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Symbolization of Generating Functions, An Application of Mullin-Rota's Theory of Binomial Enumeration

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Abstract

We have found that there are more than a dozen classical generating functions that could be suitably symbolized to yield various symbolic sum formulas by employing Mullin-Rota's theory of binomial enumeration. Various special formulas and identities involving well-known number sequences or polynomial sequences are presented as illustrative examples. The convergence of the symbolic summations is discussed.

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1 Introduction

It is known that the symbolic calculus with operators Δ (differencing), E (operation of displacement), and D (derivative) plays an important role in the Calculus of Finite Differences, which is often employed by statisticians and numerical analysts. Various well-known results can be found in some classical treatises, e.g., those by Jordan [1], Milne-Thomson [2], etc. Since all the symbolic expressions used and operated in the calculus could be formally expressed as power series in Δ (or D or E) over the real or complex number field, it is clear that the theoretical basis of the calculus may be found within the general theory of the formal power series. Worth reading is a sketch of the theory of formal series that has been given briefly in Comtet [3] (see §1.12, and § 3.2-§ 3.5) (cf. Bourbaki [4] Chap. 4-5).

This paper is a sequel to the authors with Torney paper [5], which can be considered as a special case of our results (see Remark 3.2). In this paper we shall show that a variety of formulas and identities containing famous number sequences, namely Bell, Bernoulli, Euler, Fibonacci, Genocchi, and Stirling, could be quickly derived by using a symbolic method with operators Δ, E, and D. The key idea is a suitable application of a certain symbolic substitution rule to the generating functions for those number sequences, so that a number of symbolic expressions could be obtained, which then can be used as stepping-stones to yielding particular formulas or identities of interest.

Frequently we shall get formulas or identities involving infinite series expansions. Certainly, any convergence problems, if involved in the results, should be treated separately.

2 A substitution rule and its scope of applications

As usual, we denote by C^∞ the class of real functions, infinitely differentiable in $\mathbb{R} = (-\infty, \infty)$. We will make frequent use of the operators Δ, E, and D which are known to be defined for all $f \in C^\infty$ via the relations

$$\Delta f(t) = f(t+1) - f(t), \quad Ef(t) = f(t+1), \quad Df(t) = \frac{d}{dt} f(t).$$

Consequently they satisfy some simple symbolic relations such as

$$E = 1 + \Delta, \quad E = e^D, \quad , \Delta = e^D - 1, \quad D = log(1 + \Delta), \tag{2.1}$$

where the unity 1 serves as an identity operator I such that $If(t) = f(t) = 1f(t)$, and e^D and $log(1 + D)$ are meaningful in the sense of formal power series expansions, namely

$$e^D = \sum_{k \geq 0} \frac{1}{k!} D^k, \qquad log(1 + \Delta) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \Delta^k$$

so that $e^D f(t) = \sum_{k \geq 0} D^k f(t)/k! = f(t+1) = Ef(t)$, (cf. Jordan [1]).

An operator T which commutes with the shift operator E is called a *shift-invariant operator* (see, for example, [6]), i.e.,

$$TE^\alpha = E^\alpha T,$$

where $E^\alpha f(t) = f(t + \alpha)$ and $E^1 \equiv E$. Clearly, the identity operator 1, the differentiation operator D, and the difference operator Δ are all shift-invariant operators. A shift-invariant operator Q is called a *delta operator* if Qt is a non-zero constant. Obviously, the identity operator, the differentiation operator, the difference operator, and the backward difference, the central difference, Laguerre, and Abel operators (*cf.* [6]) are all delta operators.

Note that there are two well-known operational formulas involving Stirling numbers of the first and second kinds, $S_1(m, n)$ and $S_2(m, n)$, namely the following

$$D^m f(t) = \sum_{n\geq m} \frac{m!}{n!} S_1(n,m)\Delta^n f(t) \quad (2.2)$$

$$\Delta^m f(t) = \sum_{n\geq m} \frac{m!}{n!} S_2(n,m) D^n f(t). \quad (2.3)$$

These could be derived using Newton interpolation series and Taylor series, respectively. (cf. [1] § 56 and § 67).

Certainly, according to (2.1), it is obvious that (2.2) and (2.3) may be viewed as direct consequences of the substitutions $t \to \Delta$ and $t \to D$ into the following generating functions, respectively

$$(log(1+t))^m = \sum_{n\geq m} \frac{m!}{n!} S_1(n,m) t^n$$

$$\left(e^t - 1\right)^m = \sum_{n\geq m} \frac{m!}{n!} S_2(n,m) t^n.$$

Note that certain particular identities could be deduced from (2.2) and (2.3) with particular choices of $f(t)$ (cf., for example, [1]).

The above description is an example of the following general substitution rule shown in Mullin and Rota [6] (see also in [7]).

Theorem 2.1 *[6] Let Q be a delta operator, and let F be the ring of formal power series in the variable t, over the same field, then there exists an isomorphism from F onto the ring $\sum$ of shift-invariant operators, which carries*

$$f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \text{ into } f(Q) \equiv G(t,Q) := \sum_{k\geq 0} \frac{a_k}{k!} Q^k.$$

From Theorem 2.1 we have the following general substitution rule for the formal power series expansion of the functions regarding e^t or $log(1+t)$.

Substitution Rule (w. r. t. operators D and Δ): Given a generating function or a formal power series expansion $F(t) = \sum_{k\geq 0} f_k t^k$, where $F(t)$ is expressed either in the form $G(t, e^t)$ or in the form $G(t, log(1+t))$, then a certain operational formula may be obtained as follows.

(i) For the case $F(t) = G(t, e^t)$, the substitution $t \to D$ leads to the symbolic formula

$$F(D) = G(D, 1 + \Delta) = \sum_{k\geq 0} f_n D^k. \tag{2.4}$$

(ii) For the case $F(t) = G(t, log(1 + t))$, $t \to \Delta$ leads to the formula

$$F(\Delta) = G(\Delta, D) = \sum_{k\geq 0} f_k \Delta^k. \tag{2.5}$$

Of course, (2.4) and (2.5) can be deduced from (2.1). In what follows we display a dozen generating functions for the sequences $\{W_k\}$ (Bell numbers), $\{B_k^{(n)}\}$ and $\{B_k^{(-n)}\}$ (generalized Bernoulli numbers of the orders n and $-n$), $\{E_k(t)\}$ (Euler polynomials), $\{e_k = E_k(0)\}$ (Euler numbers), $\{\alpha_k(t)\}$ (Eulerian fractions) and $\{G_k\}$ (Genocchi numbers), $\{\phi_k(t)\}$ (Bernoulli polynomials of the first kind), $\{\psi(t)\}$ (Bernoulli polynomials of the second kind), $\{b_k\}$ (Bernoulli numbers of the second kind), respectively.

(G_1) $exp(e^t - 1) = \sum_{k\geq 0} \frac{1}{k!} W_k t^k$ (Comtet [3], p. 210)

(G_2) $\left(\frac{t}{e^t-1}\right)^n = \sum_{k\geq 0} \frac{1}{k!} B_k^{(n)} t^k$ (David-Barton [8], p. 287)

(G_3) $\left(\frac{e^t-1}{t}\right)^n = \sum_{k\geq 0} \frac{1}{k!} B_k^{(-n)} t^k$ (*cf.* [8], p. 287)

(G_4) $\frac{te^{xt}}{e^t-1} = \sum_{k\geq 0} \phi_k(x) t^k$, where $\phi_k(0) = B_k^{(1)}/k!$, (Jordan [1], p. 250)

(G_5) $\frac{2e^{xt}}{e^t+1} = \sum_{k\geq 0} E_k(x) t^k$ (Jordan [1], p. 309)

(G_6) $\frac{2}{e^t+1} = \sum_{k\geq 0} e_k t^k$, where $e_k = E_k(0)$, (*cf.* [1], p. 309)

(G_7) $\frac{1}{1-xe^t} = \sum_{k\geq 0} \frac{1}{k!} \alpha_k(x) t^k$ (Wang-Hsu [9], p.24)

(G_8) $\frac{2t}{e^t+1} = \sum_{k\geq 0} \frac{1}{k!} G_k t^k$ (Comtet [3], p. 49)

(G_9) $\frac{t(1+t)^x}{\log(1+t)} = \sum_{k\geq 0} \psi_k(x) t^k$ (Jordan [1], p. 279)

(G_{10}) $\frac{t}{\log(1+t)} = \sum_{k\geq 0} b_k t^k$, where $b_k = \psi_k(0)$, (*cf.* [1], p. 279)

(G_{11}) $\frac{1}{1-\Delta-\Delta^2} = \sum_{k\geq 0} F_k \Delta^k$, where $F_0 = F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k = 2, 3, \ldots$.

(G_{12}) $\frac{1}{(1-t)e^t} = \sum_{k\ge 0} \frac{\phi(0)}{k!} t^k.$

In comparison of (G_8) with (G_6), we see that Genocchi numbers G_{k+1} are equivalent to $(k+1)!e_k$ $(k = 0, 1, 2, \ldots)$, with e_k being Euler numbers. It is also known that $G_{2m+1} = 0$ and $G_{2m} = 2\left(1 - 2^{2m}\right) B_{2m}$, where $B_{2m} \equiv B_{2m}^{(1)}$ are Bernoulli numbers given by generating function (G_2) with $n = 1$. Surely, all the generating functions shown above could be found in comprehensive books on the Calculus of Finite Differences, in particular, e.g., in Jordan [1] (§78, §85, §95-96, §109). For (G_7), see [9].

Clearly, the substitution rule is applicable to each of the generating functions (G_1)-(G_{12}), so that a dozen operational formulas could be obtained. This will be shown in the next section (§3).

3 Various symbolic operational formulas

Let us apply the substitution rule to each left-hand side (LHS) of (G_1)-(G_{12}). We easily find

$LHS(G_1)$: $exp(e^D - 1) = exp\Delta = \sum_{k\ge 0} \frac{1}{k!}\Delta^k$

$LHS(G_2)$: $\left(\frac{D}{e^D-1}\right)^n = \frac{D^n}{\Delta^n}$

$LHS(G_3)$: $\left(\frac{e^D-1}{D}\right)^n = \frac{\Delta^n}{D^n}$

$LHS(G_4)$: $\frac{D(e^D)^x}{e^D-1} = \frac{DE^x}{\Delta}$

$LHS(G_5)$: $\frac{2(e^D)^x}{e^D+1} = \frac{2E^x}{2+\Delta} = E^x \sum_{k\ge 0}(-1)^k \left(\frac{\Delta}{2}\right)^k$

$LHS(G_6)$: $\frac{2}{e^D+1} = \frac{2}{2+\Delta} = \sum_{k\ge 0}(-1)^k \left(\frac{\Delta}{2}\right)^k$

$LHS(G_7)$: $\frac{1}{1-xe^D} = \frac{1}{1-xE} = \sum_{k\ge 0} x^k E^k$

$LHS(G_8)$: $\frac{2D}{e^D+1} = \frac{2D}{2+\Delta} = D\sum_{k\ge 0}(-1)^k \left(\frac{\Delta}{2}\right)^k$

$LHS(G_9)$: $\frac{\Delta(1+\Delta)^x}{\log(1+\Delta)} = \frac{\Delta E^x}{D}$

$LHS(G_{10})$: $\frac{\Delta}{\log(1+\Delta)} = \frac{\Delta}{D}$

$LHS(G_{11})$: $\frac{1}{1-\Delta-\Delta^2} = \frac{1}{1-\Delta(\Delta+1)} = \frac{1}{1-\Delta E} = \sum_{k\ge 0} \Delta^k E^k$

$LHS(G_{12})$: $\frac{1}{(1-D)E} = E^{-1}\sum_{k\ge 0} D^k.$

Thus, by pairing each $LHS(G_i)$ with $RHS(G_i)$ $(i = 1, 2, \ldots, 12)$, we can obtain formally a dozen operational formulas for $f(t) \in C^\infty$ evaluated at $t = a$ or at $t = y$, namely

(O_1) $\sum_{k\ge 0} \frac{1}{k!}\Delta^k f(a) = \sum_{k\ge 0} \frac{W_k}{k!} D^k f(a)$

(O_2) $D^n f(a) = \sum_{k\geq 0} \frac{B_k^{(n)}}{k!} \Delta^n D^k f(a)$

(O_3) $\Delta^n f(a) = \sum_{k\geq 0} \frac{B_k^{(-n)}}{k!} D^{n+k} f(a)$

(O_4) $Df(x+y) = \sum_{k\geq 0} \phi_k(x) D^k[f(y+1) - f(y)]$

(O_5) $\sum_{k\geq 0} \left(-\frac{1}{2}\right)^k \Delta^k f(x) = \sum_{k\geq 0} E_k(x) D^k f(0)$

(O_6) $\sum_{k\geq 0} \left(-\frac{1}{2}\right)^k \Delta^k f(a) = \sum_{k\geq 0} e_k D^k f(a)$

(O_7) $\sum_{k\geq 0} f(a+k)x^k = \sum_{k\geq 0} \frac{\alpha_k(x)}{k!} D^k f(a)$

(O_8) $\sum_{k\geq 0} \left(-\frac{1}{2}\right)^k \Delta^k f(a) = \sum_{k\geq 0} \frac{G_k}{k!} D^{k-1} f(a)$

(O_9) $\Delta f(x+y) = \sum_{k\geq 0} \psi_k(x) \Delta^k D f(y)$

(O_{10}) $\Delta f(a) = \sum_{k\geq 0} b_k \Delta^k D f(a)$

(O_{11}) $\sum_{k\geq 0} \Delta^k f(k) = \sum_{k\geq 0} F_k \Delta^k f(0)$

(O_{12}) $\sum_{k\geq 0} D^k f(a-1) = \sum_{k\geq 0} \frac{\phi(0)}{k!} D^k f(a)$.

Certainly, all the series expansions shown above involve convergence problems for given functions, some of which will be considered in the next section (§4). Note that (O_4) and (O_9) are well known, and their equivalent forms with applications have been fully expounded in Jordan [1]. (O_7) appears to be an unfamiliar formula, whose finite form with certain estimable remainders has been used as a summation formula for power series (cf. [9]).

As may be predicted, a considerable variety of particular identities containing some famous number sequences (or polynomial sequences) could be obtained from the formulas (O_1)-(O_{12}) with special choices of the functions $f(t) \in C^\infty$. This will be partially justified with selective examples in the last section (§5).

Remark 3.1 For other delta operators such as the backward difference, the central difference, Laguerre, Bernoulli and Abel operators, we can construct many other symbolic sum formulas similarly, which will be shown in a later work.

Remark 3.2 We may construct symbolic sum formulas from some identities such as

$$\frac{1}{1-x(1+t)}=\frac{1}{1-x}\frac{1}{1-\frac{x}{1-x}t}.$$

The series expansion of the above identity can be written formally as

$$\sum_{k\geq 0}x^k(1+t)^k=\sum_{k\geq 0}\frac{x^k}{(1-x)^{k+1}}t^k.$$

Hence, using the substitution rule for $t\to\Delta$ and noting $1+\Delta=E$ yields formally the following sum formula

$$\sum_{k\geq 0}x^kf(k)=\sum_{k\geq 0}\frac{x^k}{(1-x)^{k+1}}\Delta^kf(0),$$

which is the generalized Euler's transformation series. The series was developed in [5], and its convergence conditions were established in [10] by authors.

4 Some theorems on Convergence

First, let us introduce a definition as follows:

Definition 4.1 *Let $\{f_k(t)\}$ and $\{g_k(t)\}$ be two sequences of functions. The commutator of $\{f_k(t)\}$ and $\{g_k(t)\}$ is defined as*

$$[f,g](x,y)\equiv\left[\{f_k\},\{g_k\}\right](x,y):=\sum_{k\geq 0}\left[f_k(x)g_k(y)-f_k(y)g_k(x)\right]. \quad (4.1)$$

If $[f,g]\equiv 0$, i.e., two sequences of functions $\{f_k(t)\}$ and $\{g_k(t)\}$ satisfy the formal equation/equality

$$\sum_{k\geq 0}f_x(x)g_k(y)=\sum_{k\geq 0}f_k(y)g_k(x), \quad (4.2)$$

we say that $\{f_k(t)\}$ and $\{g_k(t)\}$ have a symmetrical product summation property, or briefly a SPS-property.

From the definition, we immediately have $[f,g](x,y)=-[f,g](y,x)$ or $[f,g](x,y)+[f,g](y,x)=0$. Denote the Fourier transform of a function $h(t)$ as $\hat{h}(\xi)$, if it exists. If each function in sequences $\{f_k(t)\}$

and $\{g_k(t)\}$ has the Fourier transform (e.g., $f_k, g_k \in L_1,\ k \geq 0$), then $\widehat{[f,g]}(\xi,\eta) = [\hat{f},\hat{g}](\xi,\eta)$. Thus, $[f,g] = 0$ iff $[\hat{f},\hat{g}] = 0$; i.e., $\{f_k(t)\}$ and $\{g_k(t)\}$ have a SPS-property iff $\{\hat{f}(\xi)\}$ and $\{\hat{g}(\xi)\}$ have a SPS-property.

Rota's binomial-type functions (polynomials) are those characterized by the equation

$$f_n(x+y) = \sum_{k\geq 0} \binom{n}{k} f_k(x) f_{n-k}(y),$$

which may be rewritten as

$$\frac{1}{n!} f_n(x+y) = \sum_{k\geq 0} \frac{f_k(x)}{k!} \frac{f_{n-k}(y)}{(n-k)!}.$$

Thus, for fixed $n \geq 1$, the pair of sequences $\langle f_k(t)/k!, f_{n-k}(t)/(n-k)! \rangle$ has the SPS-property. Moreover, we have the following

Theorem 4.2 *The three pairs* $\left\langle \phi_k(t), \Delta D^k f(t) \right\rangle$, $\left\langle E_k(t), D^k f(t) \right\rangle$, *and* $\left\langle \psi_k(t), \Delta^k D f(t) \right\rangle$ *all have the SPS-property for* $f \in C^\infty$.

Proof. According to (O_4), (O_9), and (O_5) we have respectively

$$Df(x+y) = \sum_{k\geq 0} \phi_k(x) \Delta D^k f(y) \tag{4.3}$$

$$\Delta f(x+y) = \sum_{k\geq 0} \psi_k(x) D \Delta^k f(y) \tag{4.4}$$

$$\sum_{k\geq 0} \left(-\frac{1}{2}\right)^k \Delta^k f(x+y) = \sum_{k\geq 0} E_k(x) D^k f(y). \tag{4.5}$$

As the LHS's remain the same when x and y are interchanged, we see that the theorem is true.

In what follows we will establish convergence conditions for the series expansions of (O_4), (O_5), (O_6), (O_8), (O_9), and (O_{11}). Convergence problems for other series expansions will be left to the interested reader for consideration. A general technique yielded from the convergence theorems on sum formulas (O_4)-(O_6), (O_8), (O_9), and (O_{11}) will be described in the Remark 4.2 at the end of the section.

Theorem 4.3 *For given $f \in C^\infty$ and $x, y \in \underline{R}$, the absolute convergence of the series expansion (4.3) is ensured by the condition*

$$\overline{\lim_{k\to\infty}} \left|\Delta D^k f(y)\right|^{1/k} < 1. \tag{4.6}$$

Proof. The root test confirms that the LHS of (4.3) will be absolutely convergent provided that

$$\overline{\lim_{k\to\infty}} \left|\phi_k(x)\Delta D^k f(y)\right|^{1/k} < 1. \tag{4.7}$$

We shall show that

$$\overline{\lim_{k\to\infty}} \left|\phi(x)\right|^{1/k} \le 1. \tag{4.8}$$

so that, (4.6) plus (4.8) will imply (4.7).

Recall that the Bernoulli polynomial may be written in the form (*cf.* Jordan [1], §78-§82)

$$\phi_k(x) = \sum_{j=0}^{k} \frac{B_j}{(k-j)!j!} x^{k-j} = \sum_{j=0}^{k} \frac{x^{k-j}}{(k-j)!}\alpha_j,$$

where $\alpha_j = B_j/j! = B_j^{(1)}/j!$, and B_j are ordinary Bernoulli numbers. Note that $\alpha_0 = 1$, $\alpha_1 = -1/2$, $\alpha_{2m+1} = 0$ $(m \in bN)$ and (*cf.* [1] p. 245)

$$|\alpha_{2m}| \le \frac{1}{12(2\pi)^{2m-2}}, \ (m = 0, 1, 2, \ldots).$$

It follows that

$$\begin{aligned} |\phi_k(x)| \ &\le \ \frac{|x|^k}{k!} + \frac{|x|^{k-1}}{2(k-1)!} + \sum_{j=2}^{k} \left(\frac{1}{12(2\pi)^{j-2}}\right) \frac{|x^{k-j}|}{(k-j)!} \\ &< \ \sum_{j=0}^{k} \frac{|x|^j}{j!} \le \sum_{r=0}^{\infty} \frac{|x|^r}{r!} = e^{|x|}, \ (k \ge 2). \end{aligned}$$

Consequently we get $|\phi_k(x)|^{1/k} < exp(|x|/k) \to 1$ as $k \to \infty$, and the assertion (4.8) is proved.

Theorem 4.4 *The absolute convergence of the series expansion (4.4) is ensured by the condition*

$$\overline{\lim_{k\to\infty}} \left|\Delta^k D f(y)\right|^{1/k} < 1. \tag{4.9}$$

Proof. Given condition (4.9). Using the root test again, we have to show that

$$\overline{\lim_{k\to\infty}} \left|\psi_k(x)\Delta^k Df(y)\right|^{1/k} < 1. \tag{4.10}$$

For this it suffices to prove that

$$\overline{\lim_{k\to\infty}} \left|\psi_k(x)\right|^{1/k} \leq 1. \tag{4.11}$$

Recall that there is an integral representation of $\psi_k(x)$, namely (*cf.* [1], p. 268)

$$\psi_k(x) = \int_0^1 \binom{x+t}{k} dt. \tag{4.12}$$

For $t \in [0,1]$ and for large k we have the order estimation

$$\left|\binom{x+t}{k}\right| = \frac{|(x+t)_k|}{k!} = \frac{|(k-x-t-1)_k|}{k!} = o\left(\frac{k+[|x|])_k}{k!}\right) = o\left(k^{[|x|]}\right).$$

This means that there is a constant $M > 0$ such that

$$\max_{0\leq t\leq 1}\left|\binom{x+t}{k}\right| < Mk^{[|x|]}.$$

Thus it follows that

$$\overline{\lim_{k\to\infty}} \left|\psi_k(x)\right|^{1/k} \leq \overline{\lim_{k\to\infty}} \left(\int_0^1 \left|\binom{x+t}{k}\right| dt\right)^{1/k} \leq \overline{\lim_{k\to\infty}} \left(Mk^{[|x|]}\right)^{1/k} = 1.$$

This is a verification of (4.11), and Theorem 4.4 is proved.

We need the following lemmas for discussing the convergence of the series in (4.5), (O_6), and (O_8).

Lemma 4.5 *Let $f \in C^\infty$. Then $\overline{\lim}_{k\to\infty} \left|D^k f(y)\right|^{1/k} < a$, a positive real number, implies*

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(y)\right|^{1/k} < e^a - 1.$$

Proof. Assume $\overline{\lim}_{n\to\infty} |D^n f(y)|^{1/n} < a$. Denote $\overline{\lim}_{n\to\infty} |D^n f(y)|^{1/n} = \theta$. Then there exists a number γ such that $\theta < \gamma < a$. Thus for large enough n we have $|D^n f(y)|^{1/n} < \gamma$ or $|D^n f(y)| < \gamma^n$.

From (2.3), noting $S_2(n,m) \geq 0$ and $|D^n f(y)| < \gamma^n$ yields

$$\begin{aligned} \left|\Delta^k f(y)\right| &= \left|\sum_{n\geq k} \frac{k!}{n!} S_2(n,k) D^n f(y)\right| \leq \sum_{n\geq k} \frac{k!}{n!} S_2(n,k) \left|D^n f(y)\right| \\ &\leq \sum_{n\geq k} \frac{k!}{n!} S_2(n,k)\gamma^n = (e^\gamma - 1)^k < (e^a - 1)^k. \end{aligned}$$

Here the rightmost equality is from Jordan [1] (see p. 176). This proves the lemma.

Lemma 4.6 *Let $f \in C^\infty$. If $\overline{\lim}_{k\to\infty} \left|D^k f(y)\right|^{1/k} < a$ for some y, then for any fixed t we have*

$$\overline{\lim_{k\to\infty}} \left|D^k f(t)\right|^{1/k} < a.$$

Proof. Denote $\overline{\lim}_{k\to\infty} \left|D^k f(y)\right|^{1/k} = \theta$. Then there exists a number γ such that $\theta < \gamma < a$. Thus, for large enough k $\left|D^k f(y)\right| < \gamma^k$. Denote $x = t - y$. Hence, for large k

$$\begin{aligned} &\left|D^k f(t)\right| = \left|D^k f(y+x)\right| = \left|D^k E^x f(y)\right| = \left|D^k e^{xD} f(y)\right| \\ =& \left|\sum_{j=0}^{\infty} \frac{x^j}{j!} D^{k+j} f(y)\right| \leq \sum_{j=0}^{\infty} \frac{|x|^j}{j!} \left|D^{k+j} f(y)\right| \\ \leq& \sum_{j=0}^{\infty} \frac{|x|^j}{j!} \gamma^{k+j} = \gamma^k e^{|x|\gamma}, \end{aligned}$$

which implies

$$\left|D^k f(t)\right|^{1/k} \leq \gamma e^{|x|\gamma/k}, \ (x = t - y).$$

For given t we choose large k such that

$$k > \frac{|x|\gamma}{\log a - \log \gamma} = \frac{|t-y|\gamma}{\log a - \log \gamma}.$$

Thus,

$$\left|D^k f(t)\right|^{1/k} \le \gamma e^{|x|\gamma/k} < \gamma\frac{a}{\gamma} = a.$$

This completes the proof of the lemma.

Theorem 4.7 *The absolute convergence of the series expansions involved in (4.5) is ensured by the condition*

$$\overline{\lim_{k\to\infty}}\left|D^k f(y)\right|^{1/k} < 1. \tag{4.13}$$

Proof. By using Lemmas 4.5 and 4.6, from condition (4.13) we have $\overline{\lim}_{k\to\infty}\left|D^k f(x+y)\right|^{1/k} < 1$ and

$$\left|\Delta^k f(y+x)\right|^{1/k} < e-1 < 2.$$

That the above inequality implies the absolute convergence of the series on the LHS of (4.5) is obvious in view of the root test for convergence.

Given (4.13), the absolute convergence of the series on the RHS of (4.5) is implied by

$$\overline{\lim_{k\to\infty}}|E_k(x)|^{1/k} \le 1. \tag{4.14}$$

Let us now verify (4.14). Note that Euler polynomial $E_k(x)$ may be written in the form

$$E_k(x) = \sum_{j=0}^{k} e_j \frac{x^{k-j}}{(k-j)!},\ (e_0 = 1), \tag{4.15}$$

where $e_j = E_j(0)$, $e_{2m} = 0$ $(m = 1, 2, \ldots)$, and e_{2m-1} satisfies the inequality (*cf.* [1], p. 302)

$$|e_{2m-1}| < \frac{2}{3\pi^{2m-2}} < 1\ (m = 1, 2, \ldots). \tag{4.16}$$

Thus we have the estimation

$$|E_k(x)| \le \frac{|x|^k}{k!} + \sum_{j=1}^{k}|e_j|\frac{|x|^{k-j}}{(k-j)!} \le \frac{|x|^k}{k!} + \sum_{j=1}^{k}\frac{|x|^{k-j}}{(k-j)!} < e^{|x|}.$$

Consequently we get

$$\overline{\lim_{k\to\infty}} |E_k(x)|^{1/k} \le \lim_{k\to\infty} \left(e^{|x|}\right)^{1/k} = 1.$$

Hence (4.14) is verified.

Remark 4.1 From the LHS of (4.5), we recognize the condition

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(x+y)\right|^{1/k} < 2 \tag{4.17}$$

implies the absolute convergence of the series. In the proof of Theorem 4.7, this condition is derived from condition (4.13). Hence, the reader may propose the question: Are conditions (4.13) and (4.17) equivalent? Namely, does (4.17) also imply (4.13)? The following example shows that the answer is negative.

Consider $f(t) = 2.8^t$. Let both x and y be zero. Then,

$$\left|\Delta^k f(x+y)\right|^{1/k} = \left|\Delta^k f(0)\right|^{1/k} = \left[(2.8-1)^k\right]^{1/k} < 2.$$

However,

$$\left|D^k f(y)\right|^{1/k} = \left|D^k f(0)\right|^{1/k} = \left[(\log(2.8))^k\right]^{1/k} > 1.$$

Similar to Theorem 4.7, we obtain the following convergence results for (O_6) and (O_8).

Theorem 4.8 *The absolute convergence of the series expansions involved in (O_6) and (O_8) is ensured by the condition*

$$\overline{\lim_{k\to\infty}} \left|D^k f(a)\right|^{1/k} < 1.$$

Proof. From the given condition, by using Lemma 4.5, we have

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(a)\right|^{1/k} < e - 1 < 2.$$

Hence, the series on the LHS of (O_6) and (O_8) are absolutely convergent.

Note that $e_k = E_k(0)$ satisfies $|e_k| < e^0 = 1$ and $G_{k+1}/(k+1)! = e_k$, we immediately obtain the convergence of the series on the RHS of (O_6) and (O_8) from the given condition.

Let us consider the operational formula (O_{11}):

$$\sum_{k\geq 0} F_k \Delta^k f(0) = \sum_{k\geq 0} \Delta^k f(k), \tag{4.18}$$

where F_k has Binet expression $F_k = (\alpha^{k+1} - \beta^{k+1})/\sqrt{5}$ with $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, so that $\alpha + \beta = 1$ and $\alpha|\beta| = 1$.

Theorem 4.9 *The following condition*

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(0)\right|^{1/k} < |\beta| = \frac{\sqrt{5}-1}{2} \tag{4.19}$$

ensures the absolute convergence of the series on both sides of (4.18).

Proof. Clearly we have (*cf.* Wilf [11] Eq. (1.3.4))

$$\overline{\lim_{k\to\infty}} (F_k)^{1/k} = \alpha = \frac{\sqrt{5}+1}{2}.$$

Thus condition (4.19) implies that

$$\overline{\lim_{k\to\infty}} \left|F_k \Delta^k f(0)\right|^{1/k} < \alpha|\beta| = 1.$$

so that, the series on the LHS of (4.18) is absolutely convergent.

Rewrite (4.19) in the form

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(0)\right|^{1/k} = \theta < |\beta|.$$

Choose a number γ such that $\theta < \gamma < |\beta|$. Then for k large enough we have $\left|\Delta^k f(0)\right|^{1/k} < \gamma$, i.e., $\left|\Delta^k f(0)\right| < \gamma^k$. Consequently

$$\begin{aligned}
&\left|\Delta^k f(k)\right|^{1/k} = \left|\Delta^k E^k f(0)\right|^{1/k} = \left|\Delta^k (1+\Delta)^k f(0)\right|^{1/k} \\
= \ & \left|\sum_{j=0}^{k} \binom{k}{j} \Delta^{k+j} f(0)\right|^{1/k} \leq \left(\sum_{j=0}^{k} \binom{k}{j} \left|\Delta^{k+j} f(0)\right|\right)^{1/k} \\
< \ & \left|\sum_{j=0}^{k} \binom{k}{j} \Delta^{k+j} f(0)\right|^{1/k} \leq \left(\sum_{j=0}^{k} \binom{k}{j} \gamma^{k+j}\right)^{1/k} \\
= \ & \left(\gamma^k (1+\gamma)^k\right)^{1/k} = \gamma(1+\gamma) < |\beta|\alpha = 1.
\end{aligned}$$

It follows that

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(k)\right|^{1/k} \le \gamma(1+\gamma) < 1.$$

This shows that the series on the RHS of (4.18) is also absolutely convergent.

Remark 4.2 We may sort sum formulas (O_1)-(O_{12}) into two classes. The first class includes only either the sum $\sum \gamma_k D^k f$ or the sum $\sum \eta_k E^k f$ in the formulas such as (O_2)-(O_4), (O_9), and (O_{10}). The second class includes the sums $\sum \gamma_k D^k f$ and/or $\sum \eta_k E^k f$ on both sides such as (O_1), (O_5)-(O_8), (O_{11}), and (O_{12}). Similar to Theorems 4.3, 4.4 and 4.9, we may establish the convergence condition, $\overline{\lim}_{k\to\infty} \left|D^k f\right|^{1/k} < 1$ (or $\overline{\lim}_{k\to\infty} \left|E^k f\right|^{1/k} < 1$), for the first class series expansions if we can determine $|\gamma_k| \le 1$ (or $|\eta_k| \le 1$). We may also establish the convergence condition, $\overline{\lim}_{k\to\infty} \left|D^k f\right|^{1/k} < 1$, for the second series expansions similar to Theorems 4.7 and 4.8 by using Lemmas 4.5 and 4.6, if there exist $|\gamma_k| \le 1$ and $|\eta_k| \le (1/2)^k$.

5 Examples

Surely, the list of operational formulas (O_1)-(O_{11}) may provide a fruitful source of particular identities relating to some famous number sequences and polynomials just involved in those formulas. In this section we will present a number of particular identities or formulas as examples in which $f(x)$'s are taken to be simple elementary functions.

First, let us mention several elementary functions with simpler differences and derivatives, as a preparation for constructing various examples

(i) For $f(x) = x^m$ $(m \ge 1)$ we have (with $k \le m$)

$$\begin{aligned} \Delta^k f(0) &= \left[\Delta^k x^m\right]_{x=0} = k! S_2(m,k), \\ \left[D^k x^m\right]_{x=0} &= \left[(m)_k x^{m-k}\right]_{x=0} = \delta_{m,k} m!, \end{aligned}$$

where $S_2(m,k)$ is the Stirling number of the second kind (*cf.* [1], p. 168), and $\delta_{m,k}$ is the Kronecker symbol with $\delta_{m,k} = 1$ for $m = k$ and zero for $m \ne k$.

(ii) For $f(x) = \binom{x}{m}$ and $m \ge k \ge 0$ we have

$$\Delta^k f(0) = \binom{x}{m-k}_{x=0} = \binom{0}{m-k} = \delta_{m,k},$$
$$D^k f(0) = \left[D^k \frac{(x)_m}{m!}\right]_{x=0} = \frac{k!}{m!} S_1(m,k).$$

(iii) For $f(x) = a^x$ $(a > 1)$, we have

$$\Delta^k a^x = (a-1)^k a^x, \quad D^k a^x = (\log a)^k a^x,$$
$$\left[\Delta^k a^x\right]_{x=0} = (a-1)^k, \quad \left[D^k a^x\right]_{x=0} = (\log a)^k.$$

(iv) For $f(x) = \frac{1}{1+x}$, we have

$$\Delta^k f(x) = \frac{(-1)^k k!}{(x+k+1)_{k+1}}, \quad D^k f(x) = \frac{(-1)^k k!}{(1+x)^{k+1}},$$
$$\Delta^k f(0) = \frac{(-1)^k}{k+1}, \quad D^k f(0) = (-1)^k k!.$$

(v) For $f(x) = e^{ix}$ and $g(x) = e^{-ix}$ $(i^2 = -1)$, we have

$$\Delta^k e^{\pm ix} = e^{\pm ix}(e^{\pm i} - 1)^k, \quad D^k e^{\pm ix} = (\pm i)^k e^{\pm ix},$$
$$\left[\Delta^k e^{\pm ix}\right]_{x=0} = (e^{\pm i} - 1)^k, \quad \left[D^k e^{\pm ix}\right]_{x=0} = (\pm i)^k.$$

(vi) For $f(x) = \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ we have

$$\left[\Delta^k \cos x\right]_{x=0} = \frac{(e^i - 1)^k + (e^{-i} - 1)^k}{2} = \frac{(1 + (-1)^k e^{-ik})(e^i - 1)^k}{2},$$
$$\left[D^k \cos x\right]_{x=0} = \frac{i^k + (-i)^k}{2} = i^k \left(\frac{1 + (-1)^k}{2}\right) = i^k \delta_k,$$

where δ_k is the parity function, viz., $\delta_k = 0$ if k is an odd integer, and $\delta_k = 1$ if k is an even integer.

An immediate generalization of (O_1) is the following

$(O_1)^*$ $\sum_{k \ge 0} \frac{x^k}{k!} \Delta^k f(a) = \sum_{k \ge 0} \tau_k(x) D^k f(a),$

where $\tau_k(x)$ are known as Touchard polynomials generated by the expansion (cf. Hsu-Shiue [12], p. 186)

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} \tau_k(x)t^k.$$

Clearly, $(O_1)^*$ is obtained via the substitution $t \to D$, namely

$$e^{x\Delta} f(a) = \sum_{k\geq 0} \frac{x^k \Delta^k}{k!} f(a) = \sum_{k\geq 0} \tau_k(x) D^k f(a).$$

Example 5.1 Taking $f(t) = t^m$ $(m \geq 1)$, we see that the LHS of $(O_1)^*$ with $a = 0$ yields

$$\sum_{k\geq 0} \frac{x^k}{k!} \left[\Delta^k t^m\right]_{t=0} = \sum_{k=0}^{m} x^k S_2(m,k).$$

The RHS of $(O_1)^*$ gives

$$\sum_{k\geq 0} \tau_k(x) \left[D^k t^m\right]_{t=0} = m!\tau_m(x).$$

Hence we get an identity of the form

$$\sum_{k=0}^{m} S_2(m,k)x^k = m!\tau_m(x). \tag{5.1}$$

Usually the LHS of (5.1) is called “exponential polynomial” for Stirling numbers. Thus (5.1) shows that the exponential polynomial is precisely given by Touchard polynomial. It can be checked that the Touchard polynomial sequence is a binomial type polynomial sequence and has the SPS-property.

Example 5.2 In view of the generating function of $\{\tau_k(x)\}$ we see that Bell numbers W_k are given by $W_k = k!\tau_k(1)$. Thus (5.1) with implies the well-known relation

$$\sum_{k=0}^{m} S_2(m,k) = W_m. \tag{5.2}$$

Example 5.3 Taking $f(t) = (t)_m$ with $m \geq 1$, we find

$$\Delta^k (t)_m\Big|_{t=0} = m!\Delta^k \binom{t}{m}\Big|_{t=0} = m!\delta_{m,k},$$
$$D^k (t)_m\Big|_{t=0} = k!S_1(m,k),$$

where $S_1(m,k)$ is the Stirling number of 1st kind (*cf.* Jordan [1], p. 142). Using $(O_1)^*$ with $x=1$ and $a=0$ we see that its LHS and RHS are respectively $\sum_{k\geq 0} \frac{1}{k!} m!\delta_{mk} = 1$ and $\sum_{k\geq 0} \frac{W_k}{k!} k!S_1(m,k)$. Thus we obtain an elegant identity of the form

$$\sum_{k=0}^{m} W_k S_1(m,k) = 1.$$

Substituting (5.1) with $x=1$ into the above identity yields identity (*cf.* [1], p. 183)

$$\sum_{k=0}^{m}\sum_{j=0}^{k} S_1(m,k)S_2(k,j) = 1.$$

Example 5.4 Taking $f(t) = x_m(t)$, the mth degree Bernoulli polynomial of the 2nd kind, and recalling that (cf. [1], §89-§97)

$$x_m(t) = \int_0^1 \binom{t+x}{m} dx = \sum_{j=0}^{m} b_j \binom{t}{m-j},$$

where $b_j = x_j(0) = \int_0^1 \binom{x}{j} dx$, we have

$$\Delta^k x_m(t) = x_{m-k}(t),\ (0 \leq k \leq m),$$
$$\Delta^k x_m(0) = x_{m-k}(0) = b_{m-k},\ b_0 = 1.$$

Thus using (O_1) we get a relation involving three kinds of special numbers as follows

$$\sum_{k=1}^{m} \frac{1}{k} W_k S_1(m-1,k-1) = (m-1)! \sum_{k=1}^{m} \frac{1}{k!} b_{m-k}, \tag{5.3}$$

where b_j are known as Bernoulli numbers of the 2nd kind, defined by (G_{10}). Note that there is a known formula, namely (*cf.* [1])

$$\Delta^n = \sum_{k=0}^{\infty} n! S_2(n+k,n) \frac{D^{n+k}}{(n+k)!}.$$

Comparing this equation with (O_3) we get the well-known relation between Bernoulli numbers and Stirling numbers, viz.

$$B_k^{(-n)} = \binom{n+k}{k}^{-1} S_2(n+k,n).$$

This implies that (O_3) is another form for the expression of Δ^n. Formula (O_2) appears to be not so familiar. Of course the case $n=1$ is well-known, and it leads to the classical Euler-Maclaurin summation formula.

Remark 5.1 $x_m(t)$ can be written as $x_m(t) = J\binom{x}{m}$ symbolically, where J is the Bernoulli operator: $J : p(t) \mapsto \int_t^{t+1} p(x)dx$.

Example 5.5 Let $f(t) = t^m$ $(m > n \geq 1)$ and $a = 0$. Then

$$\Delta^n D^k f(0) = (m)_k \Delta^n t^{m-k}\big|_{t=0} = (m)_k n! S_2(m-k,n).$$

We find the RHS of (O_2) is $\sum_{k=0}^{m-n} \binom{m}{k} B_k^{(n)} n! S_2(m-k,n)$. That is

$$\sum_{k=0}^{m-n} \binom{m}{k} B_k^{(n)} n! S_2(m-k,n) = D^n t^m\big|_{t=0} = 0. \tag{5.4}$$

In particular, the case $n=1$ gives the well-known recurrence relation for Bernoulli numbers, viz.

$$\sum_{k=0}^{m-1} \binom{m}{k} B_k = (1+B)^m - B_m = 0, \tag{5.5}$$

where $(1+B)^m$ is written in the sense of umbrel calculus, in which B^i must be substituted for B_i.

Example 5.6 Take $f(t) = 2^t$, we find $D^k f(t) = 2^t(\log 2)^k$, $\Delta^k f(t) = 2^t$. Thus, formula (O_2) gives

$$D^n f(t) = 2^t(\log 2)^n = \sum_{k=0}^{\infty} \frac{B_k^{(n)}}{k!} 2^t (\log 2)^k.$$

That is

$$\sum_{k=0}^{\infty} \frac{B_k^{(n)}}{k!} \frac{(\log 2)^k}{k!} = (\log 2)^n. \tag{5.6}$$

The reader may find various examples in Jordan [1] for the equivalence of (O_4) and (O_9). Here we supplement some other examples.

Example 5.7 A case of (O_4) is the following

$$\sum_{k\geq 0} \phi_k(x) D^k[f(1) - f(0)] = Df(x).$$

Taking $f(t) = t^m$ $(m \geq 1)$ we get

$$\sum_{k=0}^{m-1} (m)_k \phi_k(x) = mx^{m-1}. \tag{5.7}$$

Example 5.8 For $f(t) = t^m$ $(m \geq 1)$ formula (O_9) with $y = 0$ yields

$$\sum_{k\geq 0} \psi_k(x) m \left[\Delta^k y^{m-1}\right]_{y=0} = \sum_{k=0}^{m-1} \psi_k(x) m \cdot k! S_2(m-1, k) = (x+1)^m - x^m.$$

This leads to the formula

$$\sum_{k=0}^{m-1} k! \psi_k(x) S_2(m-1, k) = \frac{(x+1)^m - x^m}{m}. \tag{5.8}$$

Example 5.9 Let $f(t) = \binom{t}{m}$ $(m \geq 1)$. Then (*cf.* [1], p. 64)

$$D\Delta^k f(y) = D_y \binom{y}{m-k}, \quad \Delta f(x+y) = \binom{x+y}{m-1}.$$

Thus it follows from (O_9), we obtain the closed formula

$$\sum_{k=0}^{m} \psi_k(x) \frac{d}{dy} \binom{y}{m-k} = \binom{x+y}{m-1}. \tag{5.9}$$

Example 5.10 Replacing $f(x)$ by $f(x+y)$ in (O_5), we have

$$\sum_{k\geq 0} E_k(x) D^k f(y) = \sum_{k\geq 0} \left(-\frac{1}{2}\right)^k \Delta^k f(x+y).$$

Taking $f(y)=\binom{y}{m}$ $(m\geq 1)$, we find $D^k f(y)\big|_{y=0}=\frac{k!}{m!}S_1(m,k)$ and

$$\sum_{k=0}^{m} E_k(x)\frac{k!}{m!}S_1(m,k)=\sum_{k=0}^{m}\left(-\frac{1}{2}\right)^k\binom{x}{m-k}. \tag{5.10}$$

Putting $x=m$ and $x=0$ in (5.10) respectively, we easily obtain

$$\sum_{k=0}^{m}\frac{k!}{m!}E_k(m)S_1(m,k)=\left(\frac{1}{2}\right)^m \tag{5.11}$$

and

$$\sum_{k=0}^{m} k!e_k S_1(m,k)=(-1)^m\frac{m!}{2^m}, \tag{5.12}$$

where $e_k=E_k(0)$ are Euler numbers. Evidently defined by (G_6), (5.12) may also be derived from (O_6) by setting $a=0$. Most likely, identities (5.8) - (5.12) may be new, or not easily found in classical literature.

Example 5.11 As mentioned before (see the last part of Section 2), a comparison of (G_8) with (G_6) leads to the relation $G_{k+1}=(k+1)!e_k$. Thus the equality (5.13) may also be written in terms of Genocchi numbers, viz.

$$\sum_{k=1}^{m+1}\frac{G_k}{k}S_1(m,k-1)=(-1)^m\frac{m!}{2^m}. \tag{5.13}$$

Example 5.12 In (O_7) taking $f(t)=t^m$ $(m\geq 1)$ and $a=0$, we get

$$\sum_{k\geq 0} k^m x^k=\sum_{k\geq 0}\frac{\alpha_k(x)}{k!}m!\delta_{mk}=\alpha_m(x). \tag{5.14}$$

This is the classical formula of Euler for the arithmetic-geometric series.

Example 5.13 Obviously (O_7) can be written in the form

$$\sum_{k=a}^{\infty} f(k)x^k=\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{k!}x^a f^{(k)}(a).$$

Thus for any given non-negative integers $a<b-1$, where b is a real number, we have

$$\sum_{k=a}^{b-1} f(k)x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!}\left[x^a f^{(k)}(a) - x^b f^{(k)}(b)\right]. \tag{5.15}$$

The partial sum of the *RHS* of (5.15) with a remainder can be used as a summation formula for the *LHS*. This problem has been treated much in details by Wang-Hsu's paper [9].
Example 5.14 Recall that (5.14) may be rewritten in the form (cf. Comtet [3], p. 243-5)

$$\sum_{k=0}^{\infty} k^m x^k = \alpha_m(x) = \frac{A_m(x)}{(1-x)^{m+1}},\ (|x| < 1), \tag{5.16}$$

where $A_m(x)$ is the mth degree Eulerian polynomial given by the expression

$$A_0(x) = 1 \quad \text{and} \quad A_m(x) = \sum_{k=1}^{m} A(m,k)x^k,\ (m \geq 1),$$

with $A(m, 0) = 0$ and

$$A(m,k) = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} (k-j)^m,\ (1 \leq k \leq m),$$

$A(m, k)$ is known to be the Eulerian numbers (not Euler numbers).

Now, (5.16) can be symbolized in this way: Letting x be substituted by $E = 1 + \Delta$, we have $(x-1)^{m+1} \to \Delta^{m+1}$. Thus (5.16) leads to the symbolic formula

$$(O_{13}): \ \sum_{k=0}^{\infty} k^m \Delta^{m+1} f(k) = (-1)^{m+1} A_m(E) f(0).$$

This summation formula can be used to compute the series of the form as shown on the *LHS* of (O_{13}). Thus, for instance, taking $f(t) = 1/(1+t)$, we find

$$\Delta^{m+1} f(k) = (-1)^{m+1} \frac{(m+1)!}{(m+k+2)_{m+2}} = \frac{(-1)^{m+1}}{m+2} \binom{m+k+2}{m+2}^{-1}.$$

Consequently, using formula (O_{13}) we obtain

$$\frac{1}{m+2}\sum_{k=0}^{\infty} k^m \Big/ \binom{m+k+2}{m+2} = \sum_{k=1}^{m} A(m,k)/(k+1). \tag{5.17}$$

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