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Abstract

In this short paper, we establish a family of rapidly converging series expansions for $\zeta(2n+1)$ by discretizing an integral representation given by D. Cvijović and J. Klinowski in [2]. The proofs are elementary, using basic properties of the Bernoulli polynomials.

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1 Introduction

In [1], Apéry proved the irrationality of $\zeta(3)$ by using a rapidly converging infinite series. No one has yet discovered a similar expression for $\zeta(2n+1)$ when $n \geq 2$, but there have appeared several recent papers (for example, [3], [4], and [5]) devoted to finding other types of series expansions for the zeta function at odd-integer arguments. The typical result is an exponentially convergent series, and thus insufficient to prove irrationality.

We establish a new series expansion of $\zeta(2n+1)$ by discretizing the integral representations given in [2]. Our results similiar in quality to the references cited previously, but our method has the advantage of being completely elementary. It may also be possible to use our methods in

evaluating other Dirichlet series, and we intend to return to this problem in the future.

Throughout this paper, $\log t$ denotes the natural logarithm of t, and $B_m(t)$ is the mth Bernoulli polynomial. We henceforth assume that n is a fixed positive integer, and note that implied constants may depend on n. Our main result is:

Theorem 1.1 *For any positive integer n, there is a rational number A_n such that*

$$\zeta(2n+1) = \alpha_n \left[A_n - \frac{2n+1}{2} \left(\log\frac{3}{2} + 2^{-2n} \log\frac{1}{2} \right) + Z_n \right], \tag{1.1}$$

where

$$\alpha_n = \frac{(-1)^{n+1} 2^{2n+2} \pi^{2n}}{(1-2^{-2n})(2n+1)!} \tag{1.2}$$

$$Z_n = \sum_{r=1}^{\infty} \left[\big(\zeta(2r) - 1\big)\big(2^{2r} - 1\big) \int_0^{1/2} B_{2n+1}(t) t^{2r-1} dt \right]. \tag{1.3}$$

Specifically, if Δ^k denotes the kth iterate of the ordinary forward difference operator, and

$$f_n(x) = \frac{1-(3/2)^{x+1}}{(x+1)^2} - \frac{1-2^{x+1}}{2^{2n}(x+1)^2}, \tag{1.4}$$

then

$$A_n = \int_{-1/2}^{1/2} \frac{B_{2n+1}(t)}{1-2t} dt + \frac{1}{2} \int_{1/2}^{3/2} \frac{B_{2n+1}(t)}{t} dt + n(n+1)\big[\Delta^{2n-1} f_n\big](0). \tag{1.5}$$

In light of this result, we may approximate $\zeta(2n+1)$ in terms of Z_n. We have the following result regarding the rate of convergence:

Theorem 1.2 *Let $Z_n(k)$ denote the kth partial sum of the series in* (1.3)*. Then $Z_n = Z_n(k) + O(k^{-1} 2^{-2k})$.*

2 Proofs of the Main Results

From the identity $\tan x = \cot x + 2\cot 2x$ and the well-known formula

$$\pi\cot(\pi z) = z^{-1} - 2\sum_{r=1}^{\infty}\zeta(2r)z^{2r-1}, \qquad z \notin \mathbb{Z}, \tag{2.1}$$

one readily obtains, for $0 < t < \frac{1}{2}$,

$$\tan(\pi t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(2^{2n}-1)\zeta(2n)t^{2n-1}, \tag{2.2}$$

and this remains valid when $t = 0$. From Section 4 of [1] , we have

$$\zeta(2n+1) = (-1)^{n+1}\frac{(2\pi)^{2n+1}}{(1-2^{-2n})(2n+1)!}\int_0^{1/2} B_{2n+1}(t)\tan(\pi t)dt, \tag{2.3}$$

and inserting (2.2) gives

$$\zeta(2n+1) = \alpha_n \lim_{\delta\to 1/2^-}\int_0^{\delta}\sum_{r=1}^{\infty}(2^{2r}-1)\zeta(2r)B_{2n+1}(t)t^{2r-1}dt. \tag{2.4}$$

Since $\delta < \frac{1}{2}$, the series converges uniformly on $0 \le t \le \delta$. Thus

$$\zeta(2n+1) = \alpha_n \lim_{\delta\to 1/2^-}\sum_{r=1}^{\infty}(2^{2r}-1)\zeta(2r)\int_0^{\delta} B_{2n+1}(t)t^{2r-1}dt. \tag{2.5}$$

Let $J_r(\delta)$ be the integral in (2.5), and write

$$\sum_{r=1}^{\infty}(2^{2r}-1)\zeta(2r)J_r(\delta) \tag{2.6}$$

$$= \sum_{r=1}^{\infty}2^{2r}J_r(\delta) - \sum_{r=1}^{\infty}J_r(\delta) + \sum_{r=1}^{\infty}\big[\zeta(2r)-1\big](2^{2r}-1)J_r(\delta) \tag{2.7}$$

$$= W_n(\delta) - X_n(\delta) + Y_n(\delta). \tag{2.8}$$

When $0 \leq \delta \leq \frac{1}{2}$, $J_r(\delta) = O(r^{-1}2^{-2r})$, and hence the series for X_n converges uniformly on this interval. The rth term of Y_n is dominated by $\zeta(2r) - 1$, hence this series converges uniformly on the same interval.

As we remarked previously, the infinite series $Z_n = Y_n(1/2)$ is the chief ingredient in our approximation of $\zeta(2n+1)$. The next two lemmas are devoted to evaluating W_n and X_n.

Lemma 2.1 *With W_n and X_n defined as above,*

$$\lim_{\delta \to 1/2^-} W_n(\delta) = \frac{2n+1}{2} \int_0^{1/2} B_{2n}(t) \log(1 - 4t^2) dt \tag{2.9}$$

$$X_n(1/2) = \frac{2n+1}{2} \int_0^{1/2} B_{2n}(t) \log(1 - t^2) dt. \tag{2.10}$$

Proof. Integrating $J_r(\delta)$ by parts, we have

$$W_n(\delta) = \frac{1}{2} B_{2n+1}(\delta) \sum_{r=1}^{\infty} \frac{(2\delta)^{2r}}{r} - \frac{2n+1}{2} \sum_{r=1}^{\infty} \int_0^{\delta} \frac{(2t)^{2r}}{r} B_{2n}(t) dt \tag{2.11}$$

$$= -\frac{1}{2} B_{2n+1}(\delta) \log(1 - 4\delta^2) - \frac{2n+1}{2} \sum_{r=1}^{\infty} \int_0^{\delta} \frac{(2t)^{2r}}{r} B_{2n}(t) dt. \tag{2.12}$$

The odd Bernoulli polynomials have $1/2$ as a root, hence the first term tends to zero as $\delta \to 1/2^-$. The series

$$\sum_{r=1}^{\infty} \frac{(2t)^{2r}}{r} B_{2n}(t)$$

converges uniformly when $0 \leq t \leq \delta < 1/2$. We thus have

$$\lim_{\delta \to 1/2^-} W_n(\delta) = -\frac{2n+1}{2} \lim_{\delta \to 1/2^-} \int_0^{\delta} B_{2n}(t) \sum_{r=1}^{\infty} \frac{(2t)^{2r}}{r} dt,$$

and this establishes (2.9). The proof of (2.10) is similar.

■

The domain of W_n can thus be extended so that W_n is left-continuous at $\frac{1}{2}$. With this in mind, we now simplify the above integrals.

Lemma 2.2 *There exist rational numbers* p_n *and* q_n *such that*

$$W_n(1/2) = \int_{-1/2}^{1/2} \frac{B_{2n+1}(t)}{1-2t} dt - p_n - \frac{2n+1}{2^{2n+1}} \log \frac{1}{2}. \tag{2.13}$$

$$X_n(1/2) = -\frac{1}{2} \int_{1/2}^{3/2} \frac{B_{2n+1}(t)}{t} dt - q_n + \frac{2n+1}{2} \log \frac{3}{2}. \tag{2.14}$$

Proof. From (2.9), we have

$$W_n(1/2) = \frac{2n+1}{2} \int_0^{1/2} B_{2n}(t) \log(1-2t) dt + \frac{2n+1}{2} \int_0^{1/2} B_{2n}(t) \log(1+2t) dt. \tag{2.15}$$

Again integrating by parts, the first term becomes

$$\int_0^{1/2} \frac{B_{2n+1}(t)}{1-2t} dt, \tag{2.16}$$

and the second term is

$$-\int_0^{1/2} \frac{B_{2n+1}(t)}{1+2t} dt = -\int_{-1/2}^{0} \frac{B_{2n+1}(-x)}{1-2x} dx \tag{2.17}$$

$$= \int_{-1/2}^{0} \frac{B_{2n+1}(x) + (2n+1)x^{2n}}{1-2x} dx. \tag{2.18}$$

Hence

$$W_n(1/2) = \int_{-1/2}^{1/2} \frac{B_{2n+1}(t)}{1-2t} dt + (2n+1) \int_{-1/2}^{0} \frac{x^{2n}}{1-2x} dx. \tag{2.19}$$

(2.13) follows upon letting $x = \frac{1}{2} - y$ and evaluating the resulting integral. We can write

$$p_n = \frac{n(2n+1)}{2^{2n}}\,\Delta^{2n-1}\frac{1-2^{x+1}}{(x+1)^2}\bigg|_{x=0}, \tag{2.20}$$

where Δ is the forward difference operator: $(\Delta f)(x) = f(x+1) - f(x)$. The proof of (2.14) is similar, with

$$q_n = n(2n+1)\,\Delta^{2n-1}\frac{1-(3/2)^{x+1}}{(x+1)^2}\bigg|_{x=0}. \tag{2.21}$$

■

Theorem 1.1 follows immediately upon inserting the results of this lemma, along with the given expressions for p_n and q_n, into equation (2.8).

Turning to Theorem 1.2, we have already seen that the integral appearing in (1.3) is $O(r^{-1}2^{-2r})$. To estimate $\zeta(2r) - 1$, we use the represenation

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}, \tag{2.22}$$

which is valid when $\Re(s) > 0$. The alternating series is $1 + O(2^{-2r})$, and from this we can deduce that $\zeta(2r) - 1 = O(2^{-2r})$. Hence

$$Z_n = Z_n(k) + O\Big(\sum_{r=k+1}^{\infty} r^{-1}2^{-2r}\Big) \tag{2.23}$$

and this establishes Theorem 1.2

References

[1] R. Apéry "Irrationalité de $\zeta(2)$ et $\zeta(3)$." *Astérisque* **41**(1979), 11-13.

[2] D. Cvijović and J. Klinowski "Integral representations of the Riemann zeta function for odd-integer arguments." *J. Comp. and Appl. Math.* **142**(2002), 435-439.

[3] D. Cvijović and J. Klinowski "New rapidly convergent series representations for $\zeta(2n+1)$." *Proc. Amer. Math. Soc.* **123**(1995), no. 5., 1263-1271.

[4] J. A. Ewell "On values of the Riemann zeta function at integral arguments." *Canad. Math. Bull.* **34**(1991), no. 1,. 60-66.

[5] H. M. Srivastava "Some rapidly converging series for $\zeta(2n+1)$." *Proc. Amer. Math. Soc.* **127** (1999), no. 2., 385-396.