1}^{n} d_{n,k} \frac{c_k}{k} [u^{k-1}] g'(u) \left(\frac{u}{h(u)} \right)^k = [p_n(t)] d(t) g(t). \tag{62}$$

Example 4.1 As an example of (62) with $p_n(t) = t^n / c_n$, we now consider $(d(t), h(t)) = (e^{\alpha t}, te^{\beta t})$ and $g(t) = e^{\gamma t}$. Thus,

$$\begin{aligned}
d_{n,k} &= \left[\frac{t^n}{c_n} \right] e^{\alpha t} \frac{(te^{\beta t})^k}{c_k} = \frac{c_n}{c_k} \frac{(\alpha + \beta k)^{n-k}}{(n-k)!}, \\
g_k &= \left[\frac{t^k}{c_k} \right] g(h^*(t)) = \frac{c_k}{k} [u^{k-1}] g'(u) \left(\frac{u}{h(u)} \right)^k = \frac{c_k \gamma}{k!} (\gamma - \beta k)^{k-1}, \\
[p_n(t)]\, d(t) g(t) &= c_n [t^n] e^{(\alpha+\gamma)t} = \frac{c_n}{n!} (\alpha + \gamma)^n,
\end{aligned}$$

which implies the following well-known Abel identity:

$$\frac{c_n}{c_0} \frac{(\alpha)^n}{n!} c_0 \gamma (\gamma)^{-1} + \sum_{k=1}^{n} \frac{c_n}{c_k} \frac{(\alpha + \beta k)^{n-k}}{(n-k)!} \frac{c_k \gamma}{k!} (\gamma - \beta k)^{k-1} = \frac{c_n}{n!} (\alpha + \gamma)^n,$$

or equivalently,

$$\sum_{k=0}^{n} \gamma \binom{n}{k} (\gamma - \beta k)^{k-1} (\alpha + \beta k)^{n-k} = (\alpha + \gamma)^n.$$

From (58), we can find another type identities. Let $(d(t),\ h(t)) = (d_{n,k})_{n\geq k\geq 0}$ be a (c)-generalized Riordan array with respect to a (c)-sequence

$\{c_k\}_{k\geq 0}$ with $c_0 = 1$ and $c_k \neq 0$ for all $k > 0$, where $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$, and let $f(t) = \sum_{n\geq 0} f_n t^n / c_n$. Then, there holds identity

$$\sum_{k=1}^{n} d_{n,k} \frac{f_k}{c_k} = \sum_{j=1}^{n} [t^j] f(h(t)) [t^{n-j}] d(t). \tag{63}$$

Furthermore, denote by $h^*(t)$ is the compositional inverse of $h(t)$, then (63) can be written as

$$\sum_{k=1}^{n} d_{n,k} \frac{f_k}{c_k} = \sum_{j=1}^{n} \frac{1}{j} [u^{j-1}] f'(u) \left(\frac{u}{h^*(u)} \right)^j [t^{n-j}] d(t). \tag{64}$$

Indeed, substituting $g_k = f_k$ and $g(h(t)) = f(h(t))$ into (58), we can write it as

$$\begin{aligned} & \sum_{k=0}^{n} d_{n,k} \frac{f_k}{c_k} = [t^n] d(t) f(h(t)) \\ = & \sum_{j=0}^{n} [t^j] f(h(t)) [t^{n-j}] d(t) \\ = & f_0 d_{n,0} + \sum_{j=1}^{n} [t^j] f(h(t)) [t^{n-j}] d(t), \end{aligned}$$

which implies (63). Noting the compositional inverse of $h(t)$, which exists due to $h(t) \in \mathcal{F}_1$, satisfies $u = t(u/h^*(u))$, we may use the Lagrange inverse formula to write

$$[t^j] f(h(t)) = \frac{1}{j} [u^{j-1}] f'(u) \left(\frac{u}{h^*(u)} \right)^j, \quad j \geq 1.$$

Thus (64) is derived by (63).

From the proof of (63), it is easy to see that the initial terms of both sides of (63) can be extended to $k = 0$ and $\ell = 0$, respectively, namely,

$$\sum_{k=0}^{n} d_{n,k} \frac{f_k}{c_k} = \sum_{j=0}^{n} [t^j] f(h(t)) [t^{n-j}] d(t).$$

Example 4.2 As an example, we consider $(d(t),\ h(t)) = (1/(1-t),\ t/(1-t))$. From (64). Noting formula (14) for evaluating $d_{n,k}$, from (57) we obtain the identity

$$\sum_{k=1}^{n} \binom{n}{k} \left[t^k\right] f(t) = \sum_{j=1}^{n} \frac{1}{j} [u^{j-1}] f'(u)\,(1+u)^j\,, \tag{65}$$

which yields the identity

$$\sum_{k=1}^{n} \frac{1}{k!}\binom{n}{k} = \sum_{j=1}^{n}\sum_{\ell=0}^{j-1} \frac{1}{j\ell!}\binom{j}{j-1-\ell}$$

when $f(t) = e^t$. For $f(t) = (a+t)^m$, formula (65) yields the identity

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} a^{m-k} = \sum_{l=0}^{m} \binom{m}{l}\binom{m+n-l}{m-l}(a-1)^l.$$

In fact, the right-hand side of (65) becomes

$$\begin{aligned}
& \sum_{j=1}^{n} \frac{m}{j}[u^{j-1}](a+u)^{m-1}(1+u)^j \\
= \quad & \sum_{j=1}^{n} \frac{m}{j}[u^{j-1}] \sum_{l=0}^{m-1} \binom{m-1}{l}(a-1)^l(1+u)^{m-l+j-1} \\
= \quad & m\sum_{l=0}^{m-1} \binom{m-1}{l}(a-1)^l \sum_{j=1}^{n} \frac{1}{j}\binom{m-l+j-1}{j-1} \\
= \quad & \sum_{l=0}^{m-1} \frac{m}{m-\ell}\binom{m-1}{\ell}(a-1)^\ell \sum_{j=1}^{n}\binom{m-\ell+j-1}{m-\ell-1} \\
= \quad & \sum_{l=0}^{m-1} \binom{m}{\ell}(a-1)^\ell \left(\sum_{j=0}^{n}\binom{m-\ell+j-1}{m-\ell-1} - 1\right).
\end{aligned}$$

Applying these to make further simplification, in which we have used the summation formula

$$\sum_{k=0}^{n} \binom{m+k}{m} = \binom{n+m+1}{m+1}.$$

Thus, the right-hand side of (65) can be written as

$$
\begin{aligned}
&\sum_{l=0}^{m-1}\binom{m}{\ell}(a-1)^\ell\left(\binom{m+n-\ell}{m-\ell}-1\right)\\
=&\sum_{l=0}^{m-1}\binom{m}{\ell}(a-1)^\ell\binom{m+n-\ell}{m-\ell}-\sum_{l=0}^{m-1}\binom{m}{\ell}(a-1)^\ell\\
=&\sum_{l=0}^{m}\binom{m}{\ell}(a-1)^\ell\binom{m+n-\ell}{m-\ell}-a^m,
\end{aligned}
$$

which yields the desired identity.

For $f(t)=\frac{1}{\sqrt{1+t}}$, formula (65) yields the identity (cf. (3.85) in [6])

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{(-1)^k}{4^k}=\sum_{j=0}^{n}\frac{(-3/2+j)_j}{j!}=\binom{-1/2+n}{n}$$

because that

$$f(t)=\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-t)^k}{4^k}$$

and

$$-\frac{1}{2}[u^{j-1}](1+u)^{-3/2+j}=\frac{(-3/2+j)_j}{j!}:=\binom{-3/2+j}{j}.$$

In general, for $\alpha\in\mathbb{C}$, let $f(t)=(1+t)^\alpha$, we might deduce the following identity similarly

$$\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha}{k}=\sum_{j=0}^{n}\frac{(\alpha+j-1)_j}{j!}=\binom{\alpha+n}{n}.$$

In the following example, we consider parameter $0<q<1$ and denote

$$c_n=n!_q:=\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}. \tag{66}$$

Roman [26] defines

$$\epsilon_a(t)=\sum_{k\geq 0}\frac{a^k}{c_k}t^k$$

with a non-zero constant a, and evaluates

$$\frac{1}{\epsilon_a(t)} = \sum_{k\geq 0} q^{\binom{k}{2}} \frac{(-at)^k}{c_k}.$$

Hence, we have the q-analogue (c)-Riordan array

$$\left(\frac{1}{\epsilon_a(t)}, \frac{t}{1-t}\right) = \left(\sum_{\ell=k}^{n} \binom{\ell-1}{k-1} \frac{a^{n-\ell}}{c_{n-\ell}}\right)_{n\geq k\geq 0},$$

where we use the formula

$$[t^n]f(t)g(t) = \sum_{\ell=0}^{n} [t^\ell]f(t)[u^{n-\ell}]g(y)$$

to evaluate

$$d_{n,k} = [t^n]\epsilon_a(t)\left(\frac{t}{1-t}\right)^k.$$

Therefore we obtain $q-$identities

$$\sum_{k=1}^{n} \frac{1}{k!} \sum_{\ell=k}^{n} \binom{\ell-1}{k-1} \frac{a^{n-\ell}}{c_{n-\ell}} = \sum_{j=1}^{n} \frac{a^{n-j}}{c_{n-j}} \sum_{\ell=0}^{j-1} \frac{1}{j\ell!} \binom{j}{j-1-\ell}$$

and

$$\sum_{k=1}^{n} \sum_{\ell=k}^{n} \binom{\ell-1}{k-1} \frac{a^{n-\ell}}{c_{n-\ell}} = \sum_{j=1}^{n} \frac{a^{n-j}}{c_{n-j}} \sum_{\ell=0}^{j-1} \frac{\ell+1}{j} \binom{j}{j-1-\ell}$$

from (64) with $f(t) = e^t$ and $f(t) = 1/(1-t)$, respectively, where c_k are defined by (66) and note $\binom{0}{k} = \delta_{k,0}$.

For $(d(t), h(t)) = (\epsilon_a(t), \frac{t}{1-t})$, the q–exponential function $\epsilon_a(t) = \sum_{n\geq 0} \frac{(at)^k}{c_k}$, $c_k = n!_q$. Assume further $f(t) = 1/(1+t)$. Then it is clear that

$$d_{n,k} = [t^n]\epsilon_a(t)\left(\frac{t}{1-t}\right)^k = \sum_{l=k}^{n} \binom{l-1}{k-1} \frac{a^{n-l}}{c_{n-l}}.$$

All these leads us to the following q–identity

$$\sum_{k=1}^{n} (-1)^k \sum_{l=k}^{n} \binom{l-1}{k-1} \frac{a^{n-l}}{c_{n-l}} = -\frac{a^{n-1}}{c_{n-1}}$$

since

$$\begin{aligned}&\sum_{j=1}^{n}\frac{-1}{j}[u^{j-1}](1+u)^{j-2}[t^{n-j}]\epsilon_a(t)\\ =\quad & -[t^{n-1}]\epsilon_a(t)=-\frac{(at)^{n-1}}{c_{n-1}}.\end{aligned}$$

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