Illinois Wesleyan University

From the SelectedWorks of Tian-Xiao He

Spring June, 2009

Sequence Characterization of Riordan Arrays

Tian-Xiao He, *Illinois Wesleyan University*
Renzo Sprugnoli

Available at: https://works.bepress.com/tian_xiao_he/3/

A/Z characterization of Riordan arrays

Tian-Xiao He[1] and Renzo Sprugnoli[2]

[1]Department of Mathematics and Computer Science
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA

[2]Dipartimento di Sistemi e Informatica
Viale Morgagni 65, 50134 Firenze, Italy

December 30, 2008

Abstract

In the realm of the Riordan group, we consider the characterization of Riordan arrays by means of the A- and Z-sequences. It corresponds to a horizontal construction of a Riordan array, whereas the traditional approach is through column generating functions. We show how the A- and Z-sequences of the product of two Riordan arrays are derived from those of the two factors; similar results are obtained for the inverse. Finally, we give the characterizations relative to the **some** subgroups of the Riordan group, in particular of the hitting-time subgroup.

Key words Riordan arrays, A-sequence, Z-sequence, hitting-time subgroup.

Mathematics subject Classification (2000) 05A10 - 05A15 - 05A19

1 Introduction

In the recent literature, special emphasis has been given to the concept of *proper Riordan arrays*, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [13]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli [14, 15], on subgroups of the Riordan group in Peart and Woan [8] and Shapiro [12], on some characterizations of Riordan matrices in Rogers [9] and Merlini et al. [6], and on many interesting related results in Getu et al. [4], He et al. [5], Nkwanta [7], Zhao and Wang [17], and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s.'s $f(t)$ for which the compositional inverse $\overline{f}(t)$ exists such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t),\, h(t))$ defines the *proper Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}}$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1.1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function of column k. It is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d(t),\, h(t)) * (f(t),\, g(t)) = (d(t)f(h(t)),\, g(h(t))). \tag{1.2}$$

The Riordan array $I = (1,\, t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it is easily seen that I acts as an identity for this product, that is, $(1,\, t) * (d(t),\, h(t)) = (d(t),\, h(t)) * (1, t) = (d(t),\, h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(\overline{h}(t))},\, \overline{h}(t)\right) \tag{1.3}$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays is a group. Particular subgroups of $\mathcal{R}$ are important and have been considered in the literature:

- the set $\mathcal{A}$ of *Appell arrays*, that is the Riordan arrays $D = (d(t),\, h(t))$ for which $h(t) = t$; it is an invariant subgroup and is isomorphic to the group of f.p.s.'s of order 0, with the usual product as group operation;
- the set $\mathcal{L}$ of *Lagrange arrays*, that is the Riordan arrays $D = (d(t),\, h(t))$ for which $d(t) = 1$; it is also called the *associated subgroup*; it is isomorphic with the group of f.p.s.'s of order 1, with composition as group operation;
- the set $\mathcal{B}$ of *Bell* or *renewal arrays*, that is the Riordan arrays $D = (d(t),\, h(t))$ for which $h(t) = td(t)$; it is the set originally considered by Rogers in [9];
- the set $\mathcal{C}$ of *Checkerboard arrays*, that is, the Riordan arrays $D = (d(t),\, h(t))$ for which $d(t)$ is an even function and $h(t)$ is an odd function.

Another important subgroup, called the *hitting time* subgroup, introduced by Peart and Woan in [8], will be considered in Section 4.

If $(f_k)_{k\in\mathbb{N}}$ is any sequence and $f(t) = \sum_{k=0}^{\infty} f_k t^k$ is its generating function, then for every Riordan array $D = (d(t),\, h(t))$ we have:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(h(t))$$

which relates Riordan arrays to combinatorial sums and sum inversion.

Rogers [9] introduced the concept of the *A-sequence* for Riordan arrays; Merlini et al. [6] introduced the related concept of the *Z-sequence* and showed that these two concepts, together with the element $d_{0,0}$, completely characterize a proper Riordan array. This fact is the starting point of this paper, which has therefore the following structure. In Section 2 we reconsider the A/Z characterization of Riordan arrays, proving its main properties and its relation to the previously defined subgroups; in Section 3 we show how the A/Z approach characterizes the product of two Riordan arrays; we continue in Section 4 with the inversion. In Section 5 we show how the characterization allows us to construct easily a Riordan array and finally, in Section 6, we apply the theory to the special case of the Riordan arrays in the hitting-time subgroup.

2 The A/Z characterization

In his paper [9], Rogers states that for every proper Riordan array $D = (d(t),\, h(t))$ there exists a sequence $A = (a_k)_{k\in\mathbb{N}}$ such that for every $n, k \in \mathbb{N}$ we have:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} a_j d_{n,k+j} \tag{2.4}$$

where the sum is actually finite since $d_{n,k} = 0,\ \forall k > n$. A proof of this fact was given in Sprugnoli [16], and we can reformulate it according to the modified definition of a Riordan array used in the present paper.

Theorem 2.1 *An infinite lower triangular array $D = (d_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ relation (2.4) holds*

Proof: Let us suppose that D is the Riordan array $(d(t),\, h(t))$ and let us consider the Riordan array $(d(t)h(t)/t,\, h(t))$; we define the Riordan array $(A(t),\, B(t))$ by the relation:

$$(A(t),\, B(t)) = (d(t),\, h(t))^{-1} * (d(t)h(t)/t,\, h(t))$$

or:

$$(d(t),\, h(t)) * (A(t),\, B(t)) = (d(t)h(t)/t,\, h(t)).$$

By performing the product (1.2), we find:

$$d(t)A(h(t)) = d(t)h(t)/t \qquad \text{and} \qquad \mathbf{B}(\mathbf{h}(\mathbf{t})) = \mathbf{h}(\mathbf{t}). \tag{2.5}$$

The latter identity implies $B(t) = t$. Therefore we have $(d(t), h(t)) * (A(t), t) = (d(t)h(t)/t, h(t))$. The element $f_{n,k}$ of the left hand member is $\sum_{j=0}^{\infty} d_{n,j}a_{j-k} = \sum_{j=0}^{\infty} d_{n,k+j}a_j$, if as usual we interpret a_{j-k} as 0 when $j < k$. The same element in the right hand member is:

$$[t^n]d(t)h(t)h(t)^k/t = [t^{n+1}]d(t)h(t)^{k+1} = d_{n+1,k+1}.$$

By equating these two quantities, we have the identity (2.4). We remark that the first relation in (2.5) is equivalent to $tA(h(t)) = h(t)$.

For the converse, let us observe that (2.4) uniquely defines the array D when the elements of column 0 $(d_{0,0}, d_{1,0}, d_{2,0}, \ldots)$ are given. Let $d(t)$ be the generating function of this column, $A(t)$ the generating function of the sequence A and define $h(t)$ as the solution of the functional equation $h(t) = tA(h(t))$, which is uniquely determined because of our hypothesis $a_0 \neq 0$. We can therefore consider the proper Riordan array $\widehat{D} = (d(t), h(t))$; by the first part of the theorem, $\widehat{D}$ satisfies relation (2.4), for every $n, k \in \mathbb{N}$ and therefore, by our previous observation, it must coincide with D. This completes the proof. ∎

The sequence $A = (a_k)_{k\in\mathbb{N}}$ is the A-sequence of the Riordan array $D = (d(t), h(t))$ and it only depends on $h(t)$. In fact, as we have shown during the proof of the theorem, we have:

$$h(t) = tA(h(t)) \qquad \text{or} \qquad A(y) = \left[\frac{h(t)}{t} \;\middle|\; y = h(t)\right] = \left[\frac{y}{t} \;\middle|\; y = h(t)\right] \tag{2.6}$$

and this uniquely determines A when $h(t)$ is given and vice versa, $h(t)$ is uniquely determined when A is given.

Example 2.1 *Riordan arrays were introduced as a generalization of the Pascal triangle P. Actually, we have*

$$P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right);$$

as can be easily seen:

$$P_{n,k} = [t^n]\frac{1}{1-t}\left(\frac{t}{1-t}\right)^k = [t^{n-k}](1-t)^{-k-1} = (-1)^{n-k}\binom{-k-1}{n-k} = \binom{n}{k}.$$

In order to determine the A-sequence, we set $y = h_P(t) = t/(1-t)$, that is, $t = y/(1+y)$. Formula (2.4) now gives $A_P(y) = y/t = 1+y$, which means that the A-sequence of the Pascal triangle is $(1, 1, 0, 0, \ldots)$, corresponding to the well-known recurrence:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Example 2.2 *Let us now consider a Riordan array F defined by the function $d_F(t) = (1-t-t^2)^{-1}$, so that column 0 is composed by Fibonacci numbers, shifted by one place. Besides, the A-sequence is $A_F = (1, 1, 1, 1, \ldots)$, that is, any element $F_{n+1,k+1}$ is obtained by summing all the elements in the previous row, starting from column k. The Riordan array F is easily constructed; we show its upper corner in Table 1. Formula (2.6) allows us to compute the function $h(t)$ of this array:*

$$h_F(t) = tA_F(h(t)) = \frac{t}{1-h_F(t)};$$

this equation has two solutions, but we know that $A_F(0) \neq 0$, so that we should consider the solution with the minus sign:

$$h_F(t) = \frac{1-\sqrt{1-4t}}{2} = t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + 132t^7 + \cdots$$

the well-known Catalan numbers. Therefore we have:

$$F = \left(\frac{1}{1-t-t^2}, \frac{1-\sqrt{1-4t}}{2}\right).$$

This result can be easily checked on Table 1

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	3	5	3	1			
4	5	12	9	4	1		
5	8	31	26	14	5	1	
6	13	85	77	46	20	6	1

Table 1: The Fibonacci triangle F

Although the two functions $d(t)$ and $A(t)$ completely characterize a proper Riordan array, we are mainly interested in another type of characterization. Let us consider the following result (see Merlini et al. [6]):

Theorem 2.2 *Let $(d_{n,k})_{n,k\in\mathbb{N}}$ be any infinite, lower triangular array with $d_{n,n}\neq 0$, $\forall n\in\mathbb{N}$ (in particular, let it be a proper Riordan array); then a unique sequence $Z=(z_k)_{k\in\mathbb{N}}$ exists such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, i.e.:*

$$d_{n+1,0}=z_0d_{n,0}+z_1d_{n,1}+z_2d_{n,2}+\cdots=\sum_{j=0}^{\infty}z_jd_{n,j}. \tag{2.7}$$

Proof: Let $z_0=d_{1,0}/d_{0,0}$. Now we can uniquely determine the value of z_1 by expressing $d_{2,0}$ in terms of the elements in row 1, i.e.:

$$d_{2,0}=z_0d_{1,0}+z_1d_{1,1} \qquad \text{or} \qquad z_1=\frac{d_{0,0}d_{2,0}-d_{1,0}^2}{d_{0,0}d_{1,1}}.$$

In the same way, we determine z_2 by expressing $d_{3,0}$ in terms of the elements in row 2, and by substituting the values just obtained for z_0 and z_1. By proceeding in the same way, we determine the sequence Z in a unique way. ∎

The sequence Z is called the *Z-sequence* for the (Riordan) array; it characterizes column 0, except for the element $d_{0,0}$. Therefore, we can say that the triple $(d_{0,0},A(t),Z(t))$ completely characterizes a proper Riordan array. Usually, we denote $d_{0,0}$ as δ.

To see how the Z-sequence is obtained by starting with the usual definition of a Riordan array, let us prove the following:

Theorem 2.3 *Let $(d(t),h(t))$ be a proper Riordan array and let $Z(t)$ be the generating function of the corresponding Z-sequence. We have:*

$$d(t)=\frac{\delta}{1-tZ(h(t))} \qquad \text{or} \qquad Z(y)=\left[\frac{d(t)-\delta}{td(t)}\;\Big|\; t=\overline{h}(y)\right]. \tag{2.8}$$

Proof: By the preceding theorem, the Z-sequence exists and is unique. Therefore, equation (2.7) is valid for every $n\in\mathbb{N}$, and we can pass to the generating functions. Since $d(t)h(t)^k$ is the generating function for column k, we have:

$$\frac{d(t)-d_{0,0}}{t}=z_0d(t)+z_1d(t)h(t)+z_2d(t)h(t)^2+\cdots=d(t)(z_0+z_1h(t)+z_2h(t)^2+\cdots)=d(t)Z(h(t)).$$

By solving this equation in $d(t)$, we immediately find the first relation. The second relation is an immediate consequence, when we set $y=h(t)$. ∎

Example 2.3 *Clearly, in the Pascal triangle we have $Z_P=(1,0,0,0,\ldots)$ and so $Z_P(t)=1$. Formula (2.8) gives immediately $d_P(t)=1/(1-t)$. The Z-sequence for the Fibonacci triangle is a bit more complicated. If we set $y=h_F(t)=(1-\sqrt{1-4t})/2$, we can invert the function $h_F(t)$ and find $t=y-y^2$. We now substitute this expression in:*

$$\frac{d_F(t)-\delta_F}{td_F(t)}=\left(\frac{1}{1-t-t^2}-1\right)\cdot\frac{1-t-t^2}{t}=1+t$$

and find $Z_F(t)=1+t-t^2$; the Z-sequence is $Z_F=(1,1,-1,0,0,\ldots)$, which checks with Table 1.

We can characterize the main subgroups of $\mathcal{R}$ by means of their A- and/or Z-sequences. In fact we have:

Theorem 2.4 *A Riordan array $D = (d(t), h(t))$ belongs to the Appell subgroup $\mathcal{A}$ if and only if its A-sequence satisfies $A(t) = 1$. Besides, in that case, we also have:*

$$Z(t) = \frac{d(t) - \delta}{td(t)}.$$

Proof: By formula (2.6) the two relations $h(t) = t$ and $A(t) = 1$ are equivalent. The expression for $Z(t)$ is an immediate consequence of formula (2.8). ∎

For the Lagrange subgroup we find a characterization related to the Z-sequence; the A-sequence is to be determined by inverting the general formula (2.6).

Theorem 2.5 *A Riordan array $D = (d(t), h(t))$ belongs to the Lagrange subgroup $\mathcal{L}$ if and only if its Z-sequence satisfies $Z(t) = 0$ and $\delta = 1$.*

Proof: By formula (2.8) the two relations $Z(t) = 0$ and $d(t) = \delta$ are equivalent. Therefore, we have a Lagrange array if and only if $\delta = 1$. ∎

For the Bell subgroup of renewal arrays we have a more elaborate result:

Theorem 2.6 Let $D = (d(t), h(t))$ **be** *a Riordan array in which $d(0) = h'(0) \neq 0$. Then $d(t) = h(t)/t$ if and only if: $A(y) = \delta + yZ(y)$.*

Proof: Let us assume that $A(y) = \delta + yZ(y)$ or $Z(y) = (A(y) - \delta)/y$. By formula (2.8), we have:

$$d(t) = \frac{\delta}{1 - tZ(h(t))} = \frac{\delta}{1 - (tA(h(t)) - \delta t)/h(t)} = \frac{\delta h(t)}{\delta t} = \frac{h(t)}{t},$$

because $tA(h(t)) = h(t)$ by formula (2.6). On the contrary, by the formula for $Z(y)$, we obtain from the hypothesis $d(t) = h(t)/t$:

$$\delta + yZ(y) = \left[\delta + y\left(\frac{1}{t} - \frac{\delta}{h(t)}\right) \Bigm| y = h(t)\right] =$$
$$= \left[\delta + \frac{h(t)}{t} - \frac{\delta h(t)}{h(t)} \Bigm| y = h(t)\right] = \left[\frac{h(t)}{t} \Bigm| y = h(t)\right] = A(y)$$

and this completes the proof. ∎

The reader can check this result **for the Pascal** triangle, which is a renewal array.
Finally, we give the characterization of the checkerboard subgroup $\mathcal{C}$:

Theorem 2.7 *A Riordan array $D = (d(t), h(t))$ belongs to the checkerboard subgroup $\mathcal{C}$ if and only if the generating functions $A(t)$ of its A-sequence is even and $Z(t)$ of its Z-sequence is odd.*

Proof: By formula (2.6) the oddness of $h(t)$ of a checkerboard array is equivalent to the evenness of $A(t)$, and by formula (2.8) the evenness of $d(t)$ is equivalent to the oddness of $Z(t)$. ∎

3 The product of two Riordan arrays

According to Formula (1.2), the product of two Riordan arrays is a Riordan array; therefore, a natural question is: how do the A- and Z-sequences of the product depend on the analogous sequences of the two factors? In order to answer this question, let us consider two proper Riordan arrays $D_1 = (d_1(t),\ h_1(t))$ and $D_2 = (d_2(t),\ h_2(t))$ and their product:

$$D_3 = D_1 * D_2 = (d_1(t)d_2(h_1(t)),\ h_2(h_1(t)))$$

so that:

$$d_3(t) = d_1(t)d_2(h_1(t)) \qquad \text{and} \qquad h_3(t) = h_2(h_1(t)).$$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	5	4	1				
3	13	14	6	1			
4	34	48	27	8	1		
5	89	166	111	44	10	1	
6	233	587	443	210	65	12	1

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	5	4	1				
3	12	14	6	1			
4	31	46	27	8	1		
5	85	150	108	44	10	1	
6	248	493	410	206	65	12	1

Table 2: The triangles $Q = P * F$ and $G = F * P$

Example 3.1 *Let us perform the product $Q = P * F$; we have:*

$$d_Q(t) = d_P(t)d_F(h_P(t)) = \frac{1}{1-t} \cdot \left[\frac{1}{1-u-u^2} \;\middle|\; u = \frac{t}{1-t} \right] = \frac{1-t}{1-3t+t^2}$$

corresponding to Fibonacci numbers in odd positions. Besides:

$$h_Q(t) = h_F(h_P(t)) = \left[\frac{1-\sqrt{1-4u}}{2} \;\middle|\; u = \frac{t}{1-t} \right] = \frac{1}{2}\left(1 - \sqrt{\frac{1-5t}{1-t}}\right).$$

The product is not commutative, so we can also define $G = F * P$, for which we find:

$$d_G(t) = d_F(t)d_P(h_F(t)) = \frac{1}{1-t-t^2}\left[\frac{1}{1-u} \;\middle|\; u = \frac{1-\sqrt{1-4t}}{2} \right] = \frac{1-\sqrt{1-4t}}{2t(1-t-t^2)} =$$

$$= 1 + 2t + 5t^2 + 12t^3 + 31t^4 + 85t^5 + 248t^6 + \cdots;$$

and

$$h_G(t) = h_P(h_F(t)) = \left[\frac{u}{1-u} \;\middle|\; u = \frac{1-\sqrt{1-4t}}{2} \right] = \frac{1-2t-\sqrt{1-4t}}{2t} =$$

$$= t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + \cdots$$

another version of the Catalan numbers. The upper part of these two triangles are given in Table 2. The A- and Z-sequences of these triangles can be computed by means of the formulas in the previous section; however, we will now find them in a more direct way by using the corresponding sequences of P and F.

In general, we know that $h(t) = tA(h(t))$ and consequently $\overline{h}(t) = t/A(t)$, where $\overline{h}(t)$ represents the compositional inverse of $h(t)$.

Lemma 3.1 *We have*

$$\overline{h}_3(t) = \overline{h}_1(\overline{h}_2(t)).$$

Proof: Starting with the **definition** $h_3(t) = h_2(h_1(t))$ **we** successively get $\overline{h}_2(h_3(t)) = h_1(t)$ and $\overline{h}_1(\overline{h}_2(h_3(t))) = t$. If we now set $y = h_3(t)$ or $t = \overline{h}_3(y)$ and change variable, we immediately get our assertion. ∎

We can express $\overline{h}_3(t)$ in terms of A-sequences:

Theorem 3.2 *We have*

$$\overline{h}_3(y) = \frac{y}{A_2(y)} \Big/ A_1\left(\frac{y}{A_2(y)}\right).$$

Proof: We apply the basic relation between A-sequences and the compositional inverse of $h(t)$:

$$\overline{h}_3(y) = \overline{h}_1(\overline{h}_2(y)) = \overline{h}_1\left(\frac{y}{A_2(y)}\right) = \left[\frac{z}{A_1(z)} \;\middle|\; z = \frac{y}{A_2(y)} \right] = \frac{y}{A_2(y)} \Big/ A_1\left(\frac{y}{A_2(y)}\right)$$

as desired. ∎

We can now prove the formula for the A-sequence of the product of two Riordan arrays:

Theorem 3.3 *The A-sequence $A_3(t)$ is given by:*

$$A_3(t) = A_2(t)A_1\left(\frac{t}{A_2(t)}\right).$$

Proof: By the previous theorem:

$$\overline{h}_3(y) = \frac{y}{A_3(y)} = \frac{y}{A_2(y)} \Big/ A_1\left(\frac{y}{A_2(y)}\right).$$

An easy simplification yields:

$$A_3(y) = A_2(y)A_1\left(\frac{y}{A_2(y)}\right)$$

from which we obtain the desired result by changing the variable. ∎

Example 3.2 *A simple example is given by the Pascal triangle P, for which $A_P(t) = 1+t$; the A-sequence of P^2 is:*

$$A_{P^2}(t) = (1+t)\left[1+u \;\Big|\; u = \frac{t}{1+t}\right] = (1+t)\frac{1+2t}{1+t} = 1+2t.$$

For the Riordan array Q we have:

$$A_Q(t) = \frac{1}{1-t}\left[1+u \;\Big|\; u = t(1-t)\right] = \frac{1+t-t^2}{1-t} = 1+2t+t^2+t^3+t^4+t^5+\cdots.$$

Analogously, for the Riordan array G we find:

$$A_G(t) = (1+t)\left[\frac{1}{1-u} \;\Big|\; u = \frac{t}{1+t}\right] = (1+t)^2$$

and therefore $A_G = (1,2,1,0,0,\ldots)$. All these results can be easily checked.

A **deeper result** concerns the Z-function; here we use the basic relation (2.8), valid for every Riordan array:

Theorem 3.4 *The Z-sequence Z_3 of the product $D_1 * D_2$ is given by:*

$$Z_3(t) = \left(1 - \frac{t}{A_2(t)}Z_2(t)\right)Z_1\left(\frac{t}{A_2(t)}\right) + A_1\left(\frac{t}{A_2(t)}\right)Z_2(t).$$

Proof: For $d_3(t)$ we have by definition:

$$d_3(t) = d_1(t)d_2(h_1(t)) = \frac{\delta_1\delta_2}{1 - tZ_3(h_2(h_1(t)))}$$

and we can substitute to $d_1(t)$ and $d_2(t)$ their expressions in terms of $Z_1(t)$ and $Z_2(t)$:

$$\frac{\delta_1}{1 - tZ_1(h_1(t))} \cdot \frac{\delta_2}{1 - h_1(t)Z_2(h_2(h_1(t)))} = \frac{\delta_1\delta_2}{1 - tZ_3(h_2(h_1(t)))}.$$

By simplifying, this expression is equivalent to:

$$1 - tZ_3(h_2(h_1(t))) = (1 - tZ_1(h_1(t))) \cdot (1 - h_1(t)Z_2(h_2(h_1(t)))).$$

Let us now set $y = h_2(h_1(t))$ or $t = \overline{h}_1(\overline{h}_2(t))$ so that:

$$Z_1(h_1(t)) = Z_1(\overline{h}_2(h_2(h_1(t)))) = Z_1(\overline{h}_2(y));$$

in this way, we find

$$1 - \overline{h}_1(\overline{h}_2(y))Z_3(y) = (1 - \overline{h}_1(\overline{h}_2(y))Z_1(\overline{h}_2(y))) \cdot (1 - \overline{h}_2(y)Z_2(y))$$

$$1-\overline{h}_3(y)Z_3(y)=(1-\overline{h}_3(y)Z_1(\overline{h}_2(y)))\cdot(1-\overline{h}_2(y)Z_2(y)).$$

This expression only contains the compositional inverses of $h_1(t)$, $h_2(t)$ and $h_3(t)$, and we can transform them into $A_1(y)$, $A_2(y)$, $A_3(y)$:

$$Z_3(y)=\frac{1}{\overline{h}_3(y)}-\frac{1}{\overline{h}_3(y)}\left(1-\overline{h}_3(y)Z_1(\overline{h}_2(y))\right)\cdot\left(1-\overline{h}_2(y)Z_2(y)\right)$$

$$Z_3(y)=\frac{A_3(y)}{y}-\frac{A_3(y)}{y}\left(1-\frac{y}{A_3(y)}Z_1\left(\frac{y}{A_2(y)}\right)\right)\cdot\left(1-\frac{y}{A_2(y)}Z_2(y)\right).$$

After some simplifications and by using the result of the previous theorem, we eventually arrive to the formula in the theorem statement. ∎

Example 3.3 *Let us apply the previous theorem to the Riordan array Q:*

$$Z_Q(t)=(1-t(1-t)(1+t-t^2))+\left[1+u\;\middle|\;u=t(1-t)\right](1+t-t^2)=$$

$$=(1-t+2t^3-t^4)+(1+2t-t^2-2t^3+t^4=2+t-t^2,$$

which gives $Z_Q=(2,1,-1,0,0,\ldots)$. For the array G:

$$Z_G(t)=\left(1-\frac{t}{1+t}\right)\left[1+u-u^2\;\middle|\;u=\frac{t}{1+t}\right]+\left[\frac{1}{1-u}\;\middle|\;u=\frac{t}{1+t}\right]=$$

$$=1+t+\frac{1+3t+t^2}{(1+t)^3}=2+t-2t^2+5t^3-9t^4+14t^5-20t^6+27t^7-\cdots.$$

We can find a closed form for these coefficients:

$$[t^n]\frac{1+3t+t^2}{(1+t)^3}=\binom{-3}{n}+3\binom{-3}{n-1}+\binom{-3}{n-2}=$$

$$=(-1)^n\binom{n+2}{2}+3(-1)^{n-1}\binom{n+1}{2}+(-1)^{n-2}\binom{n}{2}=(-1)^{n+1}\frac{n^2+n-2}{2},$$

and consequently:

$$Z_{G,n}=\delta_{0,n}+\delta_{1,n}+(-1)^{n+1}\frac{n^2+n-2}{2},$$

where, in this context, $\delta_{n,k}$ denotes the Kronecker δ. Finally, the reader can verify that $Z_{P^2}=(2,0,0,\ldots)$.

4 The inverse of a Riordan array

Formula (1.3) allows us to find the inverse of any Riordan array given by the pair $D=(d(t),\,h(t))$, provided we are able to find the compositional inverse $\overline{h}(t)$ of $h(t)$.

Example 4.1 *For the Pascal triangle, if we set $y=t/(1-t)$, we find $\overline{h}(y)=y/(1+y)$; therefore we have:*

$$P^{-1}=\left(\frac{1}{1+t},\frac{t}{1+t}\right)$$

the generic element of which is $(-1)^{n-k}\binom{n}{k}$. In the left part of Table 3 we give the upper part of the array P^{-1}.

For the Riordan array F^{-1} we begin by setting $y=h_F(t)$, that is

$$y=\frac{1-\sqrt{1-4t}}{2}\qquad\text{or}\qquad\overline{h}(y)=t=y-y^2.$$

Besides:

$$\frac{1}{d(\overline{h}(t))}=1-(y-y^2)-(y-y^2)^2=1-y+2y^3-y^4$$

and we conclude:

$$F^{-1}=\left(1-t+2t^3-t^4,\,t-t^2\right).$$

The initial **part of this triangle** *is shown in Table 3.*

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	−1	1					
2	1	−2	1				
3	−1	3	−3	1			
4	1	−4	6	−4	1		
5	−1	5	−10	10	−5	1	
6	1	−6	15	−20	15	−6	1

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	−1	1					
2	0	−2	1				
3	2	1	−3	1			
4	−1	2	3	−4	1		
5	0	−3	1	6	−5	1	
6	0	1	−5	−2	10	−6	1

Table 3: The inverse triangles P^{-1} and F^{-1}

The A/Z characterization of the inverse of a Riordan array $D = (d(t),\, h(t))$ can be found using a pattern similar to the one used in the previous section. Let $A(t)$ and $Z(t)$ be the generating functions of the A- and Z-sequences of D and let us denote by $d^*(t)$, $h^*(t)$, $A^*(t)$ and $Z^*(t)$ the corresponding functions for the inverse D^{-1}. We immediately observe that, by Formula (1.3), $h^*(t) = \overline{h}(t)$. Now we have:

Theorem 4.1 *The A-sequence of the inverse Riordan array D^{-1} is:*

$$A^*(y) = \left[\frac{1}{A(t)} \;\middle|\; y = \frac{t}{A(t)}\right] = \left[\frac{y}{t} \;\middle|\; y = \frac{t}{A(t)}\right].$$

Proof: By Formula (2.6) we have $h^*(t) = tA^*(h^*(t))$ and, by the previous observation, $h^*(t) = \overline{h}(t) = t/A(t)$; therefore we find

$$A^*(\overline{h}(t)) = \frac{\overline{h}(t)}{t} = \frac{1}{A(t)}.$$

Now, the formula in the assertion follows by setting $y = \overline{h}(t)$. ∎

For the Z-sequence we have a bit more complex derivation:

Theorem 4.2 *The Z-sequence of the inverse Riordan array D^{-1} is:*

$$Z^*(y) = \left[\frac{\delta - d(y)}{\delta t} \;\middle|\; y = \overline{h}(t)\left(= \frac{t}{A(t)}\right)\right] = \left[\frac{-yZ(t)}{t(1 - yZ(t))} \;\middle|\; y = \frac{t}{A(t)}\right].$$

Proof: The generating function $d^*(t)$ has two definitions, one given by Formula (1.3), the other by Formula (2.8), that is:

$$d^*(t) = \frac{\delta^*}{1 - tZ^*(h^*(t))} = \frac{1}{d(\overline{h}(t))}$$

from which we get:

$$\delta^* d(\overline{h}(t)) = 1 - tZ^*(\overline{h}(t));$$

by substituting $\delta^* = 1/\delta$ we obtain the first formula in the assertion. If we now substitute $t = \overline{h}(t)$ in Formula (2.8), we get:

$$d(\overline{h}(t)) = \frac{\delta}{1 - \overline{h}(t)Z(t)}$$

and by substituting this expression in the first formula in the assertion, we immediately obtain the second formula of Z^* in terms of $A(t)$. ∎

Example 4.2 *The Pascal triangle is a very simple example; in fact we have:*

$$A_{P^{-1}}(y) = \left[\frac{1}{1+t} \;\middle|\; y = \frac{t}{1+t}\right] = 1 - y.$$

$$Z_{P^{-1}}(y) = \left[\frac{-y}{t(1-y)} \;\middle|\; y = \frac{t}{1+t}\right] = \frac{-y(1-y)}{y(1-y)} = -1;$$

these values can be checked by the left part of Table 3.

For what concerns the Fibonacci triangle, we find:

$$A_{F^{-1}}(y) = \left[1-t \;\middle|\; y = t - t^2\right] = \left[1-t \;\middle|\; t = \frac{1-\sqrt{1-4y}}{2}\right]$$

$$A_{F^{-1}}(t) = \frac{1+\sqrt{1-4t}}{2} = 1 - t - t^2 - 2t^3 - 5t^4 - 14t^5 - 42t^6 - \cdots$$

another version of the Catalan numbers. Besides we have:

$$Z_{F^{-1}}(y) = \left[\frac{-y(1+t-t^2)}{t(1-y(1+t-t^2))} \;\middle|\; y = t - t^2\right];$$

the computation is rather complex, but changing the variable we eventually find:

$$Z_{F^{-1}}(t) = -\frac{(1+t)(1+\sqrt{1-4t})}{2(1-t-t^2)} = -1 - t + 2t^3 + 9t^4 + 30t^5 + 95t^6 + \cdots.$$

If g is an element of a group G, then the smallest positive integer n such that $g^n = e$, the identity of the group, if it exists, is called the *order* of g. If there is no such integer, then g is said to have *infinite order*. **It is well-known (see [11]) that** if we restrict all entries of a proper Riordan array to be integers, then any element of finite order in the Riordan group must have order 1 or 2, and each element of order 2 generates a subgroup of order 2. Using Theorems 4.1 and 4.2, we can find the characterization of Riordan arrays of order 2:

Theorem 4.3 *The A- and Z-sequences of a Riordan array of order 2 satisfy:*

$$A(t) = \frac{t}{h(t)} \qquad \text{and} \qquad Z(t) = \frac{\delta - d(t)}{h(t)}.$$

Proof: If $(d(t), h(t))$ is a Riordan array of order 2, then $(d(t), h(t))^{-1} = (d(t), h(t))$. It follows from Theorem 4.1 that $A = A^*$ and

$$A(\overline{h}(t)) = A^*(\overline{h}(t)) = \frac{\overline{h}(t)}{t}.$$

Substituting $t = h(t)$ into the above equation yields $A(t) = t/h(t)$. Similarly, from the first formula in Theorem 4.2, we have:

$$Z(\overline{h}(t)) = Z^*(\overline{h}(t)) = \frac{\delta - d(\overline{h}(t))}{t};$$

by substituting $t = h(t)$ we obtain the expression for $Z(t)$. ∎

Example 4.3 *The Riordan array $(1, -t)$ has order 2 and is characterized by $A(t) = -1$ and $Z(t) = 0$. Another example is $(1/(1-t), t/(t-1))$, for which we have $A(t) = t - 1$ and $Z(t) = 1$.*

In [2], an element of order 2 in the Riordan group is called a Riordan involution. Some structures of a Riordan involution were presented in [2].

5 The construction of Riordan arrays

One of the most important motivation for considering the A/Z characterization of Riordan arrays is their construction. The classical definition $D = (d(t), h(t))$ implies that we must compute the series expansion of the column generating functions $d(t)h(t)^k$ which, although obvious (especially using any Computer Algebra System), requires a large amount of operations. If we wish to construct the first $n+1$ rows of D, we should create column 0 and then perform the following operations for every $0 < k \le m \le n$:

$$d_{m,k} = [t^m]d(t)h(t)^k = \sum_{j=0}^{m}[t^j]d(t)h(t)^{k-1}[t^{m-j}]h(t) = \sum_{j=0}^{m} d_{j,k-1}h_{m-j} = \sum_{j=k-1}^{m-1} d_{j,k-1}h_{m-j}.$$

A more formal approach is given by the following algorithm, performing the vertical construction of a Riordan array:

Proc VCRA ($d(t)$, $h(t)$, n);
expand $d(t) = d_0 + d_1t + d_2t^2 + \cdots + d_nt^n$;
expand $h(t) = h_1t + h_2t^2 + \cdots + h_nt^n$;
construct column 0: $d_{0,0} = d_0,\ d_{1,0} = d_1,\ \ldots,\ d_{n,0} = d_n$;
for k **from** 1 **to** n **do**
for m **from** k **to** n **do**
$d_{m,k} = \sum_{j=k-1}^{m-1} d_{j,k-1}h_{m-j}$ **end do end do end proc**.

If we think of the Pascal triangle, we realize that this construction is rather annoying. Instead, by using the A- and Z-sequences we obtain the traditional, straight-forward method, described by the following algorithm for the horizontal construction of a Riordan array:

Proc HCRA ($A(t)$, $Z(t)$, δ, n);
expand $A(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$;
expand $Z(t) = z_0 + z_1t + z_2t^2 + \cdots + z_nt^n$;
set: $d_{0,0} = \delta$;
for m **from** 1 **to** n **do**
$d_{m,0} = \sum_{j=0}^{m} z_jd_{m-1,j}$;
for k **from** 1 **to** m **do**
$d_{m,k} = \sum_{j=k-1}^{m} a_{j-k+1}d_{m-1,j}$ **end do end do end proc**.

For the Pascal triangle, the 0-th element of every row is 1, and every other element is computed by the A-sequence (1, 1, 0, 0, ...) corresponding to the basic recurrence of binomial coefficients. This was already noted in Example 2.1, and another simple case is discussed in Example 2.2.

In the Pascal triangle the Z-sequence is particularly simple, but this is not always the case. Sometimes it is more favorable another characterization of Riordan arrays:

Theorem 5.1 *A Riordan array $D = (d(t), h(t))$ is completely characterized by the function $d(t)$ and by its A-sequence.*

Proof: By Formula 2.6, the A-sequence determines the function $h(t)$. ∎

Both the theory and the constructions relative to this characterizations are rather obvious from what we have already shown, and so we leave them to the interested reader as a simple exercise.

An important point relative to the construction of a Riordan array, is our ability in finding the compositional inverse of a *delta series*, that is a formal power series $h(t)$, having $h(0) = 0$ and $h'(0) \neq 0$. In our examples, we set $y = h(t)$ and solved this equation in $t = t(y)$, but this is not always feasible. In the simple case that $h(t)$ is a polynomial of third degree, technical problems can arise, and if the degree is greater than 4, the solution (in terms of radicals) can become at all impossible. In these cases, we can use the Lagrange Inversion Formula (LIF) to find out the coefficients of $\overline{h}(t)$. In fact, the LIF gives:

$$\overline{h}_n = \frac{1}{n}[t^{n-1}]\left(\frac{t}{h(t)}\right)^n$$

and we can compute the coefficients of the A-sequence (or vice versa):

Theorem 5.2 *Let $D = (d(t), h(t))$ be a proper Riordan array; then the coefficients of its A-sequence are given by:*

$$A_n = \frac{1}{n}[t^{n-1}]\frac{h_*'(t)}{h_*(t)^n} \qquad \text{and conversely} \qquad h_n = \frac{1}{n}[t^{n-1}]A(t)^n,$$

where $h_*(t) = h(t)/t$.

Proof: The LIF can assume the following form:

$$[t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n$$

if $w = t\phi(w)$. By Formula (2.6) we can write:

$$A(y) = \left[\frac{h(t)}{t} \;\middle|\; y = th_*(t)\right] = \left[h_*(t) \;\middle|\; t = y\frac{1}{h_*(t)}\right].$$

By the previous formula we have:

$$A_n = \frac{1}{n}[t^{n-1}]h_*'(t)\left(\frac{1}{h_*(t)}\right)^n.$$

For the converse, if we set $w = h(t)$ in Formula (2.6), we obtain $w = tA(w)$ and again, by the previous form of the LIF with $F(y) = y$, we get:

$$w_n = h_n = \frac{1}{n}[t^{n-1}]A(t)^n$$

as desired. ∎

From this theorem, it follows $h_1 = a_0$, $h_2 = a_0a_1$, $h_3 = a_0^2a_2 + a_0a_1^2$, and so on.

6 The hitting-time subgroup

Peart and Woan consider in [8] monic Riordan arrays $D = (d(t), h(t))$ with the additional property:

$$d(t) = \frac{th'(t)}{h(t)}; \tag{6.9}$$

the set of all these arrays is denoted by $\mathcal{H}$ and we can prove the well-known fact:

Theorem 6.1 *The set $\mathcal{H}$ with the usual Riordan product (1.2) is a subgroup of $\mathcal{R}$.*

Proof: Let D and E be two Riordan arrays in $\mathcal{H}$; then we have:

$$D * E = \left(\frac{th'(t)}{h(t)}, h(t)\right) * \left(\frac{tk'(t)}{k(t)}, k(t)\right) =$$

$$= \left(\frac{th'(t)}{h(t)} \cdot \frac{h(t)k'(h(t))}{k(h(t))}, k(h(t))\right) = \left(\frac{tk'(h(t))h'(t)}{k(h(t))}, k(h(t))\right);$$

since $\frac{\mathrm{d}}{\mathrm{d}t}k(h(t)) = k'(h(t))h'(t)$, this product belongs to $\mathcal{H}$. When $h(t) = t$, we obviously have $d(t) = th'(t)/h(t) = t/t = 1$, which means that $I \in \mathcal{H}$. Finally, by formula (1.3), we consider:

$$d(\overline{h}(t)) = \left[\frac{yh'(y)}{h(y)} \;\middle|\; y = \overline{h}(t)\right] = \frac{\overline{h}(t)h'(\overline{h}(t))}{h(\overline{h}(t))} = \frac{\overline{h}(t)h'(\overline{h}(t))}{t};$$

by differentiating the relation $h(\overline{h}(t)) = t$ we get $h'(\overline{h}(t))\overline{h}'(t) = 1$ and therefore:

$$D^{-1} = \left(\frac{1}{d(\overline{h}(t))}, \overline{h}(t)\right) = \left(\frac{t\overline{h}'(t)}{\overline{h}(t)}, \overline{h}(t)\right)$$

showing that $D^{-1} \in \mathcal{H}$. This concludes the proof. ∎

Peart and Woan call $(\mathcal{H}, *)$ the *hitting-time* subgroup.

Example 6.1 *The Pascal triangle P belongs to the hitting-time subgroup; in fact $h(t) = t/(1-t)$ and:*

$$d(t) = \frac{th'(t)}{h(t)} = \frac{t}{(1-t)^2} \cdot \frac{1-t}{t} = \frac{1}{1-t}.$$

In the same way, we can verify that the Catalan triangle:

$$C = \left(\frac{1}{2}\left(1 + \frac{1}{\sqrt{1-4t}}\right), \frac{1-\sqrt{1-4t}}{2}\right)$$

and the Motzkin triangle:

$$M = \left(\frac{1}{\sqrt{1-2t-3t^2}}, \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)$$

belong to $\mathcal{H}$. The upper part of these triangles **are shown in** *Table 4.*

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	3	2	1					
3	10	6	3	1				
4	35	20	10	4	1			
5	126	70	35	15	5	1		
6	462	252	126	56	21	6	1	
7	1716	924	462	210	84	28	7	1

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	3	2	1					
3	7	6	3	1				
4	19	16	10	4	1			
5	51	45	30	15	5	1		
6	141	126	90	50	21	6	1	
7	393	357	266	161	77	28	7	1

Table 4: The Catalan and the Motzkin triangles

The Riordan array on the left of Table 4 counts vertices by outdegree in ordered (planar) trees while the Riordan array on the right is counting King walks down a chess board and that $1+t+3t^2+7t^3+19t^4+\cdots = 1/\sqrt{1-2t-3t^2}$, which was discovered by Euler.

Let us consider the Riordan arrays $D = (d(t), h(t))$ in $\mathcal{H}$, in which $h(t)$ has integer coefficients. Almost all the Riordan arrays studied in Combinatorics have this property, so they are rather frequent. If we call $\mathcal{H}^*$ this set, we have the following divisibility property:

Theorem 6.2 *Let $D = (d_{n,k})_{n,k\in\mathbb{N}}$ be any Riordan array in $\mathcal{H}^*$, then n divides $kd_{n,k}$.*

Proof: Let us consider the quantity $kd_{n,k}$:

$$kd_{n,k} = k[t^n]\frac{th'(t)}{h(t)} \cdot h(t)^k = k[t^{n-1}]h'(t)h(t)^{k-1} = [t^{n-1}]\frac{\mathrm{d}}{\mathrm{d}t}h(t)^k = n[t^n]h(t)^k$$

and by our hypothesis $[t^n]h(t)^k$ is an integer number. ∎

In particular, when n is a prime number, for $0 < k < n$, n divides $d_{n,k}$; this result can be checked on the two arrays in Table 4; other examples can be found in [8]. By using the A- and Z-sequences, we can give a new interesting characterization of the hitting-time subgroup:

Theorem 6.3 *A monic Riordan array $D = (d(t), h(t))$ belongs to the hitting-time subgroup if and only if $Z(t) = A'(t)$, where $A(t)$ and $Z(t)$ are the generating functions of its A- and Z-sequences.*

Proof: First of all, let us observe that by formula 2.6 $A(t) = t/\overline{h}(t)$ so that by differentiating:

$$A'(t) = \frac{\overline{h}(t) - t\overline{h}'(t)}{\overline{h}(t)^2}.$$

Let us suppose that $D \in \mathcal{H}$, so that $d(t) = th'(t)/h(t)$; by formula (2.8) we have:

$$Z(y) = \left[\frac{1}{t} - \frac{1}{td(t)} \;\middle|\; t = \overline{h}(y)\right] = \frac{1}{\overline{h}(y)} - \frac{1}{\overline{h}(y)d(\overline{h}(y))} = \frac{1}{\overline{h}(y)} - \frac{h(\overline{h}(y))}{\overline{h}(y)^2 h'(\overline{h}(y))} =$$

$$= \frac{1}{\overline{h}(y)} - \frac{yh'(y)}{\overline{h}(y)^2} = \frac{\overline{h}(t) - t\overline{h}'(t)}{\overline{h}(t)^2} = A'(y).$$

For the converse, we start with equation (2.8) (left), in the form valid when $Z(t) = A'(t)$:

$$d(t) = \frac{1}{1 - tA'(h(t))}$$

and get:

$$A'(h(t)) = \left[\frac{\overline{h}(y) - y\overline{h}'(y)}{\overline{h}(y)^2} \;\middle|\; y = h(t)\right] = \frac{\overline{h}(h(t)) - h(t)\overline{h}'(h(t))}{\overline{h}(h(t))^2} = \frac{t - h(t)/h'(t)}{t^2}.$$

We conclude:

$$d(t) = \frac{t}{t - t + h(t)/h'(t)} = \frac{th'(t)}{h(t)}$$

showing that $D \in \mathcal{H}$. ∎

Example 6.2 *As we have already seen, in the Pascal triangle we have* $A_P(t) = 1 + t$ *and* $Z_P(t) = 1$ *and therefore* $Z_P(t) = A'_P(t)$*, as expected. For the Catalan triangle* C *we first solve in* t *the equation (2.6):*

$$y = \frac{1 - \sqrt{1-4t}}{2} \qquad \text{that is} \qquad t = \overline{h}(y) = y - y^2$$

and this implies:

$$A_C(y) = \frac{y}{y-y^2} = \frac{1}{1-y}.$$

Now, we apply equation (2.8) to find the Z*-sequence:*

$$Z_C(y) = \frac{1}{\overline{h}(y)} - \frac{1}{\overline{h}(y)d(\overline{h}(y))} = \frac{1}{y-y^2} - \frac{1}{(y-y^2)(1-y)/(1-2y)} = \frac{y}{y(1-y)^2} = \frac{1}{(1-y)^2}$$

and this proves that $Z_C(t) = A'_C(t)$*. Finally, for the Motzkin triangle, we have* $\overline{h}(y) = y/(1+y+y^2)$*, and so* $A_M(t) = 1+t+t^2$*. By some computations, formula (2.8) gives* $Z_M(t) = 1 + 2t$*, and so the expected relation* $Z_M(t) = A'_M(t)$ *is verified.*

Example 6.3 *A vast literature exists on lattice walks; in particular, the following model has received much attention. We consider lattice paths starting from the origin and composed by three kinds of step:*

1. up steps *going from* (x, y) *to* $(x+1, y+1)$*;*
2. horizontal steps *going from* (x, y) *to* $(x+1, y)$*;*
3. down steps *going from* (x, y) *to* $(x+1, y-1)$*.*

These steps can be colored, *so that up steps can be of 'a' different colors, horizontal steps can be of 'b' different colors and down steps of 'c' different colors. We assume that* a *and* c *are different from 0, while* b *can also be 0. We wish to count the paths composed of* n *steps, arriving at the point* (n, k) *and never going below the* x*-axis; let us denote by* $d_{n,k}$ *this number. If* $k \neq 0$ *it is immediate to observe that:*

$$d_{n,k} = ad_{n-1,k-1} + bd_{n-1,k} + cd_{n-1,k+1};$$

in fact we can arrive to (n, k) *from* $(n-1, k-1)$ *with* a *up steps, or from* $(n-1, k)$ *with* b *horizontal steps, or from* $(n-1, k+1)$ *with* c *down steps. This implies that the infinite triangle* $D = (d_{n,k})_{n,k\in\mathbb{N}}$ *is a Riordan array with* A*-sequence* $(a, b, c, 0, 0, \ldots)$*. It will be completely defined by defining the behavior of the paths when they arrive at a point on the* x*-axis; in fact, the condition that the path cannot go below that axis imposes to adopt another rule for going on (this condition is called* privileged access *for a reason explained in [6], but not relevant here). If we choose to define* $Z(t) = A'(t) = b + 2ct$ *we obtain a Riordan array in the hitting-time subgroup. We simply observe that, when* $a = c$*, this condition is equivalent to imagine that the array continues on the left, with negative values of* k*, as happens to the array of trinomial coefficients (see [1]). The Motzkin triangle of the previous example is the case* $a = b = c = 1$*. The Pascal triangle corresponds to* $a = b = 1$ *and* $c = 0$*.*

By using the theory developed in this paper, it is easily shown that the Riordan array is:

$$D = \left(\frac{1}{\sqrt{1-2bt+(b^2-4ac)t^2}}, \frac{1 - bt - \sqrt{1-2bt+(b^2-4ac)t^2}}{2ct} \right)$$

and the upper part of the array is:

$n\backslash k$	0	1	2	3	4	5
0	1					
1	b	a				
2	b^2+2ac	$2ba$	a^2			
3	b^3+6abc	$3a(b^2+ca)$	$3a^2b$	a^3		
4	$b^4+12ab^2c+6a^2c^2$	$4ab(b^2+3ac)$	$2a^2(3b^2+2ac)$	$4a^3b$	a^4	
5	$b^5+20ab^3c+30a^2bc^2$	$5a(b^4+6ab^2c+2a^2c^2)$	$10a^2b(b^2+2ac)$	$5a^3(2b^2+ac)$	$5a^4b$	a^5

Acknowledgments. The authors would like to thank the referees and the editor for their valuable suggestions and help.

References

[1] E. Barcucci, R. Pinzani, and R. Sprugnoli. The Motzkin family. Pure Mathematics and Applications, 2:249–279, 1991.

[2] **G.-S. Cheon, H. Kim, L. W. Shapiro, Riordan group involutions. Linear Algebra Appl. 428 (2008), no. 4, 941–952.**

[3] L. Comtet. *Advanced Combinatorics*. Reidel, Dordrecht, 1972.

[4] S. Getu, L. W. Shapiro, W.-J. Woan, and L. Woodson. How to guess a generating function. SIAM J. Discrete Math. 5:497–499, 1992.

[5] T. X. He, L. C. Hsu, P. J.-S. Shiue. The Sheffer Group and the Riordan Group. Discrete Appl. Math., 155:1895–1909, 2007.

[6] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. On some alternative characterizations of Riordan arrays. Canadian J. Math., 49:301–320, 1997.

[7] A. Nkwanta. A Riordan matrix approach to unifying a selected class of combinatorial arrays. Congressus Numerantium, 160:33–45, 2003.

[8] P. Peart, and W.-J. Woan. A divisibility property for a subgroup of Riordan matrices. Discrete Appl. Math., 98:255–263, 2000.

[9] D. G. Rogers. Pascal triangles, Catalan numbers and renewal arrays. *Discrete Mathematics*, 22:301–310, 1978.

[10] L. W. Shapiro. A survey of the Riordan Group. Talk at a meeting of the American Mathematical Society, Richmond, Virginia, 1994.

[11] L. W. Shapiro. Some open questions about random walks, involutions, limiting distributions and generating functions. Advances in Appl. Math., 27:585–596, 2001

[12] L. W. Shapiro. Bijections and the Riordan group. Theoretical Computer Science, 307:403–413, 2003.

[13] L. W. Shapiro, S. Getu, W.-J. Woan, and L. Woodson. The Riordan group. *Discrete Applied Mathematics*, 34:229–239, 1991.

[14] R. Sprugnoli. Riordan arrays and combinatorial sums. *Discrete Mathematics*, 132:267–290, 1994.

[15] R. Sprugnoli. Riordan arrays and the Abel-Gould identity. *Discrete Mathematics*, 142:213–233, 1995.

[16] R. Sprugnoli. Mathematical Methods for the Analysis of Algorithms. Available at `www.dsi.unifi.it/~resp/Handbook.pdf`.

[17] X. Zhao, and T. Wang. Some identities related to reciprocal functions. Discrete Math., 265:323–335, 2003.