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Abstract

In this short paper, we give several new formulas for $\zeta(n)$ when n is an odd positive integer. The method is based on a recent proof, due to H. Tsumura, of Euler's classical result for even n. Our results illuminate the similarities between the even and odd cases, and may give some insight into why the odd case is much more difficult.

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1 Introduction

Let $\zeta(s)$ be the Riemann zeta function. In [1], Tsumura gave an elementary proof of Euler's well-known formula

$$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!}B_{2k} \tag{1.1}$$

where k is a positive integer and $\{B_n\}$ denotes the sequence of Bernoulli numbers. In this paper, we use Tsumura's method to develop an "Euler-type" formula for $\zeta(2k+1)$ analogous to (1.1) above.

2 Preliminaries

For $d > 0$ and $u \in [1, 1+d]$, we let

$$\frac{2e^t}{e^t + u} = \sum_{n=0}^{\infty} \phi_n(u) \frac{t^n}{n!}, \tag{2.1}$$

We observe that $\phi_n(1) = E_n(1)$, where $E_n(x)$ is the nth Euler polynomial. If n is a nonegative integer and $u > 1$, we have

$$\phi_n(u) = -2 \sum_{j=1}^{\infty} (-u)^{-j} j^n. \tag{2.2}$$

When n is a negative integer, we take (2.2) as our definition of $\phi_n(u)$. It is easily shown that $\phi_{-1}(1) = 2 \ln 2$, and that

$$\phi_{-m}(1) = -2(2^{1-m} - 1)\zeta(m) \tag{2.3}$$

whenever m is an integer greater than 1. Finally, we note that for $u \geq 1$,

$$\overline{\lim_{n \to \infty}} \left(\frac{|\phi_n(u)|}{n!} \right)^{1/n} \leq \frac{1}{\pi}, \tag{2.4}$$

and thus the series in Eq. (2.1) converges absolutely for $|t| < \pi$.

For any positive integer k, we have

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} \frac{(-u)^{-n} \sin(n\pi)}{n^{2k}} \\
&= \sum_{n=1}^{\infty} \frac{(-u)^{-n}}{n^{2k}} \sum_{j=0}^{\infty} (-1)^j \frac{(n\pi)^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^{j+1} u \pi^{2j+1}}{2(2j+1)!} \phi_{2j+1-2k}(u) \\
&= \sum_{j=0}^{k-1} \frac{(-1)^{j+1} u \pi^{2j+1}}{2(2j+1)!} \phi_{2j+1-2k}(u) + \sum_{j=k}^{\infty} \frac{(-1)^{j+1} u \pi^{2j+1}}{2(2j+1)!} \phi_{2j+1-2k}(u) \\
&= \sum_{j=0}^{k-1} \frac{(-1)^{j+1} u \pi^{2j+1}}{2(2j+1)!} \phi_{2j+1-2k}(u) + \sum_{m=0}^{\infty} \frac{(-1)^{m+k+1} u \pi^{2m+2k+1}}{2(2m+2k+1)!} \phi_{2m+1}(u).
\end{aligned}$$

In light of (2.4), we can now let $u \to 1^+$, obtaining

$$0 = \sum_{j=0}^{k-1} \frac{(-1)^{j+1}\pi^{2j+1}}{2(2j+1)!}\phi_{2j+1-2k}(1) + \sum_{m=0}^{\infty} \frac{(-1)^{k+1}\pi^{2k+1} f_m}{2(2m+2k+1)!}, \quad (2.5)$$

where $f_m = (-1)^m \pi^{2m} E_{2m+1}(1)$.

Setting $k = 1$ in (2.5) and recalling that $\phi_{-1}(1) = 2\ln 2$, we have the curious formula

$$\ln 2 = \frac{\pi^2}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m}}{(2m+3)!} E_{2m+1}(1). \quad (2.6)$$

3 Main Results

We can use (2.3) and (2.5) to deduce the following theorem, which gives $\zeta(2k+1)$ recursively in terms of $\ln 2, \zeta(3), \ldots, \zeta(2k-1)$:

Theorem 3.1 *For any positive integer k,*

$$(1-2^{-2k})\zeta(2k+1) = \sum_{j=1}^{k-1} \frac{(-1)^j \pi^{2j}}{(2j+1)!}\left(2^{2j-2k}-1\right)\zeta(2k-2j+1) - \frac{(-1)^k \pi^{2k}\ln 2}{(2k+1)!} + \frac{(-1)^k \pi^{2k+2}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} E_{2m+1}(1)}{(2m+2k+3)!}. \quad (3.1)$$

This result may be regarded as the analogue of equation (5) of [1], in the sense that the infinite series above reduces to a single term if the E_{2m+1} is replaced by an E_{2m}. This provides some perspective on the difficulty of evaluating $\zeta(2k+1)$ as opposed to $\zeta(2k)$.

When $k = 1, 2,$ and 3, Theorem 3.1 yields the respective formulas:

$$\zeta(3) = \frac{\pi^2}{9}\ln 4 - \frac{2\pi^4}{3}\sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} E_{2m+1}(1)}{(2m+5)!},$$

$$\zeta(5) = \frac{2\pi^2}{15}\zeta(3) - \frac{\pi^4}{225}\ln 4 + \frac{8\pi^6}{15}\sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} E_{2m+1}(1)}{(2m+7)!},$$

$$\zeta(7) = \frac{10\pi^2}{63}\zeta(5) - \frac{2\pi^4}{315}\zeta(3) + \frac{\pi^6}{19845}\ln 16 - \frac{32\pi^8}{63}\sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} E_{2m+1}(1)}{(2m+9)!}.$$

If desired, we may express the infinite series in these formulas in terms of the Bernoulli numbers, via the identities

$$E_{2m+1}(1) = -E_{2m+1}(0) = \frac{2\left(2^{2m+2}-1\right)}{2m+2} B_{2m+2}.$$

It is well-known that

$$|B_{2m+2}| < \frac{2(2m+2)!}{(2\pi)^{2m+2}\left(1-2^{-2m-1}\right)},$$

and this implies that the mth term of the series in (3.1) is $O(m^{-2k-2})$. Hence our result gives slightly faster convergence than the standard series for $\zeta(2k+1)$.

Building on Theorem 3.1, we now develop the following Euler-type formula for $\zeta(2k+1)$:

Theorem 3.2 *Let $f_m = \pi^{2m}(-1)^m E_{2m+1}(1)$. For any positive integer k,*

$$\zeta(2k+1) = \frac{(-1)^{k+1}\pi^{2k+2}}{(1-2^{-2k})} \sum_{m=0}^{\infty} \frac{P_k(m) f_m}{(2m+2k+3)!}, \tag{3.2}$$

where $P_k(m)$ is a polynomial in m with rational coefficients, having degree at most $2k$. For $k \geq 0$, we have the recurrence:

$$(-1)^{k+1} P_k(m) = \frac{1}{2(m+k+2)} \sum_{l=0}^{k-1} (-1)^l \binom{2m+2k+4}{2m+2l+3} P_l(m) - \frac{1}{2}. \tag{3.3}$$

Proof. For ease of notation, we set $Z(s) = \frac{1}{2}\phi_{-(2s+1)}(1)$, noting that $Z(0) = \ln 2$ and $Z(n) = (1-2^{-2n})\zeta(2n+1)$ for any positive integer n. We may rewrite (3.1) as

$$Z(k) = -\sum_{j=1}^{k} \frac{(-1)^j \pi^{2j}}{(2j+1)!} Z(k-j) + \frac{(-1)^k \pi^{2k+2}}{2} \sum_{m=0}^{\infty} \frac{f_m}{(2m+2k+3)!}. \tag{3.4}$$

If $Z(k) = \pi^{2k+2} \sum_{m=0}^{\infty} P_k(m) f_m / ((2m+2k+3)!)$, we see from (2.6) that $P_0(m) = 1/2$. For $k > 0$ we have

$$\begin{aligned}
Z(k) = &-\sum_{j=1}^{k}\frac{(-1)^j\pi^{2j}}{(2j+1)!}\pi^{2k-2j+2}\sum_{m=0}^{\infty}\frac{P_{k-j}(m)f_m}{(2m+2k-2j+3)!}\\
&\qquad\qquad + \frac{(-1)^k\pi^{2k+2}}{2}\sum_{m=0}^{\infty}\frac{f_m}{(2m+2k+3)!}\\
&= -\pi^{2k+2}\sum_{j=1}^{k}\frac{(-1)^j}{(2j+1)!}\sum_{m=0}^{\infty}\frac{P_{k-j}(m)f_m}{(2m+2k-2j+3)!}\\
&\qquad\qquad + \frac{(-1)^k}{2}\sum_{m=0}^{\infty}\frac{f_m}{(2m+2k+3)!}\\
&= -\pi^{2k+2}\sum_{l=0}^{k-1}\frac{(-1)^{k-l}}{(2k-2l+1)!}\sum_{m=0}^{\infty}\frac{P_l(m)f_m}{(2m+2l+3)!} + \sum_{m=0}^{\infty}\frac{f_m}{2(2m+2k+3)!}\\
&= (-1)^{k+1}\pi^{2k+2}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{k-1}\frac{(-1)^l P_l(m)f_m}{(2k-2l+1)!(2m+2l+3)!} - \frac{f_m}{2(2m+2k+3)!}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
(-1)^k P_k(m) &= \frac{1}{2} - \sum_{l=0}^{k-1}\frac{(-1)^l P_l(m)(2m+2k+3)!}{(2k-2l+1)!(2m+2l+3)!}\\
&= \frac{1}{2} - \frac{1}{2(m+k+2)}\sum_{l=0}^{k-1}(-1)^l P_l(m)\binom{2m+2k+4}{2m+2l+3}. \qquad (3.5)
\end{aligned}$$

■

Equation (3.3) can be written in the following closed form as a partition of unity:

$$\frac{1}{m+k+2}\sum_{\ell=0}^{k}(-1)^\ell P_\ell(m)\binom{2m+2k+4}{2m+2\ell+3} = 1.$$

The first few $P_k(m)$ are given by

$$\begin{aligned}
P_1(m) &= \frac{(m+1)(2m+7)}{6},\\
P_2(m) &= \frac{(m+1)(m+2)(28m^2+224m+465)}{180},\\
P_3(m) &= \frac{(m+1)(m+2)(m+3)(248m^3+3348m^2+15346m+24003)}{3780},
\end{aligned}$$

and we can use these to obtain formulas for $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$, respectively. The pattern suggested by the above formulas holds in general, and we have:

Theorem 3.3 *Let $P_k(m)$ be defined as in Theorem 3.2. For any integers k and n with $1 \le n \le k$, $P_k(-n) = 0$.*

Proof. We first establish

$$P_{n-1}(-n) = \frac{(-1)^{n-1}}{2} \tag{3.6}$$

for any positive integer n. From (3.3) we have

$$(-1)^{n+1} P_n(-n-1) = \frac{1}{2}\sum_{\ell=0}^{n-1}(-1)^\ell \binom{2}{2\ell-2n+1} P_\ell(-n-1) - \frac{1}{2} = -\frac{1}{2},$$

which implies (3.6).

We now proceed by induction on k. Assuming that $P_\ell(-n) = 0$ when $1 \le n \le \ell \le k$, we must show that $P_{k+1}(-n) = 0$ when $1 \le n \le k+1$. From (3.3) we have

$$\begin{aligned}
(-1)^{k+2} P_{k+1}(-n) &= \frac{1}{2(k-n+3)}\sum_{\ell=0}^{k}(-1)^\ell \binom{2k-2n+6}{2\ell-2n+3} P_\ell(-n) - \frac{1}{2} \\
&= \frac{1}{2(k-n+3)}\sum_{\ell=n-1}^{k}(-1)^\ell \binom{2k-2n+6}{2\ell-2n+3} P_\ell(-n) - \frac{1}{2}.
\end{aligned}$$

Since $P_\ell(-n) = 0$ if $n \le \ell \le k$, the right-hand side is

$$\frac{1}{2(k-n+3)}(-1)^{n-1}(2k-2n+6)P_{n-1}(-n) - \frac{1}{2} = 0. \tag{3.7}$$

■

In light of Theorem 3.3 and computational evidence, we propose the following conjecture.

Conjecture 1 *For any positive integer k, the polynomial $P_k(m)$ has simple roots $m = -1, -2, \ldots, -k$, and no other rational roots if $k \ge 2$.*

Although the Euler-type formulas from Theorem 3.2 are more compact, they converge very slowly as compared to Theorem 3.1. As a compromise, we give:

Theorem 3.4 *Let $f_m = (-1)^m \pi^{2m} E_{2m+1}(1)$. For any positive integer k,*

$$\zeta(2k+1) = \frac{(-1)^k \pi^{2k}}{1-2^{-2k}} \left[a_k \ln 2 - \pi^2 \sum_{m=0}^{\infty} \frac{Q_k(m) f_m}{(2m+2k+3)!} \right], \tag{3.8}$$

where a_k is a constant and $Q_k(m)$ is a polynomial in m with rational coefficients, having degree at most $2k-2$. Recursive formulas for a_k and $Q_k(m)$ are given by

$$a_k = -\sum_{\ell=1}^{k-1} \frac{a_\ell}{(2k-2\ell+1)!} - \frac{1}{(2k+1)!} \tag{3.9}$$

and

$$Q_k(m) = -\frac{1}{2(m+k+2)} \sum_{\ell=1}^{k-1} \binom{2m+2k+4}{2m+2\ell+3} Q_\ell(m) - \frac{1}{2}. \tag{3.10}$$

The proof of this result is similar to that of Theorem 3.2, and hence is omitted.

Eqs. (3.9) and (3.10) can be written in closed form as

$$-(2k+1)! \sum_{\ell=1}^{k} \frac{a_\ell}{(2k-2\ell+1)!} = 1$$

and

$$-\frac{1}{m+k+2} \sum_{\ell=1}^{k} Q_\ell(m) \binom{2m+2k+4}{2m+2\ell+3} = 1,$$

respectively. For $k = 1, 2,$ and 3, we obtain $a_1 = -1/6$, $a_2 = 7/360$, $a_3 = -31/15120$, $Q_1(m) = -1/2$, $Q_2(m) = (m+2)(2m+9)/6$, and $Q_3(m) = -(m+2)(m+3)(28m^2+280m+717)/180$. Hence,

$$\begin{aligned}
\zeta(3) &= \frac{\pi^2}{9} \ln 4 - \frac{2\pi^4}{3} \sum_{m=0}^{\infty} \frac{f_m}{(2m+5)!}, \\
\zeta(5) &= \frac{7\pi^4}{675} \ln 4 - \frac{8\pi^6}{45} \sum_{m=0}^{\infty} \frac{(m+2)(2m+9)}{(2m+7)!} f_m, \\
\zeta(7) &= \frac{62\pi^6}{59535} \ln 4 - \frac{16\pi^8}{2835} \sum_{m=0}^{\infty} \frac{(m+2)(m+3)(28m^2+280m+717)}{(2m+9)!} f_m.
\end{aligned}$$

Additionally, we have the following result on the roots of $Q_k(m)$:

Theorem 3.5 *Let k and n be integers. Then the polynomials $Q_k(m)$ satisfy $Q_n(-n-1) = -\frac{1}{2}$ for all $n \geq 1$ and $Q_k(-n) = 0$ for all $k \geq n \geq 2$.*

We may also make the following:

Conjecture 2 *For $k \geq 2$, the polynomial $Q_k(m)$ has simple roots $-2, -3, \ldots, -k$, and $Q_k(m)$ has no other rational roots if $k \geq 3$.*

References

[**1**] H. Tsumura, An elementary proof of Euler's formula for $\zeta(2m)$, *American Mathematical Monthly*, **111**(2004), 430-431.