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Abstract

In this paper a connective study of Gould's annihilation coefficients and Abel-Gontscharoff polynomials is presented. It is shown that Gould's annihilation coefficients and Abel-Gontscharoff polynomials are actually equivalent to each other under certain linear substitutions for the variables. Moreover, a pair of related expansion formulas involving Gontscharoff's remainder and a new form of it are demonstrated, and also illustrated with several examples.

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1 Introduction

Recently, as motivated by some special identities of Abel-type, Gould [5] has investigated a kind of general algebraic identity of the form

$$\sum_{k=0}^{n} \binom{n}{k} c(k)(x-\beta_k)^{n-k} = x^n, \tag{1.1}$$

where $\beta_k \in \mathbb{C}$ (complex number field), $\beta_0 \neq 0$, $c(0)=1$, and the uniquely determined coefficients $\binom{n}{k}c(k)$ are called (by Gould) "annihilation coefficients" and satisfy the recurrence relations

$$c(n) = \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} c(j)\beta_j^{n-j}, \quad n \geq 1, \tag{1.2}$$

Evidently (1.2) is implied by (1.1) with $x=0$. A few $c(k)$-polynomials can be readily found by using (1.2), i.e.,

$$\begin{aligned} c(1) &= \beta_0, \quad c(2) = 2\beta_0\beta_1 - \beta_0^2, \\ c(3) &= 6\beta_0\beta_1\beta_2 - 3\beta_0^2\beta_2 - 3\beta_0\beta_1^2 + \beta_0^3. \end{aligned}$$

Actually it is easily observed that $c(k) \equiv c(k;\beta) \equiv c(k;\beta_0,\cdots,\beta_{k-1})$ $(k \geq 1)$ is a certain kind of homogeneous polynomial of degree k in $\beta_0,\cdots,\beta_{k-1}$. Also, it is obvious that (1.1) implies $c(k;\alpha,\cdots,\alpha)=\alpha^k$ by setting $\beta_k=\alpha$.

Recall that an alternative form of Abel's identity can be written as

$$\sum_{k=0}^{n} \binom{n}{k} z(z-kt)^{k-1}(x-z+kt)^{n-k} = x^n. \tag{1.3}$$

This example corresponds to Gould's identity (1.1) with $\beta_k = z-kt$ and $c(k;\beta) = c(k;z,z-t,\cdots,z-(k-1)t) = z(z-kt)^{k-1}$.

Denote by $\Gamma \equiv (\Gamma;+,\cdot)$ the commutative ring of formal power series over $\mathbb{C}$, in which formal differentiation and integration of power series are defined as usual (cf. e.g. Comtet [1], Section 1.12). Let $\alpha_k \in \mathbb{C}$, $\beta_k \in \mathbb{C}$ $(k=0,1,2\cdots)$. Then for any $f \in \Gamma$ we have formally (cf. Hsu [7])

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha_k)}{k!} c(k; z-\alpha_0,\cdots,z-\alpha_{k-1}), \tag{1.4}$$

and

$$f(z)=\sum_{k=0}^{\infty}\frac{c(k;\beta)}{k!}\sum_{m=0}^{\infty}\frac{f^{(k+m)}(0)}{m!}(z-\beta_k)^m, \tag{1.5}$$

where $c(0)=1$, $c(k,\beta)=c(k;\beta_0,\cdots,\beta_{k-1})$ $(k\geq 1)$, and $f^{(k)}(a)$ denotes the kth formal derivative of $f(z)$ at $z=a$.

Both (1.4) and (1.5) could be formally verified by using the identity (1.1) and the substitutions $\beta_j=z-\alpha_j$ $(j\geq 0)$ as follows. From the right-hand side of Eq. (1.4), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha_k)}{k!}c(k;z-\alpha_0,\cdots,z-\alpha_{k-1})\\
=&\sum_{k=0}^{\infty}\frac{c(k;\beta)}{k!}\sum_{m=0}^{\infty}\frac{f^{(k+m)}(0)}{m!}(\alpha_k)^m\\
=&\sum_{k=0}^{\infty}\frac{c(k;\beta)}{k!}\sum_{n=k}^{\infty}\frac{f^{(n)}(0)}{(n-k)!}(z-\beta_k)^{n-k}\\
=&\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}c(k;\beta)(z-\beta_k)^{n-k}\\
=&\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=f(z).
\end{aligned}$$

Comparing (1.3) with (1.1) in which $\beta_k=z-kt$, we see that the assignment $\alpha_k=kt$ in (1.4) yields the Abel series expansion (cf. [1])

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(kt)}{k!}z(z-kt)^{k-1}. \tag{1.6}$$

This may be the reason why Gontscharoff called (1.4) "generalized Abelian Series." Surely (1.4) may be more properly called Abel-Gontscharoff interpolation series, which is a type of Hermite-Birkhoff interpolation (see [8] and [9]). To describe Hermite-Birkhoff interpolation problems, we define the following interpolation (or incidence) matrix.

$$E=[e_{i,k}]_{i=1,k=0}^{m\ \ n},\quad m\geq 1, n\geq 0, \tag{1.7}$$

where elements $e_{i,k}$ are 0 or 1 and the number of $1'$s in E is equal to $N+1$. Denote $|E|=\sum_{i,k}e_{i,k}$. Thus, $|E|=N+1$. In addition, we do

not allow empty rows in E, i.e., an i for which $e_{i,k} = 0$, $k = 0, 1, \ldots, n$. A set of nodes $T = \{t_1, \ldots, t_m\}$ consists of m distinct points of the set A that is either an interval or a circle. The elements E, T, and the data $c_{i,k}$ (defined for $e_{i,k} = 1$) determine a Hermite-Birkhoff interpolation problem which consists in finding a polynomial $p = p(t) \in \pi_N(\mathbb{C})$, the collection of all polynomials of degrees $\leq N$, satisfying

$$p^{(k)}(t_i) = c_{i,k}, \quad e_{i,k} = 1. \tag{1.8}$$

It is easily seen that problem (1.8) has a unique solution if and only if the determinant

$$D(E;T) := det \left[\frac{t_i^{\alpha-k}}{(\alpha-k)!} \right]_{(i,k)\in \mathbf{e}}^{0\leq\alpha\leq n-1} \tag{1.9}$$

is nonzero, where $\mathbf{e} := \{(i,k) : e_{i,k} = 1\}$.

Matrix E is said to be poised with respect to the node set T if the corresponding equations (1.8) have a unique solution for each given set of $c_{i,k}$ (i.e., $p^{(k)}(t_i) = 0$, $e_{i,k} = 1$, implies $p = 0$). E is said to be conditionally poised if there is a set T of distinct nodes in $\mathbb{C}$ with respect to which it is poised. E is said to be real poised (or complex poised) if it is poised with respect to all sets of distinct nodes in $\mathbb{R}$ (or $\mathbb{C}$).

More in details about the generalized Abelian series and various representations for the $c(k)$-polynomials involved in (1.4) and (1.5) will be expounded in the next two sections.

Remark 1. Let $S = \{a_1, a_2, \cdots, a_m\}$ and $T = \{b_1, b_2, \cdots, b_n\}$, $m \geq 1$, $n \geq 0$, and let R be a relation from S to T. Then the adjacency matrix of R is the $m \times n$ matrix $M(R) = (\alpha_{ij})$ defined by

$$\alpha_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \text{ or } a_i R b_j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $E = [e_{i,k}]_{i=1,k=0}^{m\ \ n}$, $m \geq 1$, $n \geq 0$, can be regarded as the adjacency matrix of relation from $S = \{a_1, a_2, \cdots, a_m\}$ to $T = \{b_1, b_2, \cdots, b_n\}$.

Remark 2. Consider the set π of all polynomials, with real (complex) number as coefficients, as a vector space with respect to usual addition and scalar multiplication. It is obvious that $B_1 = \{x^0, x^1, \ldots, x^n, \ldots\}$ is a natural basis of π. It is also easy to check that $B_2 = \{(x-\beta_0)^0, (x-\beta_1)^1, \ldots, (x-\beta_n)^n, \ldots\}$ is another basis of π, where $\beta_0 \neq 0$.

It is known that Gould [5] has defined "dot product annihilation coefficients" $\alpha\,(n,k;\beta_0,\cdots,\beta_n)$ by the expansion

$$\sum_{k=0}^{n}\alpha(n,k;\beta)\,(x-\beta_k)^k = x^n.$$

The inverse expansion is given by

$$\sum_{k=0}^{n}\binom{n}{k}(-\beta_n)^{n-k}\,x^k = (x-\beta_n)^n\,.$$

Consequently, the transformation matrix $A=[\alpha(n,k;\beta)]_{0\le k\le n<\infty}$ is an inverse of the matrix $\left[\binom{n}{k}(-\beta_n)^{n-k}\right]$, i.e.,

$$A=\left[\binom{n}{k}(-\beta_n)^{n-k}\right]^{-1}.$$

In terms of inverse relations we have the reciprocal pair

$$f_n=\sum_{k=0}^{n}\alpha(n,k;\beta)g_k \Longleftrightarrow g_n=\sum_{k=0}^{n}\binom{n}{k}(-\beta_n)^{n-k} f_k. \tag{1.10}$$

Evidently, the binomial inversion is just a simple particular case when β_n =const.$\neq 0$, $(n=0,1,2,\ldots)$.

An identity can be derived from the reciprocal pair shown as in (1.10). Substituting the second equation in (1.10) to the first one yields

$$\begin{aligned} f_n &= \sum_{k=0}^{n}\alpha(n,k;\beta)\sum_{\ell=0}^{k}(-\beta_k)^{k-\ell}\binom{k}{\ell}f_\ell \\ &= \sum_{\ell=0}^{n}\left[\sum_{k=\ell}^{n}\alpha(n,k;\beta)(-\beta_k)^{k-\ell}\binom{k}{\ell}\right]f_\ell \end{aligned}$$

Thus, we obtain the following identity

$$\sum_{k=\ell}^{n}\alpha(n,k;\beta)(-\beta_k)^{k-\ell}\binom{k}{\ell}=\delta_{n\ell}. \tag{1.11}$$

where $n, \ell \in \mathbb{N} \cup \{0\}$, and $\delta_{n\ell}$ is the Kronecker symbol; i.e., $\delta_{n\ell}$ equals to 1 if $n = \ell$ and 0 otherwise. In particular, if $\beta_0 = \beta_1 = \cdots = \beta_n = 1$, then $\alpha(n, k; \beta) = \binom{n}{k}$, and the identity (1.11) is reduced to

$$\sum_{k=\ell}^{n} (-1)^k \binom{n}{k} \binom{k}{\ell} = (-1)^\ell \delta_{n\ell}.$$

From Eq. (1.11), we obtain a recursion formula for calculating $\alpha(n, k; \beta)$.

$$\alpha(n, n-\ell; \beta) = -\sum_{k=n-\ell+1}^{n} \alpha(n, k; \beta_k) \left(-\beta_k\right)^{k+\ell-n} \binom{k}{n-\ell},$$

where $\ell = 1, 2, \ldots, n$ and $\alpha(n, n; \beta) = 1$. Using the above formula of $\alpha(n, k; \beta)$ we can find implicitly the expression of

$$\left[\binom{n}{k} \left(-\beta_n\right)^{n-k}\right]^{-1} = A.$$

As an unsolved problem, we propose to investigate whether there is any simple and explicit expression for $\left[\binom{n}{k} \left(-\beta_n\right)^{n-k}\right]^{-1}$.

2 Statement of Propositions

Since all the propositions to be presented are closely related to each other, we shall state them collectively in this section and then give proofs for them connectively in Section 3.

Proposition 2.1 *Gould's polynomial $c(k) \equiv c(k; z - \alpha_0, \ldots, z - \alpha_{k-1})$ involved in (1.4) is identical with Abel-Gontscharoff polynomial $Q_k(z) \equiv Q(z; \alpha_0, \ldots, \alpha_{k-1})$ of the form (cf. Davis [2], p. 46-47)*

$$c(k) = Q_k(z) = \begin{vmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^k \\ 0 & 1 & \binom{2}{1}\alpha_1 & \cdots & \binom{k}{1}\alpha_1^{k-1} \\ 0 & 0 & 1 & \cdots & \binom{k}{2}\alpha_2^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \binom{k}{k-1}\alpha_{k-1} \\ 1 & z & z^2 & \cdots & z^k \end{vmatrix}, \qquad (2.1)$$

where $Q_0(z) = 1$, $Q_1(z) = z - \alpha_0$ *and the* k*th row of the determinant is given by the row-vector* $(0, \ldots, 0, 1, \binom{k}{k-1}\alpha_{k-1})$, *so that* $Q_k(z)$ *is a polynomial with highest power term* z^k.

Note that the jth time differentiation of the determinant with respect to z at $z = \alpha_j$ $(0 \le j \le k-1)$ will make the last row-vector $(1, z, z^2, \ldots, z^k)$ change into a row-vector that is just $j!$ times (product) of the $(j+1)$th row of the determinant. Thus, we have

$$\begin{cases} Q_k^{(j)}(\alpha_j) = 0, & j = 0, \ldots, k-1, \\ Q_k^{(k)}(\alpha_k) = k!. \end{cases} \tag{2.2}$$

Certainly, these $k+1$ equations suffice to determine the polynomial $Q_k(z)$ uniquely, and they can also be used as a definition to define the kth degree Abel-Gontscharoff polynomial. Due to Proposition 2.1, in the following we will call polynomials $c(k;\beta)$ the Abel-Gontscharoff-Gould's polynomials.

Proposition 2.2 *(i) Abel-Gontscharoff-Gould's polynomials* $c(k;\beta) \equiv c(k;\beta_0, \ldots, \beta_{k-1})$ *as defined by (1.1) can be represented by the determinant*

$$c(k;\beta) = (-1)^k \begin{vmatrix} -\beta_0 & (-\beta_0)^2 & \cdots & (-\beta_0)^k \\ 1 & \binom{2}{1}(-\beta_1) & \cdots & \binom{k}{1}(-\beta_1)^{k-1} \\ 0 & 1 & \cdots & \binom{k}{2}(-\beta_2)^{k-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \binom{k}{k-1}(-\beta_{k-1}) \end{vmatrix}, \tag{2.3}$$

where the last row of the determinant is $\left(0, 0, \cdots, 1, \binom{k}{k-1}(-\beta_{k-1})\right)$.

(ii) Moreover, as an expansion of the determinant (2.3) there is an explicit expression of the form (cf. Hsu [7])

$$\begin{aligned} c(k;\beta) &= (-1)^{k-1}\beta_0^k \\ &+ \sum_{(j)} \frac{(-1)^{k-r-1}k!}{j_1!(j_2-j_1)!\cdots(k-j_r)!}\beta_0^{j_1}\beta_{j_1}^{j_2-j_1}\beta_{j_2}^{j_3-j_2}\cdots\beta_{j_r}^{k-j_r}, \end{aligned} \tag{2.4}$$

where $(j) \equiv \{j_1, \cdots, j_r\}$ *denotes an ordered subset of the ordered set* $\{1, 2, \cdots, k-1\}$ *with* $1 \le j_1 < j_2 < \cdots < j_r \le k-1$, $(1 \le r \le k-1)$, *and*

the summation is taken over all the ordered subsets of $\{1, 2, \cdots, k-1\}$ *for* $r = 1, \cdots, k-1$, *so that the right-hand of (2.4) consists of* 2^{k-1} *terms.*

Remark 3. Eq. (2.3) can also be written as follows.

$$\begin{aligned} c(k;\beta) &= (-1)^k \left| \begin{pmatrix} \binom{1}{0} & \binom{2}{0} & \cdots & \binom{k}{0} \\ \binom{1}{1} & \binom{2}{1} & \cdots & \binom{k}{1} \\ \binom{1}{2} & \binom{2}{2} & \cdots & \binom{k}{2} \\ \vdots & \vdots & & \vdots \\ \binom{1}{k-1} & \binom{2}{k-1} & \cdots & \binom{k}{k-1} \end{pmatrix} \right. \\ & \qquad \left. * \begin{pmatrix} -\beta_0 & (-\beta_0)^2 & \cdots & (-\beta_0)^k \\ 1 & -\beta_1 & \cdots & (-\beta_1)^{k-1} \\ 0 & 1 & \cdots & (-\beta_2)^{k-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\beta_{k-1} \end{pmatrix} \right|, \end{aligned}$$

where $(*)$ is the Hadamard product, and the last row of the second matrix is $(0, 0, \cdots, 1, -\beta_{k-1})$.

Note that (2.3) yields an alternative form of (2.1) when taking $\beta_j = z - \alpha_j$ $(j = 0, 1, \cdots, k-1)$. Of course one may also make use of (2.4) to express $Q_k(z) = c(k; z - \alpha_0, \cdots, z - \alpha_{k-1})$ as a sum by taking $\beta_j = z - \alpha_j$. Besides the determinantal form (2.1) Gontscharoff had also given an integral form for $Q_k(z)$, namely (cf. Davis [2])

$$Q_k(z) = k! \int_{\alpha_0}^{z} dz' \int_{\alpha_1}^{z'} dz'' \cdots \int_{\alpha_{k-1}}^{z^{(k-1)}} dz^{(k)}, \quad k \geq 1. \tag{2.5}$$

For $k = 0$, we define $Q_0(z) = 1$. Obviously, the polynomial $Q_k(z)$ defined by (2.5) satisfies all the conditions given by (2.2).

Observe that the coefficients appearing in the summation of (2.4) are divisible by an odd prime number p when $k = p$. Also, Fermat's little theorem asserts that $\beta^p \equiv \beta \,(mod\, p)$ for $\beta \in \mathbb{Z}$. Thus we have a consequence from Proposition 2.2.

Corollary 2.3 *Let* p *be an odd prime and* $\beta_j \in \mathbb{Z}$. *Then there holds the congruence relation*

$$c(p;\beta) = c(p;\beta_0, \cdots, \beta_{p-1}) \equiv \beta_0 \,(mod\, p).$$

Of course Corollary 2.3 is also easily deduced from (2.3) since $\binom{p}{j} \equiv 0\,(mod\,p)$, $(1 \le j \le p-1)$. In what follows we state two propositions for expansion formulas involving remainder expressions.

Proposition 2.4 *(Gontscharoff) Let $f(z) \in \Gamma$ and let $\alpha_k \in \mathbb{C}$ $(k = 0, 1, \cdots, n)$. Then we have an expansion formula of the form (cf. [3])*

$$f(z) = f(\alpha_0) + \sum_{k=1}^{n} \frac{f^{(k)}(\alpha_k)}{k!} Q_k(z) + R_n(z), \tag{2.6}$$

where $Q_k(z)'s$ are given by (2.1) or (2.5), and the remainder $R_n(z)$ can be written in the form

$$R_n(z) = \int_{\alpha_0}^{z} dz' \int_{\alpha_1}^{z'} dz'' \cdots \int_{\alpha_n}^{z^{(n)}} f^{(n+1)}(z^{(n+1)}) dz^{(n+1)}. \tag{2.7}$$

Proposition 2.5 *Let $f(z) \in \Gamma$ and denote $c(k;\beta) = c(k;\beta_0, \cdots, \beta_{k-1})$ with $\beta_j \in \mathbb{C}$, $(0 \le j \le n)$, and $c(0;\beta) = 1$. Then there holds formally the $(n+1)$-point expansion formula*

$$f(z) = \sum_{k=0}^{n} \frac{c(k;\beta)}{k!} \sum_{m=0}^{\infty} \frac{f^{(k+m)}(0)}{m!} (z - \beta_k)^m + \rho_n(z), \tag{2.8}$$

where the remainder $\rho_n(z) = \rho_n(f;z)$ is given by

$$\rho_n(z) = \int_{z-\beta_0}^{z} dz' \int_{z-\beta_1}^{z'} dz'' \cdots \int_{z-\beta_n}^{z^{(n)}} f^{(n+1)}(z^{(n+1)}) dz^{(n+1)}. \tag{2.9}$$

Note that the case $\beta_0 = 0$ leads to $c(k;\beta) = 0$ $(k \ge 1)$ and $\rho_n(z) = 0$. So in this particular case (2.8) reduces to the classical Taylor-Maclaurin expansion.

By using Peano's Theorem (cf. Davis [2]), we obtain another form of the remainder for expansion (1.4).

Proposition 2.6 *Let $f(x) \in C^{n+1}[a,b]$ and denote $c(k;\beta) = c(k;\beta_0, \cdots, \beta_{k-1})$ with $\beta_j = x - \alpha_j$, $x \in \mathbb{R}$ and $\alpha_j \in [a,b]$, $(0 \le j \le n)$, and $c(0;\beta) = 1$. Then there holds formally the $(n+1)$-point expansion formula*

$$f(x) = \sum_{k=0}^{n} \frac{c(k;\beta)}{k!} \sum_{m=0}^{\infty} \frac{f^{(k+m)}(0)}{m!}(x-\beta_k)^m + \rho_n(x), \tag{2.10}$$

where the remainder $\rho_n(x) = \rho_n(f;x)$ is given by

$$\rho_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} c(k;\beta) \int_{\alpha_k}^{x} (\alpha_k - t)^{n-k} f^{(n+1)}(t)dt, \tag{2.11}$$

which is equivalent to expression (2.9) when $z = x \in \mathbb{R}$.

It is obvious that remainder shown as in Eq. (2.11) can be written as a derivative form:

$$\rho_n(x) = \frac{1}{(n+1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} c(k;\beta)(x-\alpha_k)^{n-k+1} f^{(n+1)}(\xi_{n,k}), \tag{2.12}$$

where $\xi_{n,k}$ is between x and α_k, $k = 0, 1, \cdots, n$. In addition, the remainder (2.11) also has the following expansion.

$$\begin{aligned} \rho_n(x) &= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \frac{1}{(n-i)!} \binom{n-i}{k} c(k;\beta)(\alpha_k - x)^{n-k-i} f^{(n-i)}(x) \\ &\quad - \sum_{k=0}^{n} \frac{1}{k!} c(k;\beta) f^{(k)}(\alpha_k). \end{aligned}$$

Remark 4. If all $\alpha_k = \alpha$ for $k = 0, 1, \cdots, n$, then Eq. (2.10) reduced to the Taylor expansion of f at point α, and Eqs. (2.11) and (2.12) become respectively the integral form remainder and the derivative form remainder of the expansion.

We consider the generalized Abelian series (1.4) as the Abel-Gontscharoff interpolation for the following Hermite-Birkhoff interpolation problem defined on a set of nodes $T = \{t_1, \ldots, t_m\}$ consists of m distinct points of the set A that is either an interval or a circle:

$$p^{(j)}(t_i) = c_{i,j}, \quad j = k_{i-1}, k_{i-1}+1, \cdots, k_i - 1,$$

where $k_i - k_{i-1} \in \mathbb{N}$ is the multiplicity of node t_i, $i = 1, 2, \cdots, m$, and $k_0 = 0$ and $k_m = N + 1$. Hence, the corresponding interpolation matrix (see Eq. (1.7)) is

$$E = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}, \tag{2.13}$$

where the number of $1'$s in the ith row is $k_i - k_{i-1}$ $(i = 1, 2, \cdots, m)$, i.e., the multiplicity of node t_i. Consequently, the determinant, defined as in (1.9),

$$D(E; T) = det \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ 0 & 1 & \frac{2!}{1!}t_1 & \cdots & \frac{n!}{(n-1)!}t_1^{n-1} \\ 0 & 0 & \frac{2!}{0!} & \cdots & \frac{n!}{(n-2)!}t_1^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n!}{0!} \end{pmatrix} \tag{2.14}$$

is $1!2! \cdots n!$. Therefore, we obtain the following result.

Proposition 2.7 *Abel-Gontscharoff interpolation is complex poised; it is poised with respect to all sets of distinct nodes in* $\mathbb{C}$.

3 Proof of Propositions with Remarks

Proof of Proposition 2.1. For proving Proposition 2.1, it suffices to show that the polynomial $c(k) \equiv c(k; z - \alpha_0, \cdots, z - \alpha_{k-1})$ satisfies the condition

$$\left(\frac{d}{dz}\right)^j c(k; z - \alpha_0, \cdots, z - \alpha_{k-1}) \mid_{z=\alpha_j} = j!\delta_{jk}, \tag{3.1}$$

where $0 \leq j \leq k$ and δ_{jk} is the Kronecker symbol, i.e., $\delta_{jk} = 1$ for $j = k$, and $\delta_{jk} = 0$ for $j \neq k$. In fact, (3.1) is consistent in form with (2.2) so that it must imply $c(k) = Q_k(z)$.

Taking $f(z) = z^n$ we see that (1.4) gives

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha_k)}{k!} c(k; z-\alpha_0, \cdots, z-\alpha_{k-1}).$$

This is actually an alternative form of Gould's algebraic identity involving independent variables $\alpha_0, \cdots, \alpha_n$ and z. By differentiation with respect to z we may get identities of the form

$$f^{(j)}(\alpha_j) = \sum_{k=j}^{n} \frac{f^{(k)}(\alpha_k)}{k!} \left(\frac{d}{dz}\right)^j c(k; z-\alpha_0, \cdots, z-\alpha_{k-1}) \left.\right|_{z=\alpha_j}. \quad (3.2)$$

Here the RHS of (3.2) is a linear combination of distinct functionals $f^{(j)}(\alpha_k)$. Hence, (3.1) follows from the Eq. (3.2) by comparison of its both sides. ■

Proof of Proposition 2.2. For obtaining (2.3) of Proposition 2.2, it suffices to make substitutions $z = 0$ and $\alpha_j = -\beta_j$ $(0 \le j \le k-1)$ in (2.1).

The explicit formula (2.4) is an alternative form of the second expression given by Gould [5]. Here we give a new and shorter proof for it.

Clearly the determinant (2.3) contains the term $(-1)^{k-1}\beta_0^k$ Thus what we need to determine is the coefficient for the general term

$$\beta_0^{\lambda_1}\beta_{j_1}^{\lambda_2}\cdots\beta_{j_r}^{\lambda_r} \quad (1 \le j_1 < j_2 < \cdots < j_r \le k-1),$$

where $1 \le r \le k-1$ and the exponents λ_i's are also to be determined subject to the condition $\lambda_1 + \cdots + \lambda_r = k$.

Let us rewrite the recurrence formula (1.2) in the form

$$c(k) = \sum_{j_r=0}^{k-1} \binom{k}{j_r} (-1)^{k-j_r-1} c(j_r) \beta_{j_r}^{k-j_r}. \quad (3.3)$$

Similarly, we denote

$$c(j_s) = \sum_{j_{s-1}=0}^{j_s-1} \binom{j_s}{j_{s-1}} (-1)^{j_s-j_{s-1}-1} c(j_{s-1}) \beta_{j_{s-1}}^{j_s-j_{s-1}}, \quad (3.4)$$

where $s = 1, \cdots, r$. Then substituting the equations given by (3.4) into (3.3) successively, we may obtain the coefficient for the general term $\beta_{j_0}^{j_1}\beta_{j_1}^{j_2-j_1}\cdots\beta_{j_r}^{k-j_r}$ with $j_0 = 0$, namely

$$\begin{aligned}&\binom{k}{j_r}(-1)^{k-j_r-1}\Pi_{s=1}^{r}\binom{j_s}{j_{s-1}}(-1)^{j_s-j_{s-1}-1}\\ = \quad & \frac{(-1)^{k-1-r}k!}{j_1!(j_2-j_1)!\cdots(j_r-j_{r-1})!(k-j_r)!}.\end{aligned}$$

Hence we have the complete expression as given by (2.4). ■

Remark 5. An alternative proof of formula (2.3) in Proposition 2.2 can be done as follows: From (1.1), by substituting $n = 0, 1, \cdots, k$ successively and $x = 0$, we obtain system

$$\begin{cases} \binom{1}{1}c(1)(-\beta_1)^0 = -(-\beta_0) \\ \binom{2}{1}c(1)(-\beta_1) + \binom{2}{2}c(2)(-\beta_2)^0 = -(-\beta_0)^2 \\ \vdots \\ \binom{k}{1}c(1)(-\beta_1)^{k-1} + \binom{k}{2}(-\beta_2)^{k-2} + \cdots + \binom{k}{k-1}c(k-1)(-\beta_{k-1}) \\ \qquad\qquad\qquad\qquad + \binom{k}{k}c(k)(-\beta_k)^0 = -(-\beta_0)^k. \end{cases}$$

We note that the determinant of the coefficient matrix is 1, thus the above system in $c(1)$, $c(2)$, $\cdots$, $c(k)$ can be solved by Cramer's rule, and the formula (2.3) follows.

Remark 6. It may be a good exercise to show that formula (2.4) could also be obtained by means of a suitable expansion of the determinant in (2.3). We can also use formula (2.4) to evaluate the determinant in (2.3). For instance, if $\beta_0 = \cdots = \beta_{k-1} = 1$, we obtain that the corresponding determinant (without $(-1)^k$) equals to

$$-1 + \sum_{(j)} \frac{(-1)^{r+1}k!}{j_1!(j_2-j_1)!\cdots(k-j_r)!}.$$

In addition, some identities can be derived from (2.4) (see Section 4 and [7]).

Proof of Proposition 2.4. Proposition 2.4 is easily proved by starting with (2.5) and making use of (2.5). Indeed, $R_n(z)$ can be rewritten as follows.

$$\begin{aligned} R_n(z) &= \int_{\alpha_0}^{z} dz' \cdots \int_{\alpha_{n-1}}^{z^{(n-1)}} dz^{(n)} \left[f^{(n)}(z^{(n)}) - f^{(n)}(\alpha_n) \right] \\ &= R_{n-1}(z) - Q_n(z) \frac{f^{(n)}(\alpha_n)}{n!} = \cdots \\ &= R_0(z) - \sum_{k=1}^{n} \frac{f^{(k)}(\alpha_k)}{k!} Q_k(z). \end{aligned}$$

Note that $R_0(z) = f(z) - f(\alpha_0)$, so that (2.6) is inferred from (2.7). ■

Remark 7. Both (2.6) and (2.7) were constructed and investigated in details by Gontscharoff before 1930's. The RHS of (2.6) without the remainder term R_n is known as the classical Gontscharoff interpolation polynomial of degree n. Clearly, the condition $R_n(z) \to 0$ $(n \to \infty)$ implies the convergence of Abel-Gontscharoff interpolation series (1.4). Proposition 2.6 shows another form of the remainder. For more about the convergence problems the reader may be referred to the earlier work of Gontscharoff [4].

Proof of Proposition 2.5. It suffices to observe that for $f \in \Gamma$ the RHS of (2.8) can be expressed in the form

$$\sum_{k=0}^{n} \frac{c(k;\beta)}{k!} f^{(k)}(z - \beta_k) + \rho_n(z).$$

Clearly, this is equivalent to the RHS of (2.6) with the substitutions $\beta_j = z - \alpha_j$. This shows that (2.8) and (2.9) just follow from (2.6) and (2.7), i.e., Proposition 2.4. ■

Remark 8. Worth noticing is that (2.6) of Proposition 2.5 is merely a consequence from the repeated integration, (2.7), of the formal derivative $f^{(n+1)}(\cdot)$, so that (2.6) is just an operational identity implied by the fundamental theorem of integral calculus (Newton-Leibniz fundamental formula). Recall that the generalized differentiation (giving Fréchet derivatives) and integration defined usually in the Nonlinear Functional Analysis are all in consistence with the ordinary concepts and also satisfy Newton-Leibniz formula (in general terms). So it is clear that (2.6)-(2.7) with $Q_k(z)$ defined by (2.5) could be correspondingly extended to the case where $f^{(n+1)} \in (X \to \mathbb{C})$ with X denoting a Banach space, or in other words, $f^{(n+1)}$ is an $(n+1)$th order Fréchet derivative defined on a Banach space X and taking values in $\mathbb{C}$. Surely, a generalization of

Proposition 2.4 can be precisely formulated with the Nonliear Functional Analysis.

Proof of Proposition 2.6. Denote functional

$$L(f) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(\alpha_k)}{k!} c(k;\beta), \tag{3.5}$$

where $c(k;\beta) = c(k;\beta_1, \cdots, \beta_{k-1})$, $\beta_j = x - \alpha_j$ $(j \geq 0)$. It is obvious that $L(x^k) = 0$ for all integers $k = 0, 1, \cdots, n$. Thus, from the Peano theorem (see e.g., Davis [2]), for all $f \in C^{n+1}[a,b]$, $\{\alpha_k\}_{k=0}^n \subset [a,b]$,

$$L(f) = \int_a^b f^{(n+1)}(t) K(t) dt, \tag{3.6}$$

where

$$K(t) = \frac{1}{n!} L_x \left[(x-t)_+^n\right] \tag{3.7}$$

and

$$(x-t)_+^n = \begin{cases} (x-t)^n & \text{if } x \geq t, \\ 0 & \text{if } x < t. \end{cases} \tag{3.8}$$

The notation $L_x \left[(x-t)_+^n\right]$ means that the functional L is applied to $(x-t)_+^n$ considered as a function of x. For $x \geq t$, we have

$$\begin{aligned}
& n! K(t) = L_x \left[(x-t)_+^n\right] \\
= \ & (x-t)^n - \sum_{k=0}^{n} \frac{\left[(x-t)_+^n\right]^{(k)} |_{x=\alpha_k}}{k!} c(k;\beta) \\
= \ & \sum_{k=0}^{n} \binom{n}{k} (x - t - \beta_k)^{n-k} c(k;\beta) - \sum_{k=0}^{n} \frac{\left[(x-t)_+^n\right]^{(k)} |_{x=\alpha_k}}{k!} c(k;\beta) \\
= \ & \sum_{k=0}^{n} \left[(\alpha_k - t)^{n-k} - \frac{(n-k)!}{n!} \left[(x-t)_+^n\right]^{(k)} |_{x=\alpha_k} \right] \binom{n}{k} c(k;\beta).
\end{aligned} \tag{3.9}$$

Therefore,

$$
\begin{aligned}
& n!\int_a^b f^{(n+1)}(t)K(t)dt \\
= & \int_a^b L_x\left[(x-t)_+^n\right] f^{(n+1)}(t)dt \\
= & \int_a^x \left[(x-t)^n - \sum_{k=0}^n \frac{[(x-t)^n]^{(k)}\mid_{x=\alpha_k}}{k!}c(k;\beta)\right] f^{(n+1)}(t)dt \\
& + \sum_{k=0}^n \int_{\alpha_k}^x \frac{[(x-t)^n]^{(k)}\mid_{x=\alpha_k}}{k!}c(k;\beta)f^{(n+1)}(t)dt \\
= & \sum_{k=0}^n \int_a^x \left[(\alpha_k - t)^{(n-k)} - \frac{(n-k)!}{n!}\left[(x-t)^n\right]^{(k)}\mid_{x=\alpha_k}\right]\binom{n}{k} \\
& \times c(k;\beta)f^{(n+1)}(t)dt + \sum_{k=0}^n \int_{\alpha_k}^x \binom{n}{k}(\alpha_k - t)^{n-k}c(k;\beta)f^{(n+1)}(t)dt.
\end{aligned}
$$

Since the first term on the right-hand side of the last equation equals to zero, we obtain

$$
\begin{aligned}
L(f) &= \int_a^b f^{(n+1)}(t)K(t)dt \\
&= \frac{1}{n!}\sum_{k=0}^n \binom{n}{k} c(k;\beta)\int_{\alpha_k}^x (\alpha_k - t)^{n-k} f^{(n+1)}(t)dt.
\end{aligned}
$$

Hence, Eqs. (2.10) and (2.11) are established.

We can also prove Eqs. (2.10) and (2.11) more directly as follows. First, we have

$$\begin{aligned}
\rho_n(x) &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}c(k;\beta)\int_{\alpha_k}^{x}(\alpha_k-t)^{n-k}f^{(n+1)}(t)dt \\
&= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}c(k;\beta)\left[(\alpha_k-t)^{n-k}f^{(n)}(t)\mid_{\alpha_k}^{x}\right. \\
&\quad \left.+(n-k)\int_{\alpha_k}^{x}(\alpha_k-t)^{n-k-1}f^{(n)}(t)dt\right] \\
&= \frac{f^{(n)}(x)}{n!}\sum_{k=0}^{n}\binom{n}{k}c(k;\beta)(\alpha_k-x)^{n-k}-\frac{f^{(n)}(\alpha_n)}{n!}c(n;\beta) \\
&\quad +\frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}c(k;\beta)\int_{\alpha_k}^{x}(\alpha_k-t)^{n-k-1}f^{(n)}(t)dt \\
&= \rho_{n-1}(x)-\frac{f^{(n)}(\alpha_n)}{n!}c(n;\beta),
\end{aligned}$$

where the last step is due to

$$\sum_{k=0}^{n}\binom{n}{k}c(k;\beta)(\alpha_k-y)^{n-k}=0,$$

which is a special case of (1.1) with $x=0$ and $\beta_k=y-\alpha_k$. Repeating the above process yields

$$\begin{aligned}
\rho_n(x) &= \rho_0(x)-\sum_{k=1}^{n}\frac{f^{(k)}(\alpha_k)}{k!}c(k;\beta) \\
&= \int_{\alpha_0}^{x}f'(t)dt-\sum_{k=1}^{n}\frac{f^{(k)}(\alpha_k)}{k!}c(k;\beta) \\
&= f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(\alpha_k)}{k!}c(k;\beta).
\end{aligned}$$

Noting the proof of Proposition 2.5, we infer (2.10) from (2.11). In addition, by using mathematical induction, we can obtain the equivalence of remainders (2.9) and (2.11) for all $z=x\in\mathbb{R}$. ∎

4 Illustrative Examples

Using the notation for multinomial coefficient

$$\binom{k}{x_1, \cdots, x_r} = \frac{k!}{x_1! \cdots x_r!}, \quad (k = x_1 + \cdots + x_r)$$

we rewrite the expression (2.4) for $c(k;\beta) = c(k;\beta_0, \cdots, \beta_{k-1})$ in the form

$$c(k;\beta) = \sum_{(k;r\geq 1)} (-1)^{k+r} \binom{k}{x_1, \cdots, x_r} \beta_0^{x_1} \beta_{x_1}^{x_2} \beta_{x_1+x_2}^{x_3} \cdots \beta_{x_1+\cdots+x_{r-1}}^{x_r}, \tag{4.1}$$

where the summation is taken over the set, denoted by $(k, r \geq 1)$, of all positive integer compositions $(x_1, \cdots, x_r)$ of k into r parts with $r \geq 1$, or in other words, over all the positive integer solutions of the equation $x_1 + \cdots + x_r = k$ for $r = 1, \cdots, k$.

Example 1. For any number $z, t \in \mathbb{C}$, we have the identity

$$\sum_{(k;r\geq 1)} (-1)^r \binom{k}{x_1, \cdots, x_r} z^{x_1} (z + x_1 t)^{x_2} \cdots (z + (x_1 + \cdots + x_{r-1})t)^{x_r} = (-1)^k z(z + kt)^{k-1}. \tag{4.2}$$

Actually this is a consequence of (4.1) for the case $\beta_k = z + kt$ which leads to $c(k;\beta) = z(z + kt)^{k-1}$ (cf. (1.3)). In particular, two special identities may be worth mentioning, namely

$$\sum_{(k;r\geq 1)} (-1)^r \binom{k}{x_1, \cdots, x_r} = (-1)^k \tag{4.3}$$

and

$$\sum_{(k;r\geq 1)} (-1)^r \binom{k}{x_1, \cdots, x_r} (x_1 + 1)^{x_2} \cdots (x_1 + \cdots + x_{r-1} + 1)^{x_r} = (-1)^k (k + 1)^{k-1}. \tag{4.4}$$

Clearly, these identities follow from (4.2) by taking $z = 1$, $t = 0$ and $z = t = 1$, respectively. In fact, both (4.3) and (4.4) have been noted previously, see, e.g. [5] and [7].

Surely the Abel series expansion (1.6) is an important particular case of the expansion (1.4), of which some convergence conditions could be investigated via the remainder formula (2.7) due to Gontscharoff (cf. [4]). As regards some intensive study of the convergence problem for (1.6), we have to mention two rather earlier papers by G. H. Halphen [6] (1882) and S. Pincherle [10] (1904), respectively.

In what follows we will only consider real functions $f(x)$ defined on $\mathbb{R}$ and always assume that $f(x)$ is infinitely differentiable. What we would like to show in the next example is that a simple use of Stirling's formula for large $n!$ will yield an easily available condition of convergence for Abel's series (considered as a generalization of Taylor's expansion).

Example 2. For given $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $b \neq 0$, we have the convergent Abel series expansion (cf. (1.6))

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a+bk)}{k!}(x-a)(x-a-bk)^{k-1} \tag{4.5}$$

provided the following order condition is satisfied

$$f^{(k)}(a+bk) = O\left((|b|e)^{-k}\right), \quad (k \to \infty). \tag{4.6}$$

Note that for $\beta_k = x - a - bk$ we have

$$c(k;\beta) = c(k; x-a, \cdots, x-a-(k-1)b) = (x-a)(x-a-bk)^{k-1}.$$

Thus (4.5) follows from (1.4) with $\alpha_k = a + bk$. Now, using Stirling's formula for large $k!$ we easily find the following asymptotic estimates

$$\begin{aligned} & \frac{(x-a-bk)^{k-1}}{k!} \\ \sim\ & (-1)^{k-1}(bk)^{k-1}\left(1+\frac{a-x}{bk}\right)^{k-1} \Big/ \left(\left(\frac{k}{e}\right)^k \sqrt{2\pi k}\right) \\ \sim\ & (-1)^{k-1}(be)^k e^{(a-x)/b} \big/ \left(bk^{3/2}(2\pi)^{1/2}\right) \\ =\ & O\left(k^{-3/2}(|b|e)^k\right). \end{aligned}$$

Thus it is clear that, for every given x, condition (4.6) suffices to ensure the absolute convergence of the series (4.5).

As immediate consequences of (4.5)-(4.6) we have some special examples as follows.

(i) Letting $f(x) = e^{-x}$, $(x \in \mathbb{R})$, and $b > 0$, we have

$$f^{(k)}(a+bk) = (-1)^k e^{-a-bk} = O\left(e^{-bk}\right) = O\left((be)^{-k}\right), \quad (k \to \infty),$$

where the elementary inequality $ex \leq e^x$, $0 < x < \infty$, is utilized. Thus (4.5)-(4.6) imply the convergent series expansion

$$e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-a-bk}}{k!}(x-a)(x-a-bk)^{k-1}. \tag{4.7}$$

In particular, putting $x = 0$ we get the well-known identity

$$\sum_{k=0}^{\infty} \frac{a(a+bk)^{k-1}}{k!} e^{-(a+bk)} = 1, \tag{4.8}$$

where $a \neq 0$ and $b \geq 0$. Similar procedures could yield the following instances (some details are here omitted).

(ii) Given $f(x) = \ln(1+x)$, $x > -1$, and $a > -1$ and $b > 0$. We have the convergent expansions

$$\ln(1+x) = \ln(1+a) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-a)(x-a-bk)^{k-1}}{k(1+a+bk)^k} \tag{4.9}$$

and

$$\ln(1+a) = \sum_{k=1}^{\infty} \frac{a(a+bk)^{k-1}}{k(1+a+bk)^k} \tag{4.10}$$

(iii) Given $f(x) = \sin x$ or $f(x) = \cos x$, $x \in \mathbb{R}$, and suppose that $a \in \mathbb{R}$ and $|b| \leq 1/e$. Then we have the convergent expansions

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{k!} \sin\left(a + bk + \frac{k\pi}{2}\right)(x-a)(x-a-bk)^{k-1}, \tag{4.11}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{1}{k!} \cos\left(a + bk + \frac{k\pi}{2}\right)(x-a)(x-a-bk)^{k-1}, \tag{4.12}$$

and

$$\frac{\sin x}{x} = \sum_{k=1}^{\infty} \frac{1}{k!} \sin k \left(b + \frac{\pi}{2}\right) (x - bk)^{k-1}, \quad x \neq 0. \tag{4.13}$$

In fact, it is easily verified that (4.11)-(4.13) are all uniformly convergent for x in any finite interval including 0. Thus putting $x = 0$ in (4.12) and letting $x \to 0$ in (4.13) yields a pair of identities

$$\sum_{k=0}^{\infty} \frac{a(a+bk)^{k-1}}{k!} \cos\left(a + bk - \frac{k\pi}{2}\right) = 1 \tag{4.14}$$

and

$$\sum_{k=1}^{\infty} \frac{(k\alpha)^{k-1}}{k!} \sin k\left(\frac{\pi}{2} - \alpha\right) = 1, \quad 0 < \alpha \leq 1/e. \tag{4.15}$$

These identities are comparable with (4.8). In particular, (4.15) implies the following special identity with $\alpha = \pi/10$.

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{k\pi}{10}\right)^{k-1} \sin \frac{2k\pi}{5} = 1. \tag{4.16}$$

We have mentioned that (4.5)-(4.6) could be also applied to some hypergeometric functions under suitable conditions for parameters involved.

As may be observed the remainder formulas (2.7) and (2.9) become particularly useful when $\{\alpha_j\}$ and $\{\beta_j\}$ are decreasing and increasing sequences, respectively. More precisely, we have the following examples.
Example 3. Let $\{\alpha_j\}$ be a nonincreasing sequence, i.e., $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq \cdots$. Then for any given $x > \alpha_0$ we have the convergent series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha_k)}{k!} Q_k(x), \tag{4.17}$$

involving Abel-Gontscharoff-Gould's polynomials $Q_k(x) = c(k; x - \alpha_0, \cdots, x - \alpha_{k-1})$'s given by (2.1) with $z = x$, provided that the following condition is satisfied

$$\lim_{n\to\infty} \frac{(x-\alpha_n)^{n+1}}{(n+1)!} \max_{x\geq t\geq \alpha_n} \left|f^{(n+1)}(t)\right| = 0. \tag{4.18}$$

In particular, if $f(x)$ is defined on a finite interval (a,b), and if $\alpha_k \in (a,b)$ with $\alpha_k \geq \alpha_{k+1}$ $(k=0,1,2,\cdots)$, then for $x \in (a,b)$ with $x > \alpha_0$ we have the convergent series (4.17) provided that

$$\lim_{n\to\infty} \frac{(x-b)^{n+1}}{(n+1)!} \left\|f^{(n+1)}\right\|_{(a,b)} = 0, \tag{4.19}$$

where $\|f^{(n+1)}\|_{(a,b)} = \max\limits_{a\leq t\leq b} |f^{(n+1)}(t)|$.

It may be observed that for the nonincreasing sequence $\{\alpha_k\}$ and given $x > \alpha_0$ we have the inequalities

$$0 < \int_{\alpha_0}^{x} dx' \int_{\alpha_1}^{x'} dx'' \cdots \int_{\alpha_n}^{x^{(n)}} dx^{(n+1)} \leq \frac{1}{(n+1)!}(x-\alpha_n)^{n+1}, \tag{4.20}$$

where $n = 0,1,2,\cdots$. Indeed, (4.20) is easily verified by induction on n. Consequently, the remainder term given by (2.7) with $z = x$ has the estimate

$$0 \leq |R_n(x)| \leq \max_{\alpha_n\leq t\leq x} \left|f^{(n+1)}(t)\right| \frac{(x-\alpha_n)^{n+1}}{(n+1)!}.$$

Hence, we see that condition (4.18) implies the convergence of the series in (4.17).

Similarly, as a consequence of Proposition 2.5, we state the following example.

Example 4. Let $\{\beta_k\}$ be a nondecreasing sequence, i.e., $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \cdots$. Then for any given entire function $f(z)$ we have the convergent series expansion for $z = x \in \mathbb{R}$ of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{c(k;\beta)}{k!} \sum_{m=0}^{\infty} \frac{f^{(k+m)}(0)}{m!}(x-\beta_k)^m \tag{4.21}$$

provided that $\lim\limits_{n\to\infty} \beta_n > 0$ and that

$$\lim_{n\to\infty} \frac{\beta_n^{n+1}}{(n+1)!} \max_{x-\beta_n\leq t\leq x} \left|f^{(n+1)}(t)\right| = 0. \tag{4.22}$$

Clearly, condition (4.22) can be inferred from (2.9) and (4.20) with $\alpha_k = x - \beta_k$. In particular, if $\lim_{n\to\infty} \beta_n = M$, then (4.22) can be replaced by

$$\lim_{n\to\infty} \frac{M^{n+1}}{(n+1)!} \max_{x-M\le t\le x} \left|f^{(n+1)}(t)\right| = 0. \tag{4.23}$$

Particular instances are easily constructed to illustrate Examples 3 and 4. Here we just take $f(x) = \sin x$ as a very simple instance serving for illustration.

(iv) Given $\alpha_k = a - bk$, $(b > 0)$. Note that

$$\left|f^{(k)}(x)\right| = \left|\sin\left(x + \frac{k\pi}{2}\right)\right| \le 1.$$

It is easily found via Stirling's formula (see Example 2) that

$$\frac{(x-a+bn)^{(n+1)}}{(n+1)!} = O\left(\frac{1}{\sqrt{n}}(be)^{n+1}\right), \quad \text{as } n \to \infty.$$

Thus, it is clear that condition (4.18) will be satisfied by taking b such that $0 < be \le 1$, i.e., $0 < b \le 1/e$. This is essentially the same condition as that given in (iii), so that we arrive at (4.11) and (4.13) again, replacing b by $-b$.

(v) Applying (4.21)-(4.22) to $f(x) = \sin x$, we obtain the convergent series expansion

$$\sin x = \sum_{k=0}^{\infty} \frac{c(k;\beta)}{k!} \sum_{m=0}^{\infty} \frac{\sin\left(\frac{\pi}{2}(k+m)\right)}{m!} (x - \beta_k)^m \tag{4.24}$$

provided that $\{\beta_k\}$ is a nondecreasing sequence such that $\lim_{n\to\infty} \beta_n > 0$ and that

$$\lim_{n\to\infty} \frac{\beta_n^{n+1}}{(n+1)!} = 0. \tag{4.25}$$

Finally, let us add an example which is actually deducible from Example 2 by verifying condition (4.6).

Example 5. In (4.5) taking $f(x) = (1+x)^{\lambda}$ $(0 < \lambda < 1)$ and putting $a = 0$, we obtain Abel's generalized binomial series expansion

$$(1+x)^{\lambda} = \sum_{k=0}^{\infty} \binom{\lambda}{k} (1+bk)^{\lambda-k} x(x-bk)^{k-1}, \tag{4.26}$$

where $|x| \le 1$. To establish the convergence of the series, let us note that Stirling's asymptotic formula for gamma function $\Gamma(k-\lambda)$ (k being large) is available for getting the order estimate (with $(\lambda)_k = \lambda(\lambda - 1)\cdots(\lambda - k + 1)$)

$$\left|f^{(k)}(bk)\right| = \left|(\lambda)_k (1+bk)^{\lambda-k}\right| = O\left(\frac{1}{\sqrt{k}} \left(|b|e\right)^{-k}\right). \tag{4.27}$$

Actually, the verification of (4.27) involves only routine analysis and may be here omitted. Therefore, condition (4.6) is fulfilled, and we see that the series (4.26) is convergent for every $b \in \mathbb{R}$.

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