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Parametric Catalan Numbers and Catalan Triangles

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Abstract

Here presented a generalization of Catalan numbers and Catalan triangles associated with two parameters based on the sequence characterization of Bell-type Riordan arrays. Among the generalized Catalan numbers, a class of large generalized Catalan numbers and a class of small generalized Catalan numbers are defined, which can be considered as an extension of large Schröder numbers and small Schröder numbers, respectively. Using the characterization sequences of Bell-type Riordan arrays, some properties and expressions including the Taylor expansions of generalized Catalan numbers are given. A few combinatorial interpretations of the generalized Catalan numbers are also provided. Finally, a generalized Motzkin numbers and Motzkin triangles are defined similarly. An interrelationship among parametrical Catalan triangle, Pascal triangle, and Motzkin triangle is presented based on the sequence characterization of Bell-type Riordan arrays.

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1 Introduction

Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$, form a sequence of integers that occur in the solutions of many counting problems. The book Enumerative Combinatorics: Volume 2 [27] by Stanley contains a set of exercises of Chapter 6 which describe 66 different interpretations of the Catalan numbers. A complementary materials of the exercises of chapter 6 are collected in [28]. The small and large Schröder numbers are defined by $\{1, 1, 3, 11, 45, 197, \ldots\}$ and $\{1, 2, 6, 22, 90, 394, \ldots\}$, respectively. A survey regarding those numbers can be found in [29] by Stanley. Like Catalan numbers, Schröder numbers occur in various counting problems, often involving recursively defined objects, such as dissections of a convex polygon, certain polyominoes, various lattice paths, Lukasiewicz words, permutations avoiding given patterns, and, in particular, plane trees (see, for example, [9, 20]). Our intention is to consider the Catalan numbers and Schröder numbers not only as sequences, but also as belonging to infinite (lower) triangles, called the Riordan arrays, in which they form the first columns. Then, we use the sequence characterization of the arrays to extend the Catalan numbers, Schröder numbers, and Motzkin numbers in a more general setting associate with parameters and to present some expressions and properties of those numbers.

Riordan arrays are infinite, lower triangular matrices defined by the generating functions of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [24]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli[25, 26], on subgroups of the Riordan group in Peart and Woan [18] and Shapiro [21], on some characterizations of Riordan matrices in Rogers [19], Merlini et al. [16], and He et al. [15], and on many interesting related results in Cheon et al. [2, 3], Gould et al. [10], He [12, 11], He et al. [14, 13], Nkwanta [17], Shapiro [22, 23], and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ $(f_k \in \mathbb{R})$, is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $\bar{f}(t)$ exists such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines

the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k.

It is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t)) * (d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{2}$$

The Riordan array $I = (1,\, t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it is easily seen that I acts as an identity for this product, that is, $(1,\, t) * (d(t),\, h(t)) = (d(t),\, h(t)) * (1, t) = (d(t),\, h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))},\, \bar{h}(t)\right) \tag{3}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays is a group.

Several subgroups of $\mathcal{R}$ are important: (1) the set $\mathcal{A}$ of *Appell arrays*, that is the set of Riordan arrays $D = (d(t),\, h(t))$ for which $h(t) = t$; it is an invariant subgroup and is isomorphic to the group of f.p.s.'s of order 0, with the usual product as group operation; (2) the set $\mathcal{L}$ of *Lagrange arrays*, that is the set of Riordan arrays $D = (d(t),\, h(t))$ for which $d(t) = 1$; it is also called the *associated subgroup*; it is isomorphic with the group of f.p.s.'s of order 1, with composition as group operation. In particular, a subgroup denoted by $\mathcal{B}$ is the set of of *Bell-type arrays* or *renewal arrays*, that is the Riordan arrays $D = (d(z),\, h(z))$ for which $h(z) = zd(z)$, which was considered in the literature [19]. It is clear that there exists a semidirect product decomposition for Riordan group $\mathcal{R}$.

$$\mathcal{R} \simeq \mathcal{A} \rtimes \mathcal{B}\, since\, (d(t), h(t)) = \left(\frac{td(t)}{h(t)}, t\right)\left(\frac{h(t)}{t}, h(t)\right).$$

From [19], an infinite lower triangular array $[d_{n,k}]_{n,k\in\mathbb{N}_0} = (d(t), h(t))$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}_0$ there holds

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n}, \tag{4}$$

which is shown to be equivalent to

$$h(t) = tA(h(t)) \tag{5}$$

in [15], by He and Sprugnoli. Here, $A(t)$ is the generating function of A-sequence. [15, 16] also show that a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ exists such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{6}$$

or equivalently,

$$d(t) = \frac{d_{1,1}}{1 - tZ(h(t))}. \tag{7}$$

From Theorem 2.5 of [15], a Riordan array is a Bell-type Riordan arrays; i.e., $h(t) = td(t)$, if and only if its A-sequence and Z-sequence satisfy $A(t) = d_{0,0} + tZ(t)$. Thus, a sequence characterization of the numbers of Lukasiewicz path (or abbreviated as L-path) including the Catalan and Motzkin numbers is obtained by using the characterization of Bell-type Riordan arrays. Here, a L-path is a lattice path that starts at the origin with steps $(1, a)$ $(a \le 1)$ that can not go below the x-axis. It is known (see, for example, [4]) that the number of L-paths from $(0, 0)$ to (n, k) is the quasi-Catalan number

$$d_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n}, \quad 0 \le k \le n, \tag{8}$$

which is the (n, k)-entry of a Bell-type Riordan array $(C(t), tC(t))$ shown in Table 1, where

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t}.$$

Formula (7) will be extended to a parametric form in Corollary 3.6.

Since the sum of nth row is nth Catalan number c_n and the first column of the Catalan triangle is Catalan number sequence, from the sequence characterization of Bell-type Riordan array $(C(t), tC(t))$, we

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	5	5	3	1			
4	14	14	9	4	1		
5	42	42	28	14	5	1	

Table 1: The Catalan triangle C of L-path

have $Z(t) = 1/(1-t)$. Thus, the relation $A(t) = d_{0,0}+tZ(t) = 1+tZ(t)$ implies $A(t) = 1/(1-t)$.

From Theorems 2.3 and 2.5 of [15], we immediately obtain

Proposition 1.1 *Let $(d(t), h(t))$ be a Bell-type Riordan array satisfying $h(t) = td(t)$, where $d(0) \neq 0$. Denote the compositional inverse of $h(t)$ by $\bar{h}(t)$. If one of $d(t)$ and $h(t)$, i.e., the A- and Z-sequence characterizations of $(d(t), h(t))$, is given, then other three among $d(t)$, $h(t)$, $A(t)$, and $Z(t)$ can be determined uniquely from the relations $h(t) = td(t)$,*

$$Z(t) = \frac{d(\bar{h}(t)) - d(0)}{\bar{h}(t)d(\bar{h}(t))}, \quad \textit{and} \quad A(t) = d(0) + tZ(t).$$

In particular, for any given $d(t)$, there exist unique $h(t)$, $Z(t)$, and $A(t)$ shown above such that $(d(t), td(t))$ is a Bell-type triangle.

Proof. One may gives a proof based on the definition of Bell-type Riordan arrays, Theorems 2.1, 2.3, and 2.5 of [15]. Here, we omitted the details.

■

Example 1.1 Let

$$C(t) = \frac{1-\sqrt{1-4t}}{2t}$$

be the generating function of Catalan numbers $\{c_n = \binom{2n}{n}/(n+1)\}_{n\geq 0}$. Considering the Bell-type Riordan array $(d(t), h(t)) := (C(t), tC(t))$, the Catalan triangle, we find the compositional inverse of $h(t) := tC(t)$ is

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	4	5	3	1			
4	9	12	9	4	1		
5	21	30	25	14	5	1	
6	51	76	69	44	20	6	1

Table 2: The Motzkin triangle M

$$\bar{h}(t) = t(1-t),$$

which, from Proposition 1.1, implies the generating functions of A- and Z-sequences of $(C(t), tC(t))$ are

$$Z(t) = \frac{1}{1-t}, \quad A(t) = \frac{1}{1-t}.$$

Similarly, for Motzkin function

$$M(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2},$$

the generating function of Motzkin numbers $\{1, 1, 2, 4, 9, 21, 51, \ldots\}$, we can find the Motzkin triangle $(M(t), tM(t))$ with characterization

$$Z(t) = 1+t, \quad A(t) = 1+t+t^2,$$

by using the compositional inverse of $tM(t)$ as $t/(1+t+t^2)$ and Proposition 1.1. Hence, we may evaluate the first few entries of the Motzkin triangle shown in the first column of Table 2.

As for the large Schröder numbers $\{1, 2, 6, 22, 90, \ldots\}$, we have its generating function

$$G(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2t}.$$

Thus, the Schröder triangle $(G(t), tG(t))$ has the generating functions of its A- and Z-sequences

$$Z(t) = \frac{2}{1-t}, \quad A(t) = \frac{1+t}{1-t},$$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	6	4	1				
3	22	16	6	1			
4	90	68	30	8	1		
5	394	304	146	48	10	1	

Table 3: The Schröder triangle S

respectively, since the compositional inverse of $tG(t)$ is $t(1-t)/(1+t)$. Therefore $Z = \{2, 2, 2, \ldots\}$ and $A = \{1, 2, 2, \ldots\}$. We may evaluate the first few entries of the large Schröder triangle as those shown in the first column of Table 3.

It is known that Catalan numbers and Motzkin numbers have the generating equations

$$C(t) = 1 + tC(t)^2 \quad \textit{and} \quad M(t) = 1 + tM(t) + t^2M(t)^2,$$

respectively.

In the next section, we will use the sequence characterization of Bell-type Riordan array shown in Proposition 1.1 to define (c, e)-(generalized or parametric) Catalan numbers with a parameters c and e, and we call such Bell-type Riordan arrays the (c, e)-(generalized or parametric) Catalan triangles. The Taylor expansion and some properties of the generalized Catalan numbers and generalized Catalan triangles will be presented. In addition, we will give some combinatorial interpretations for the Bell-type Riordan arrays including the generalized Catalan triangles. Furthermore, a similar argument is used to extend classical Motzkin numbers to a parametrical Motzkin numbers. In Section 3, we shall discuss the inverse of the generalized Catalan triangles and inverse Motzkin triangles, from which the expressions of the parametric Catalan numbers and triangles and parametric Motzkin numbers in terms of classical Catalan numbers are given .

2 Generalized Catalan numbers and their Taylor expansion

In this section, we will develop a type of numbers that generates Catalan numbers and Schröder numbers. We will also give their generating functions and present the matrices related to the numbers.

Let $d(t)$ be the generating function of a type of numbers $\{d_n\}_{n\geq 0}$ defined by

$$d(t) = \frac{1}{1 - tZ(td(t))},$$

where $Z(t) = c\sum_{n\geq 0} e^n t^n$ is an infinite series $c/(1-et)$ or $Z(t) = c/(1-et)$, we obtain the corresponding $d(t)$ as

$$d_{c,e}(t) = \frac{1 - etd_{c,e}(t)}{1 - ct - etd_{c,e}(t)},$$

or equivalently,

$$d_{c,e}(t) = 1 + td_{c,e}(t)(c - e + ed_{c,e}(t)), \tag{9}$$

which is the generating equation of $d_{c,e}(t)$. From the above equation we solve

$$d_{c,e}(t) = \frac{1 - (c-e)t - \sqrt{1 - 2(c+e)t + (c-e)^2t^2}}{2et}. \tag{10}$$

In particular, when $(c, e) = (1, 1), (2, 1)$, and $(1, 2)$, we obtain $d_{1,1}(t) = C(t)$, the classical Catalan function, $d_{2,1}(t) = G(t)$, the large Schröder function, and

$$d_{1,2}(t) = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t},$$

the small Schröder function, respectively. From Proposition 1.1, the generating function of A-sequence of $(d_{c,e}(t), td_{c,u}(t))$ is $A(t) = 1 + tZ(t) = 1 + ct/(1-et)$.

Definition 2.1 *For any $(c, e) \in \mathbb{Z}^2$, $c, e \neq 0$, we call the coefficients, $\{c_n\}_{n\geq 0}$, of $d_{c,e}(t)$ defined by (9) or (10) the generalized Catalan numbers associated with (c, e), or simply, $(c, e)-$Catalan numbers. In particular, $(c, 1)$ and $(1, e)$-Catalan numbers are called the large and small*

$n\backslash k$	0	1	2	3	4
0	1				
1	c	1			
2	c^2+c	$2c$	1		
3	c^3+3c^2+c	$3c^2+2c$	$3c$	1	
4	$c^4+6c^3+6c^2+c$	$4c^3+8c^2+2c$	$6c^2+3c$	$4c$	1

Table 4: The $(c,1)-$Catalan number triangle C

generalized Catalan numbers, respectively. The corresponding Bell-type Riordan arrays, from which the generalized Catalan numbers are defined, are called the generalized Catalan triangles, or (c,e)*-Catalan triangles.*

Remark 2.1 The series inverse of $d_{c,1}(t)$, $c \in bN$, is studied in [1] using a different approach. Constants c and e in Definition 2.1 can be extended to non-zero real numbers. In that setting, (c,e)-Catalan numbers become functions, which will be studied in another paper.

From Proposition 1.1, we obtain the following recurrence relations of entries $d_{n,k}$ of generalized Catalan triangle $(d_{c,e}(t), td_{c,e}(t))$ by using its Z and $A-$sequences.

$$d_{n,0} = c\sum_{k=0}^{n-1} e^k d_{n-1,k} \tag{11}$$

and for $k \geq 1$

$$d_{n,k} = d_{n-1,k-1} + c\sum_{j=0}^{n-k-1} e^j d_{n-1,k+j} \tag{12}$$

Based on (11) and (12), the first few $(c,1)$, $(1,e)$, and (c,e)-Catalan numbers are shown in the first column of the $(c,1)$, $(1,e)$, and (c,e)-Catalan triangle in Tables 4, 5, and 6, respectively.

(c,e)-Catalan triangles can bring many properties of (c,e)-Catalan numbers and functions. In the following, we will present an expansion and a combinatorial interpretation of (c,e)-Catalan numbers.

Generating equation (9) can be considered as the zeroth order Taylor expansion of $d_{c,e}(t)$, i.e., the generalized Catalan numbers $\{c_n\}_{n\geq 0}$. We now give the nth Taylor expansion of generalized Catalan numbers using Riordan arrays.

$n\backslash k$	0	1	2	3	4
0	1				
1	1	1			
2	$e+1$	2	1		
3	e^2+3e+1	$2e+3$	3	1	
4	e^3+6e^2+6e+1	$2e^2+8e+4$	$3e+6$	4	1

Table 5: The $(1,e)-$Catalan number triangle C

$n\backslash k$	0	1	2	3	4
0	1				
1	c	1			
2	c^2+ce	$2c$	1		
3	$c^3+3c^2e+ce^2$	$3c^2+2ce$	$3c$	1	
4	$c^4+6c^3e+6c^2e^2+ce^3$	$4c^3+8c^2e+2ce^2$	$6c^2+3ce$	$4c$	1

Table 6: The $(c,e)-$Catalan number triangle C

Theorem 2.2 *Let $ce\neq 0$. Denote by $(d_{n,k})_{n\geq k\geq 0}$ the Riordan array of $(d_{c,e}(t), td_{c,e}(t))$ and $\{c_n=d_{n,0}\}_{n\geq 0}$ the coefficients of $d_{c,e}(t)$. Then there holds the following Taylor's expansion of the generating function of (c,e)-Catalan numbers, $\{c_n\}_{n\geq 0}$,*

$$d_{c,e}(t)=\sum_{j=0}^{n-1}d_{j,0}t^j+t^n\sum_{k=1}^{n+1}e^{k-1}(d_{n,k-1}-ed_{n,k})(d_{c,e}(t))^k, \qquad (13)$$

where $d_{j,0}=c_j$.

Proof. We prove (13) by using the sequence characterization of Riordan array C . First, we establish an expression of power of $d_{c,e}(t)$,

$$(d_{c,e}(t))^k=1+t\left((c-e)d_{c,e}(t)+c\sum_{j=2}^{k}(d_{c,e}(t))^j+e(d_{c,e}(t))^{k+1}\right) \qquad (14)$$

for $k\geq 2$. (14) is obvious for $k=2$. In deed, from (9), we have

$$\begin{aligned}(d_{c,e}(t))^2 &= d_{c,e}(t)\left(1+td_{c,e}(t)(c-e+ed_{c,e}(t))\right)\\ &= 1+td_{c,e}(t)(c-e+ed_{c,e}(t))+t(d_{c,e}(t))^2(c-e+ed_{c,e}(t))\\ &= 1+t\left((c-e)d_{c,e}(t)+c(d_{c,e}(t))^2+e(d_{c,e}(t))^3\right).\end{aligned}$$

Assume (14) holds for $k-1$. Then

$$\begin{aligned}
& (d_{c,e}(t))^k = d_{c,e}(t)(d_{c,e}(t))^{k-1} \\
= \;& d_{c,e}(t) + t\left((c-e)(d_{c,e}(t))^2 + c\sum_{j=2}^{k-1}(d_{c,e}(t))^{j+1} + e(d_{c,e}(t))^{k+1}\right) \\
= \;& 1 + t\left((c-e)d_{c,e}(t) + e(d_{c,e}(t))^2\right) \\
& \quad + t\left((c-e)(d_{c,e}(t))^2 + c\sum_{j=3}^{k}(d_{c,e}(t))^{j} + e(d_{c,e}(t))^{k+1}\right) \\
= \;& 1 + t\left((c-e)d_{c,e}(t) + c\sum_{j=2}^{k}(d_{c,e}(t))^{j} + e(d_{c,e}(t))^{k+1}\right).
\end{aligned}$$

Secondly, we prove (13) using mathematical induction. Noting $d_{0,0} = 1$, $d_{1,0} = c$, $d_{1,1} = 1$ and $d_{n,k} = 0$ for all $k \geq n$, we know (13) holds for $n = 1$. To prove the induction step, we need to show

$$\sum_{k=1}^{n} e^{k-1}\left(d_{n-1,k-1} - ed_{n-1,k}\right) = d_{n-1,0}. \tag{15}$$

In fact, from the recurrence relations (11) and (12) we obtain

$$d_{n-1,0} - ed_{n-1,1} = (c-e)d_{n-2,0}$$

and

$$d_{n-1,k-1} - ed_{n-1,k} = d_{n-2,k-2} + (c-e)d_{n-2,k-1}$$

for all $k \geq 2$. Thus

$$
\begin{aligned}
& \sum_{k=1}^{n} e^{k-1}\left(d_{n-1,k-1}-ed_{n-1,k}\right) \\
= \;& (c-e)d_{n-2,0}+\sum_{k=2}^{n} e^{k-1}\left(d_{n-1,k-1}-ed_{n-1,k}\right) \\
= \;& \sum_{k=2}^{n} e^{k-1}d_{n-2,k-2}+(c-e)\sum_{k=1}^{n-1} e^{k-1}d_{n-2,k-1} \\
= \;& c\sum_{k=1}^{n-1} e^{k-1}d_{n-2,k-1},
\end{aligned}
$$

which implies (15).

Substituting (14) into the $(n-1)$th remainder of (13) and noting (15), we obtain

$$
\begin{aligned}
& \sum_{k=1}^{n} e^{k-1}(d_{n-1,k-1}-ed_{n-1,k})(d_{c,e}(t))^{k} \\
= \;& \sum_{k=1}^{n} e^{k-1}(d_{n-1,k-1}-ed_{n-1,k})\left[1+t\left(-ed_{c,e}(t)\right.\right. \\
& \qquad \left.\left.+c\sum_{j=1}^{k}(d_{c,e}(t))^{j}+e(d_{c,e}(t))^{k+1}\right)\right] \\
= \;& d_{n-1,0}+t\left[-ed_{n-1,0}d_{c,e}(t)+c\sum_{k=1}^{n}\sum_{j=1}^{k} e^{k-1}(d_{n-1,k-1}-ed_{n-1,k})(d_{c,e}(t))^{j}\right. \\
& \quad \left.+\sum_{k=1}^{n} e^{k}(d_{n-1,k-1}-ed_{n-1,k})(d_{c,e}(t))^{k+1}\right] \\
= \;& d_{n-1,0}+t\left[-ed_{n-1,0}d_{c,e}(t)+c\sum_{j=1}^{n}\left(\sum_{k=j}^{n} e^{k-1}(d_{n-1,k-1}-ed_{n-1,k})\right)(d_{c,e}(t))^{j}\right. \\
& \quad \left.+\sum_{k=2}^{n+1} e^{k-1}(d_{n-1,k-2}-ed_{n-1,k-1})(d_{c,e}(t))^{k}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= d_{n-1,0} + t\left[-ed_{n-1,0}d_{c,e}(t) + c\sum_{k=1}^{n} e^{k-1}d_{n-1,k-1}(d_{c,e}(t))^k\right.\\
&\qquad \left.+\sum_{k=2}^{n+1} e^{k-1}(d_{n-1,k-2} - ed_{n-1,k-1})(d_{c,e}(t))^k\right]\\
&= d_{n-1,0} + t\left[(c-e)d_{n-1,0}d_{c,e}(t) + \sum_{k=2}^{n} e^{k-1}\left(d_{n-1,k-2}\right.\right.\\
&\qquad \left.\left.+(c-e)d_{n-1,k-1}\right)(d_{c,e}(t))^k + e^n(d_{c,e}(t))^{n+1}\right]\\
&= d_{n-1,0} + t\left[(d_{n,0} - ed_{n,1})d_{c,e}(t) + \sum_{k=2}^{n+1} e^{k-1}(d_{n,k-1} - ed_{n,k})(d_{c,e}(t))^k\right],
\end{aligned}
$$

where the last step is due to the following facts coming from the sequence characterization of (c,e)-Catalan triangle.

$$
d_{n,0} - ed_{n,1} = c\sum_{j=0}^{n-1} e^j d_{n-1,j} - e\left(d_{n-1,0} + c\sum_{j=1}^{n-1} e^{j-1}d_{n-1,j}\right) = (c-e)d_{n-1,0}.
$$

Similarly, there hold

$$
\begin{aligned}
& d_{n,k-1} - ed_{n,k}\\
&= d_{n-1,k-2} + c\sum_{j=k-1}^{n-1} e^{j-k+1}d_{n-1,j} - e\left(d_{n-1,k-1} + c\sum_{j=k}^{n-1} e^{j-k}d_{n-1,j}\right)\\
&= d_{n-1,k-2} + (c-e)d_{n-1,k-1}
\end{aligned}
$$

for $n \geq k \geq 2$ and $d_{n,k-1} - d_{n,k} = 1$ for $k = n+1$, which completes the proof of the theorem.

■

For the case of $c = e = 1$, the Taylor expansion of $d_{1,1}(t)$ is given in [8] using a sequence approach.

Remark 2.2 Equation (13) gives a sequence of identities of $d_{c,e}(t)$ in terms of n.

Inspired by [4], we give a combinatorial interpretation of the entries $d_{n,k}$ of $(d_{c,e}(t), td_{c,e}(t))$, the c-(generalized) Catalan triangle. First, we

consider a lattice path that starts at the origin, can not go below the $x-$axis and above $y = x$, and has a possible steps $S_r = (1, r)$ with $r \leq -1$, which is called a Lukasiewicz path (or simply L-path). The numbers of L-paths from $(0, 0)$ to (n, k) is $(k+1)\binom{2n-k}{n}/(n+1)$, which is the entry of the Catalan triangle shown in Table 1. Secondly, we associate a weight to each step S_r of an L-path, which is denoted by $\omega(S_r)$. The weight of an L-path, denoted by $\omega(S_r)$, is defined as the product of weights of its steps.

Theorem 2.3 *$(d(t), h(t)) = \{d_{n,k}\}_{0\leq k\leq n}$, $d(0) = 1$, is a Riordan array with sequence characterizations $Z = (z_0, z_1, \cdots,)$ and $A = (1, z_1, z_2 \cdots,)$ if and only if $d_{n,k}$ is the sum of weights of weighted L-paths from the origin to the point (n, k) using the weights*

$$\omega(S_r) = \begin{cases} z_r, & if\ S_r\ touches\ the\ x-axis, \\ z_{r+1}, & otherwise, \end{cases} \tag{16}$$

where $r \geq 0$ and $\omega(S_{-1}) = 1$.

Theorem **??** is exactly the same as Theorem 2.2 of [4] when $a_0 = 1$, $a_i = z_i$ for $i \geq 1$, and $k = z_0 - 1$. Hence, its proof is omitted.

From Theorem 2.3 we obtain a combinatorial interpretation of the entries of the generalized Catalan triangles.

Corollary 2.4 *Let $ce \neq 0$. The entry $d_{n,k}$ of a (c, e)-Catalan triangle $(d_{c,e}(t),\ td_{c.e}(t))$ is the sum of weights of weighted L-paths from the origin to the point (n, k) with the weights defined in Theorem 2.3, where $z_j = ce^j$ for all $j \geq 0$.*

Proof. Noting the A-sequence and Z-sequence have the generating functions $A(t) = 1 + (ct/(1 - et))$ and $Z(t) = c/(1 - et)$, one my use Theorem 2.3 to establish Corollary 2.4.

■

The entries of a (c, e)-Catalan triangle $(d_{c,e}(t),\ td_{c.e}(t))$ will be given in Corollary 3.6.

The combinatorial interpretations of some special cases of generalized Catalan numbers, i.e., the first columns of the corresponding generalized Catalan triangle, were studied individually. For instance, [5] presents the first column sequence $\{d_{n,0}\}$ of the generalized Catalan

triangle $(d_{4,1}(t), td_{4,1}(t))$ are the numbers of lattice paths from $(0,0)$ to $(n+1, n+1)$ that consist of steps $(i,0)$ and $(0,j)$ with $i, j \geq 1$ and that stay strictly below the diagonal line y=x except at the endpoints.

In the beginning of this section, the sequence characteristic of the Catalan triangle, a Bell-type Riordan array, is presented by $Z(t) = c/(1-et)$. We now extend it to a more general form as follows, using which a type of parametric Motzkin numbers and Motzkin triangles are defined. Denote

$$Z(t) = c\frac{1-(et)^n}{1-et} \tag{17}$$

where $n \in \mathbb{N}_0$. If $n = 0$, we have the generating function of the Z-sequence of Catalan triangle, which has been discussed. If $n = 1$, then $Z(t) = c$, and the corresponding triangle is the parametric Pascal triangle $(1/(1-ct), t/(1-ct))$ with parameter c. We now consider the case of $n = 2$, $Z(t) = c(1+et)$. Then the generating function of A-sequence for Bell-type Riordan array $(\tilde{d}_{c,e}(t), t\tilde{d}_{c,e}(t))$ is $A(t) = 1 + ct(1+et)$, where $\tilde{d}_{c,e}(t)$ satisfies

$$\tilde{d}_{c,e}(t) = \frac{1}{1 - tZ(t\tilde{d}_{c,e}(t))},$$

or equivalently,

$$\tilde{d}_{c,e}(t) = 1 + ct\tilde{d}_{c,e}(t)(1 + et\tilde{d}_{c,e}(t)). \tag{18}$$

Thus

$$\tilde{d}_{c,e}(t) = \frac{1 - ct - \sqrt{1 - 2ct + c(c-4e)t^2}}{2cet^2} \tag{19}$$

for $ce \neq 0$. Therefore, we have reason to establish the following definition.

Definition 2.5 *The lower triangle $(\tilde{d}_{c,e}(t), t\tilde{d}_{c,e}(t))$ with $ce \neq 0$ is called the (c,e)-Motzkin triangle, where $d_{c,e}(t)$ presented in (19) is called (c,e)-Motzkin function. The coefficients of $\tilde{d}_{c,e}(t)$ are called (c,e)-Motzkin numbers or simply generalized Motzkin numbers.*

From Definition 2.5, when $(c,e) = (1,1)$, $\tilde{d}_{1,1}(t) = M(t)$, the generating function of classical Motzkin numbers. The first four rows of the generalized Motzkin triangle is shown in Table 7, in which the first column gives first four generalized Motzkin numbers.

$n\backslash k$	0	1	2	3	4
0	1				
1	c	1			
2	c^2+ce	$2c$	1		
3	c^3+3c^2e	$3c^2+2ce$	$3c$	1	
4	$c^4+6c^3e+2c^2e^2$	$4c^3+8c^2e$	$6c^2+3ce$	$4c$	1

Table 7: The $(c,e)-$Motzkin number triangle M

3 The inverses of Bell-type Riordan arrays

It is well-known that the inverse of a Riordan array $D:=(d(t),h(t))$ is $D^{-1}=(1/d(\bar{h}(t)),\bar{h}(t))$, where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $h(\bar{h}(t))=\bar{h}(h(t))=t$. From Theorems 4.1 and 4.2 of [15], we have

Theorem 3.1 *[15] The $A-$ and $Z-$sequences, denoted by A^*- and Z^*, respectively, of the inverse Riordan array D^{-1} are*

$$A^*(y)=\frac{1}{A(t)}\quad \text{and}\quad Z^*(y)=\frac{d(0)-d(y)}{d(0)t}, \tag{20}$$

respectively, where $y=t/A(t)=\bar{h}(t)$.

From the proofs of Theorems 2.1, 2.3, 4.1, and 4.3 of [15], we also have the following corrollary

Corollary 3.2 *The A- and Z-sequences of $(d(t),h(t))$ and its inverse $(1/(d(\bar{h}(t))),\bar{h}(t))$ are*

$$\begin{aligned} A(t)&=\frac{t}{\bar{h}(t)},\quad Z(t)=\frac{d(\bar{h}(t))-d(0)}{\bar{h}(t)d(\bar{h}(t))},\\ A^*(t)&=\frac{t}{h(t)},\quad Z^*(t)=\frac{d(0)-d(t)}{d(0)h(t)}. \end{aligned} \tag{21}$$

In particular, for Appell arrays $(d(t),t)$, we have

$$A(t)=1,\quad Z(t)=\frac{d(t)-d(0)}{td(t)},\quad A^*(t)=1,\quad Z^*(t)=\frac{d(0)-d(t)}{d(0)t}.$$

For Lagrange arrays,

$$A(t) = \frac{t}{\bar{h}(t)}, \quad Z(t) = 0, \quad A^*(t) = \frac{t}{h(t)}, \quad Z^*(t) = 0.$$

For Bell-type arrays, there hold $h(t) = td(t)$, $\bar{h}(t) = t/d(\bar{h}(t))$, *and*

$$\begin{aligned} A(t) &= d(\bar{h}(t)), \quad Z(t) = \frac{d(\bar{h}(t)) - d(0)}{t}, \\ A^*(t) &= \frac{1}{d(t)}, \quad Z^*(t) = \frac{d(0) - d(t)}{td(0)d(t)}. \end{aligned} \tag{22}$$

Proof. (22) is straightforward from Theorems 4.1 and 4.2 of [15]. Others are obvious from Theorem 3.1 and the definition of various subgroups.

■

From Corollary 3.2, we may obtain the following formula to evaluate the entries of Bell-type Riordan arrays.

Corollary 3.3 *Let* $(d_{n,k})_{n\geq k\geq 0} = (d(t), td(t))$ *be a Bell-type Riordan array. Then its inverse is* $(d^*_{n,k})_{n\geq k\geq 0} = (1/d(\bar{h}(t))), \bar{h}(t))$, *where* $\bar{h}(t)$ *is the compositional inverse of* $h(t)$. *Their entries satisfy*

$$d_{n,k} = \frac{k+1}{n+1}\left[t^{n-k}\right]\left(\frac{\bar{h}(t)}{t}\right)^{-n-1} = \frac{k+1}{n+1}\left[t^{n-k}\right](A(t))^{n+1} \tag{23}$$

$$d^*_{n,k} = \frac{k+1}{n+1}\left[t^{n-k}\right]\left(\frac{h(t)}{t}\right)^{-n-1} = \frac{k+1}{n+1}\left[t^{n-k}\right](A^*(t))^{n+1}, \tag{24}$$

where $A(t)$ *and* $A^*(t)$ *are defined in Theorem 3.1.*

Proof. From the definition of $d_{n,k}$, we have

$$\begin{aligned} d_{n,k} &= [t^n]d(t)(h(t))^k = [t^{n+1}](h(t))^{k+1} \\ &= \frac{k+1}{n+1}[t^{n-k}]\left(\frac{\bar{h}(t)}{t}\right)^{-n-1}, \end{aligned}$$

where the last step is an application of Lagrange inversion formula (see, for example, [8b] in [6]), which gives the first formula of (23). The rightmost formula of (23) yields from the first formula of (21) in Corollary 3.2. (24) can be proved similarly.

■

Proposition 3.4 *Let $(d(t), h(t))$ be a Bell-type Riordan array satisfying $h(t) = td(t)$, where $d(0) \neq 0$. If one of $d(t)$, $h(t)$, the A^*- and Z^*-sequence characterizations of D^{-1} is given, then other three can be determined uniquely from the relations $h(t) = td(t)$ and*

$$Z^*(t) = \frac{d(0) - d(t)}{d(0)td(t)} \quad and \quad A^*(t) = \frac{1}{d(0)} + tZ^*(t). \tag{25}$$

In addition, there hold the following relationships for evaluating $d(t)$ or $\bar{h}(t)$ if one of them is given:

$$\begin{aligned} [t^n]\, d(t) &= \frac{1}{n+1}\, [t^n] \left(\frac{\bar{h}(t)}{t}\right)^{-n-1} = \frac{1}{n+1}\, [t^n]\, (A(t))^{n+1}, \\ [t^n]\, \bar{h}(t) &= \frac{1}{n} \left[t^{n-1}\right] (d(t))^{-n}. \end{aligned} \tag{26}$$

Proof. The first equation has been proved in Corollary 3.2. Comparing two expressions of A^* and Z^* shown in (22), we immediately obtain the second equation of (25). The first formula of (26) can be considered as the case of $k = 0$ of (23). To prove it directly, we only need to note

$$[t^n]\, d(t) = [t^n]\, \frac{h(t)}{t} = \left[t^{n+1}\right] h(t)$$

and use the Lagrange inversion formula and the first formula of (21) in Corollary 3.2. Similarly, we have the second formula of (26).

■

We now use the generating function $A(t)$ to find the expressions of $(c, e)-$Catalan numbers and the entries of $(c, e)-$Catalan triangles.

Corollary 3.5 *For the (c, e)-generalized Catalan number $[t^n]\, d_{c,e}(t)$ defined by (9), there holds*

$$[t^n]\, d_{c,e}(t) = \sum_{j=0}^{n} \binom{2n-j}{j} c_{n-j} e^{n-j} (c-e)^j, \tag{27}$$

where $c_j = \binom{2j}{j}/(j+1)$ is the jth Catalan number. In particular,

$$\begin{aligned}[t^n]\, d_{c,1}(t) &= \sum_{j=0}^{n} \binom{2n-j}{j} c_{n-j}(c-1)^j \\ &= \sum_{j=0}^{n} \binom{2n-j}{n-j} c_j (c-1)^{n-j}\end{aligned}$$

and

$$\begin{aligned}[t^n]\, d_{1,e}(t) &= \sum_{j=0}^{n} \binom{2n-j}{j} c_{n-j} e^{n-j} (1-e)^j \\ &= \sum_{j=0}^{n} \binom{2n-j}{n-j} c_j e^j (1-e)^{n-j}.\end{aligned}$$

Proof. Consider Bell-type Riordan array $(d_{c,e}(t), h(t)) = (d_{c,e}(t), td_{c,e}(t))$, where $d_{c,e}(t)$ is defined by (9), and find the compositional inverse of $h(t)$ as

$$\bar{h}(t) = \frac{t - et^2}{1 + (c-e)t}.$$

Thus the corresponding sequence characteristic function

$$A(t) = \frac{t}{\bar{h}(t)} = \frac{1 + (c-e)t}{1 - et}.$$

By using the above expression of $\bar{h}(t)$ and the first formula of (26), (27) can be proved as follows.

$$
\begin{aligned}
[t^n]\, d_{c,e}(t) &= \frac{1}{n+1}\,[t^n]\left(\frac{1+(c-e)t}{1-et}\right)^{n+1}\\
&= \frac{1}{n+1}\,[t^n]\sum_{k=0}^{n+1}\sum_{j\geq 0}\binom{n+1}{k}\binom{n+j}{j}(c-e)^{n+1-k}e^j t^{n-k+j+1}\\
&= \frac{1}{n+1}\sum_{j\geq 0}\binom{n+1}{j+1}\binom{n+j}{j}e^j(c-e)^{n-j}\\
&= \sum_{j=0}^{n}\frac{1}{j+1}\binom{n}{j}\binom{n+j}{j}e^j(c-e)^{n-j}\\
&= \sum_{j=0}^{n}\frac{1}{n-j+1}\binom{n}{j}\binom{2n-j}{n}e^{n-j}(c-e)^j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{n}\frac{1}{n-j+1}\binom{2n-j}{j}\binom{2n-2j}{n-j}e^{n-j}(c-e)^j\\
&= \sum_{j=0}^{n}\binom{2n-j}{j}c_{n-j}e^{n-j}(c-e)^j.
\end{aligned}
$$

Substituting $e=1$ into the rightmost equation, we obtain the special case, the $(c,1)$-Catalan numbers. Similarly, we have the expressions of $(1,e)$-Catalan numbers.

■

If $(c,e)=(2,1)$, (27) gives the well-known expression of large Schröder numbers (see, for example, [7])

$$[t^n]\, d_{2,1}(t) = \sum_{j=0}^{n}\binom{2n-j}{j}c_{n-j}, \quad j\geq 0.$$

Similarly, the small Schröder numbers have expression

$$[t^n]\, d_{1,2}(t) = \sum_{j=0}^{n}(-1)^j 2^{n-j}\binom{2n-j}{j}c_{n-j}, \quad j\geq 0.$$

A similar argument can be used to derive a formula of the entries $d_{n,k}$ of the (c,e)-Catalan triangle, i.e., the entries $d_{n,k}$ of $(d_{c,e}(t), td_{c,e}(t))$ for $c, e \in \mathbb{N}$ (Hence $ce \neq 0$), which can be considered as the parametric extension of the numbers of L-paths shown in (8).

Corollary 3.6 *For the generalized Catalan triangle* $(d_{n,k})_{n\geq k\geq 0} = (d_{c,e}(t), td_{c,e}(t))$*, where* $d_{c,e}(t)$ *is defined by (9), there holds*

$$d_{n,k} = \sum_{j=0}^{n-k} \frac{k+1}{n-j+1}\binom{n}{j}\binom{2n-k-j}{n} e^{n-k-j}(c-e)^j, \qquad (28)$$

or equivalently,

$$d_{n,k} = (k+1)\sum_{j=0}^{n} \frac{[n-j]_k}{[2n-j]_k}\binom{2n-j}{j} c_{n-j} e^{n-k-j}(c-e)^j, \qquad (29)$$

where $c_j = \binom{2j}{j}/(j+1)$ *is the* j*th Catalan number,* $[\ell]_0 = 1$*, and* $[\ell]_k = \ell(\ell-1)\cdots(\ell-k+1)$ *for all* $k \geq 1$*.*

Proof. From (23) in Corollary 3.3, we obtain

$$\begin{aligned} d_{n,k} &= \frac{k+1}{n+1}[t^{n-k}](A(t))^{n+1} = \frac{k+1}{n+1}\left[t^{n-k}\right]\left(\frac{1+(c-e)t}{1-et}\right)^{n+1} \\ &= \frac{k+1}{n+1}\sum_{j\geq 0}\binom{n+1}{k+j+1}\binom{n+j}{j} e^j (c-e)^{n-k-j} \\ &= \sum_{j=0}^{n-k} \frac{k+1}{k+j+1}\binom{n}{k+j}\binom{n+j}{j} e^j (c-e)^{n-k-j} \\ &= \sum_{j=0}^{n-k} \frac{k+1}{n-j+1}\binom{n}{j}\binom{2n-k-j}{n} e^{n-k-j}(c-e)^j \\ &= (k+1)\sum_{j=0}^{n-k} \frac{[n-j]_k}{[2n-j]_k}\frac{1}{n-j+1}\binom{2n-j}{j}\binom{2n-2j}{n-j} e^{n-k-j}(c-e)^j, \end{aligned}$$

which implies (29).

■

It can be found that

$$(d_{c,e}(t), td_{c,e}(t))^{-1} = \left(\frac{1-et}{1+(c-e)t}, \frac{t(1-et)}{1+(c-e)t} \right). \tag{30}$$

Let $(\alpha(t), \phi(t))$ and $(\beta(t), \psi(t))$ be a pair of inverse matrices i.e.,

$$(\alpha(t), \phi(t)) * (\beta(t), \psi(t)) = (1, t)$$

We denote $(\alpha(t), \phi(t))$ and its inverse by $(d_{n,k})$ and $(\bar{d}_{n,k})$, respectively. Then a pair of inverse matrices $(d_{n,k})$ and $(\bar{d}_{n,k})$ can be used to generalize the combinatorial sum inversion

$$f_n = \sum_{k=0}^{n} d_{n,k} g_k \Leftrightarrow g_n = \sum_{k=0}^{n} \bar{d}_{n,k} f_k, \tag{31}$$

or equivalently,

$$F(t) = \alpha(t) G(\phi(t)) \Leftrightarrow G(t) = \beta F(\psi(t)),$$

where $F(t)$ and $G(t)$ are the generating functions of sequences $\{f_n\}$ and $\{g_n\}$, respectively. It is known that

$$(d_{n,k})^{-1} \equiv (d(t), h(t))^{-1} \equiv (d^*(t), \bar{h}(t)) \equiv (\bar{d}_{n,k}), \tag{32}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$ and

$$d^*(t) = \frac{1}{d(\bar{h}(t))}. \tag{33}$$

As an example, for the Pascal triangle $(d(t), h(t)) = (1/(1-t), t/(1-t)) = (\binom{n}{k})$ from (33) there holds

$$(d^*(t), \bar{h}(t)) = \left(\frac{1}{1+t}, \frac{t}{1+t} \right) = \left((-1)^{n-k} \binom{n}{k} \right),$$

which yields the well known sum inversion

$$f_n = \sum_{k=0}^{n} \binom{n}{k} g_k \Leftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k.$$

Another example can be found in Catalan matrix $(C(t), tC(t)) = (d^c_{n,k})$, where

$$C(t) = \frac{1-\sqrt{1-4t}}{2t}$$

and the entries of the Catalan matrix are (see, for example, [4])

$$d^c_{n,k} = [t^n]t^k C(t)^{k+1} = \frac{k+1}{n+1}\binom{2n-k}{n}, \quad 0 \le k \le n.$$

It is easy to find

$$(C(t), tC(t))^{-1} = (1-t, t(1-t)) = (\bar{d}^c_{n,k}),$$

where the matrix entries are

$$\bar{d}^c_{n,k} = [t^n]t^k(1-t)^{k+1} = (-1)^{n-k}\binom{k+1}{n-k}, \quad 0 \le k \le n.$$

Thus we have the sum inversion

$$f_n = \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{2n-k}{n} g_k \Leftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k}\binom{k+1}{n-k} f_k.$$

Therefore, from (30), we have the sum inversion (need $d_{n,k}$ and $\bar{d}_{n,k}$.)

A similar argument can also be applied to give an expression of generalized Motzkin numbers in terms of Classical Catalan numbers.

Corollary 3.7 *For the generalized Catalan number* $[t^n]\,\tilde{d}_{c,e}(t)$ *defined by (19), there holds*

$$[t^n]\,\tilde{d}_{c,e}(t) = \sum_{j=0}^{n} \binom{n}{2j-n} c_{n-j} c^j e^{n-j}, \tag{34}$$

where $c_j = \binom{2j}{j}/(j+1)$ *is the* j*th Catalan number. In particular, for the Motzkin number* M_n

$$M_n = [t^n]\,\tilde{d}_{1,1}(t) = \sum_{j=0}^{n} \binom{n}{2j-n} c_{n-j}.$$

Proof. From Corollary 3.3 and noting the generating function of A-sequence of Motzkin triangle $(\tilde{d}_{c,e}(t), t\tilde{d}_{c,e}(t))$ is $A(t) = 1 + ct(1 + et)$, we have

$$
\begin{aligned}
& [t^n]\,\tilde{d}_{c,e}(t) = \frac{1}{n+1}\,[t^n]\,(A(t))^{n+1} = \frac{1}{n+1}\,[t^n]\,(1+ct(1+et))^{n+1} \\
= \quad & \frac{1}{n+1}\,[t^n]\sum_{k=0}^{n+1}\binom{n+1}{k}c^k t^k (1+et)^k \\
= \quad & \frac{1}{n+1}\,[t^n]\sum_{k=0}^{n+1}\sum_{j=0}^{k}\binom{n+1}{k}\binom{k}{j}c^k e^j t^{k+j} \\
= \quad & \frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{n-j}\binom{n-j}{j}c^{n-j}e^j \\
= \quad & \sum_{j=0}^{n}\frac{1}{n-j+1}\binom{n}{n-j}\binom{j}{n-j}c^j e^{n-j} \\
= \quad & \sum_{j=0}^{n}\frac{1}{n-j+1}\binom{n}{2j-n}\binom{2n-2j}{n-j}c^j e^{n-j} \\
= \quad & \sum_{j=0}^{n}\binom{n}{2j-n}c_{n-j}c^j e^{n-j},
\end{aligned}
$$

completing the proof.

■

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