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Simple Economies with Multiple Equilibria*

Theodore C. Bergstrom, Ken-Ichi Shimomura, and Takehiko Yamato

Abstract

We study a general class of pure exchange economies that have multiple equilibria. This class generalizes an example presented by Shapley and Shubik. For such economies, we find easily verified conditions that determine whether there are multiple equilibria. We also provide simple methods for constructing economies in which arbitrary pre-specified sets of prices are equilibria. These economies have strong comparative statics properties, since prices at interior competitive equilibrium depend on the parameters of utility but not on the endowment quantities. We believe that this easily manipulated special case is a valuable addition to the class of simple general equilibrium economies that can be used as testing grounds in economic theory.

KEYWORDS: competitive equilibrium, multiple equilibria, comparative statics, quasi-linear utility, Shapley-Shubik economy

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1 Introduction

Lloyd Shapley and Martin Shubik (1977) used a clever trick to construct a simple exchange economy that has three distinct competitive equilibria. There are two consumers and two goods. Both consumers have quasi-linear utility functions, but their utilities are linear in opposite goods, and each consumer's initial endowment includes only the good in which his own utility is linear. This trick works because strong income effects act in the opposite direction from substitution effects for both consumers, and consequently the aggregate excess demand function is increasing in price over part of its domain.

The Shapley-Shubik paper consists of a single specific example, in which the nonlinear parts of the utility functions are exponential. Although their example does not admit a closed form solution, Shapley and Shubik exhibit numerical solutions for three distinct competitive equilibria.

In this paper, we explore several families of two-person[1] economies of the Shapley-Shubik type, where the nonlinear portions of utility functions take alternative special forms. We find general conditions that are necessary and sufficient for the existence of multiple equilibria and we show how to solve for equilibrium prices, given specified utility functions.

In order to create examples to illustrate a theoretical point or to use in an experiment, it is desirable to "work backwards" from desired equilbrium prices to parameters of the utility function. To do this, one chooses a set of distinct prices that are to be the equilibria and then solves for parameters of the utility functions for which these prices are the competitive equilibria. This is often much easier than finding the equilibrium prices corresponding to pre-specified parameters.

We believe that Shapley-Shubik economies are a useful addition to the collection of tractable, but non-trivial special-case economies for which general equilibrium comparative statics is relatively easy. This collection includes economies with Cobb-Douglas utilities, economies with standard quasi-linear utility, and economies with production satisfying the conditions of Samuelson's non-substitution theorem. Such special economies provide instructive testing grounds on which to explore theories that apply in more general environments. While the other three special theories imply unique competitive equilibrium prices, Shapley-Shubik economies allow study of behavior in economies with multiple equilibria.

Other papers that have explored the problem of equilibrium selection in

[1]This discussion applies equally well to "replica economies" with two types of consumers and equal numbers of each type.

economies with multiple equilibria include Huber, Shubik, and Sunder (2009) and Shimomura and Yamato (2009). Gjerstad (1996) constructs examples of relatively simple economies with multiple equilibria in which consumers have differing homothetic constant-elasticity-of-substitution utility functions. Kumar and Shubik (2003) use computational methods to explore the geometry of the core as the number of replicas of the two participants in the Shapley-Shubik example is increased.

2 Shapley-Shubik Economies

We define the family of *Shapley-Shubik economies* as follows. There are two consumers and two goods. Consumer 1 is endowed with a positive amount $\bar{x}$ of good X and zero units of good Y. Consumer 2 is endowed with a positive amount $\bar{y}$ of Y and zero units of X. Consumers 1 and 2 have utility functions

$$\begin{aligned} U_1(x, y) &= x + f_1(y) \\ U_2(x, y) &= f_2(x) + y, \end{aligned} \tag{1}$$

where the $f_i(\cdot)$ are strictly concave, continuously differentiable functions from $[0, \infty)$ to the extended real line and where $\lim_{z\to\infty} f_i'(z) \leq 0$. For $i = 1, 2$, let $B_i = \lim_{z\to 0} f_i'(z)$.[2]

Excess demand depends only on relative prices. Therefore we can make good X the *numeraire* and set its price equal to one. Define $x_i(p)$ and $y_i(p)$ to be consumer i's demands for goods X and Y when the price of X is 1 and the price of Y is p. In a two-commodity economy, Walras' law implies that if excess demand is zero in one market it will also be zero in the other. Therefore if we denote excess demand for good Y by $E(p)$, a price $\bar{p}$ will be an equilibrium price if and only if $E(\bar{p}) = 0$. An equilibrium price $\bar{p}$ will be stable or unstable under the usual competitive dynamics, depending on whether $E'(\bar{p}) < 0$ or $E'(\bar{p}) > 0$.

Since $f_i'(\cdot)$ is a strictly decreasing function, we can define the *inverse marginal utility function* $\phi_i(p) = f_i'^{-1}(p)$ on $(0, B_i]$.[3] If B_i is finite, it must be that $\phi_i(B_i) = 0$ and for all $p \in (0, B_i)$, $\phi_i(p) > 0$ and $\phi_i'(p) < 0$.

[2]We will consider Shapley-Shubik economies in which $B_i = f_i'(0)$ is finite and positive as well cases in which $f_i'(z)$ becomes arbitrarily large as z becomes small, in which case $B_i = \infty$.

[3]Conversely, if $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are positive valued strictly decreasing functions, we can recover utility functions $f_i(\cdot)$ such that $\phi_i(p) = f_i'^{-1}(p)$.

Consumer 1's budget constraint is

$$x_1 + py_1 = \bar{x}. \tag{2}$$

If Consumer 1 demands a positive amount of both goods at price p, her marginal rate of substitution between y and x must be equal to p. This implies that $f_1'(y_1(p)) = p$ and hence

$$y_1(p) = f_1'^{-1}(p) = \phi_1(p). \tag{3}$$

Consumer 2's budget equation is

$$x_2 + py_2 = p\bar{y}. \tag{4}$$

At an interior solution for Consumer 2, it must be that $f_2'(x_2(p)) = 1/p$ and hence

$$x_2(p) = f_2'^{-1}(\frac{1}{p}) = \phi_2(\frac{1}{p}). \tag{5}$$

If $x_2(p)$ is given by Equation 5, then Consumer 2's budget equation implies that

$$y_2(p) - \bar{y} = -\frac{1}{p}x_2(p) = -\frac{1}{p}\phi_2(1/p) \tag{6}$$

From Equations 3 and 6, it follows that: if both consumers choose positive quantities of both goods at price p, then excess demand for Y at price p is

$$E(p) = \phi_1(p) - (1/p)\phi_2(1/p). \tag{7}$$

Definition 1. *Consumer i's demands are* interior at price p *if i demands positive amounts of both goods at price p. A price p is an* interior competitive equlibrium price *if demands of both consumers are interior at price p and if $E(p) = 0$.*

From Equation 7 and the definition of interior competitive equilibrium, it is immediate that:

Remark 1. *The price p is an interior competitive equilibrium price if and only if demands of both consumers are interior at p and*

$$\phi_1(p) = \frac{1}{p}\phi_2(1/p). \tag{8}$$

A consumer's demand can fail to be interior in two different ways. One possibility is that he demands none of the good in which his utility is nonlinear. If $B_1 = \infty$, then at every price p, Consumer 1 will buy some positive amount of Y. But if $B_1 = f_1'(0)$ is finite, then at any price $p \geq B_i$, Consumer 1 will buy none of Y. Consumer 2's utility is nonlinear in X and the price of X relative to Y is $1/p$. If $B_2 = \infty$, Consumer 2 will buy a positive amount of X at any price. If $B_2 = f_2'(0)$ is finite, then Consumer 2 will demand none of X if $B_2 \leq 1/p$. The lowest price at which a consumer demands none of a good is commonly known as its "choke price."

Consumer 1's demand is linear in X and he will demand a positive amount of X only if after purchasing $\phi(p)$ units of Y at price p, he has income left to spend on X. Since Consumer 1's income is $\bar{x}$, this is the case if $p\phi(p) < \bar{x}$. Consumer 2's demand is linear in Y and he consumes positive amounts of Good Y at price p only if after purchasing $\phi(1/p)$ units of X, he has income left to spend on Y. Since Consumer 2's income is $p\bar{y}$, this is the case if $\phi_2(1/p) < p\bar{y}$.

These considerations motivate the following *interiority conditions*:

Definition 2. *For a Shapley-Shubik economy, the* interiority conditions *are satisfied at $p > 0$ if $1/B_2 < p < B_1$, $p\phi_1(p) < \bar{x}$, and $\phi_2(1/p) < p\bar{y}$.*

and the following remark:

Remark 2. *In a Shapley-Shubik economy, demands of both consumers are interior at price p if and only if the interiority conditions of Definition 2 are satisfied at p.*

We have the following result:

Lemma 1. *In a Shapley-Shubik economy where $B_i = \lim_{z \to 0} f_i'(z)$ for $i = 1, 2$ and where $B_1 > 1/B_2$, excess demand for Y is negative for all prices $p \geq B_1$, and positive at all prices $p \leq 1/B_2$.*

Proof. Let $E_i(p)$ denote excess demand for Y by Consumer i. For all prices $p \geq B_1$, $E_1(p) = 0$. Consumer 2's budget is $x_2 + py_2 = p\bar{y}$. Rearranging the budget constraint we see that $E_2(p) = -x_2(p)/p$, where $x_2(p)$ is the amount of good 2 demanded by Consumer 2. Therefore $E_2(p)$ will be negative if $x_2(p)$ is positive. Consumer 2 will demand a positive amount of X so long as $1/p < B_2$. But $p \geq B_1 > 1/B_2$ and hence $1/p < B_2$. It follows that $x_2(p) > 0$ and hence $E_2(p) < 0$. Therefore $E(p) = E_1(p) + E_2(p) < 0$ for all $p \geq B_1$. A similar argument shows that if $p \leq 1/B_2$, then $E_2(p) = 0$ and $E_1(p) > 0$. Therefore $E(p) > 0$ for all $p \leq 1/B_2$.

□

Applying Lemma 1, we can show that if the choke prices B_1 and B_2 are finite and if initial endowments are sufficiently large, then there exists an interior competitive equilibrium.

Theorem 1. *In a Shapley-Shubik economy where $B_1 = f_1'(0)$ and $B_2 = f_2'(0)$ are finite and $B_1 > 1/B_2$, there exists at least one competitive equilibrium $\bar{p} \in (1/B_2, B_1)$. The price $\bar{p}$ is an interior competitive equilibrium if and only if $\bar{p}\phi_1(\bar{p}) < \bar{x}$, $\phi_2(1/\bar{p}) < \bar{p}\bar{y}$, and $\phi_1(\bar{p}) = (1/\bar{p})\phi_2(1/\bar{p})$.*

Proof. In a Shapley-Shubik economy, utility functions are strictly quasi-concave and continuous, and incomes are positive whenever prices are positive. Therefore, by a well-known result on the continuity of demand functions, the excess demand function $E(p)$ is continuous over $(1/B_2, B_1)$. According to Lemma 1, $E(p) > 0$ for all $p \leq 1/B_2$ and $E(p) < 0$ for all $p \geq B_1$. Continuity implies that $E(p) = 0$ for some $\bar{p} \in (1/B_2, B_1)$.

□

Shapley-Shubik economies yield a strong comparative statics result that is polar to Paul Samuelson's Non-substitution Theorem Samuelson (1951). Samuelson constructs an economy in which commodities are produced from other commodities with constant returns to scale, with only one non-produced input. In Samuelson's economy, competitive equilibrium prices are determined by the "supply-side" alone, and are not affected by changes in the demand functions. For a Shapley-Shubik economy, interior competitive equilibrium prices are determined by the "demand-side" alone and are not affected by changes in the supply of Goods X and Y so long as endowments remain sufficiently large to satisfy the interiority conditions. The following "neutrality theorem" follows from Remark 1 and Theorem 1:

Theorem 2. *If $\bar{p}$ is an interior competitive equilibrium price for a Shapley-Shubik economy with initial endowments $\bar{x}$ and $\bar{y}$, then $\bar{p}$ will also be an interior competitive equilibrium price for endowments x and y, so long as $\bar{p}\phi_1(\bar{p}) < x$ and $\phi_2(1/\bar{p}) < \bar{p}y$.*

2.1 Mirror-symmetric Shapley-Shubik Economies

We define a *mirror-symmetric* Shapley-Shubik economy as one in which $U_1(x, y) = U_2(y, x)$ for all relevant (x, y) and where $\bar{x} = \bar{y}$. This requires that $f_1(x) = f_2(x) = f(x)$ for some function f. The consumers' utility functions are:

$$\begin{aligned} U_1(x, y) &= x + f(y) \\ U_2(x, y) &= f(x) + y. \end{aligned} \tag{9}$$

Let $B = \lim_{z \to} f'(0)$ and let $\phi_1(p) = \phi_2(p) = \phi(p)$ for all $p \in (0, B]$.

Remark 3. *In a mirror-symmetric Shapley-Shubik economy where $B > 1$ and $\phi(1) < \min\{\bar{x}, \bar{y}\}$, there is an interior competitive equilibrium at price $p = 1$.*

Proof. When $p = 1$, $E(p) = \phi(p) - (1/p)\phi(1/p) = \phi(1) - \phi(1) = 0$. From Theorem 1, it follows that $p = 1$ will be an interior competitive equilibrium price if the interiority conditions are satisfied. We see that $1/B < 1 < B$. The remaining interiority conditions are satisfied if $\phi(1) < \min\{\bar{x}, \bar{y}\}$. □

Notice also that in a mirror-symmetric Shapley-Shubik economy, $E(1/p) = \phi(1/p) - p\phi(p) = -pE(p)$. Therefore $E(p) = 0$ if and only if $E(1/p) = 0$, and so:

Remark 4. *In a mirror-symmetric Shapley-Shubik economy, if p is an interior competitive equilibrium and if the interiority conditions are satisfied at $1/p$, then $1/p$ is also an interior competitive equilibrium.*

A simple mathematical result allows us to identify mirror-symmetric economies that have multiple equilibria.

Lemma 2. *Let F be a continuous real-valued function on the interval $[x_1, x_2]$, and assume that $F(x_1) > 0$ and $F(x_2) < 0$. If for some $p^* \in (x_1, x_2)$, $F(p^*) = 0$ and $F'(p^*) > 0$, there exist at least three distinct solutions of the equation $F(p) = 0$ in the interval (x_1, x_2).*

Proof. Since $F(p^*) = 0$ and $F'(p^*) > 0$, there must be some $\hat{p} \in (x_1, p^*)$ such $F(\hat{p}) < 0$. Since $\lim_{p \to x_1} F(p) > 0$, continuity of F implies that there is some $p' \in (x_1, \hat{p})$ such that $F(p') = 0$. A similar argument establishes the existence of $p'' > p^*$ such that $F(p'') = 0$. □

Theorem 3. *In a mirror-symmetric Shapley-Shubik economy, if $\phi(1) < min\{\bar{x}, \bar{y}\}$ and $\phi(1) + 2\phi'(1) > 0$, there must be at least three competitive equilibria; one at price $p = 1$, one at a price $\bar{p} > 1$ and one at price $1/\bar{p} < 1$.*

Proof. From Remark 3 it follows that $E(1) = 0$. Direct calculation shows that

$$E'(p) = \phi'(p) + \frac{1}{p^2}\phi(1/p) + \frac{1}{p^3}\phi'(1/p) \tag{10}$$

and therefore $E'(1) = \phi(1) + 2\phi'(1)$. Applying Lemma 2 with $p^* = 1$ we know that there are at least three equilibria. Remark 4 informs us that if $\bar{p}$ is an equilibrium, so is $1/\bar{p}$. □

It is noteworthy that in a mirror-symmetric Shapley-Shubik economy with multiple equilibria, the "fair" equilibrium outcome in which $p = 1$ and where the two consumption bundles are mirror images is unstable. The only stable competitive equilibria divide the gains from trade unequally.

Remark 5. *In a mirror-symmetric Shapley-Shubik economy with three distinct equilibria, the symmetric outcome with $p = 1$ is a competitive equilibrium, but it is unstable. The two stable equilibria are at a reciprocal pair of prices $\bar{p} > 1$ which favors Consumer 2 and $1/\bar{p} < 1$ which favors Consumer 1.*

2.2 Rescaling Commodities

It is often convenient to reduce a Shapley-Shubik economy to a simpler canonical form by linear transformations of the variables x and y. This is equivalent to changing the units in which commodities X and Y are measured.

Let preferences be represented by the utility functions in Equations 1. Where $k_x > 0$ and $k_y > 0$, we can restate these utilities in terms of variables $x' = x/k_x$ and $y' = y/k_y$ as follows:

$$\begin{aligned} U_1(x', y') &= k_x x' + f_1(k_y y') \\ U_2(x', y') &= f_2(k_x x') + k_y y'. \end{aligned} \tag{11}$$

Since preferences are invariant to monotone transformations of the utility function, the preferences represented by the utilities in 11 can also be represented by the utility functions:

$$\tilde{U}_1(x', y') = x' + \frac{1}{k_x} f_1(k_y y')$$

$$\tilde{U}_2(x', y') = \frac{1}{k_y} f_2(k_x x') + y', \tag{12}$$

Thus a Shapley-Shubik economy with the functions $f_1(y)$ and $f_2(x)$ can be transformed into an equivalent Shapley-Shubik economy with the functions $\tilde{f}_1(y') = f_1(k_y y')/k_x$ and $\tilde{f}_2(x') = f_2(k_x x')/k_y$. Equilibrium price p' for commodity Y relative to commodity X in the rescaled economy corresponds to a price $p = (k_x/k_y)p'$ where goods are scaled as in the original economy.

3 Quadratic Shapley-Shubik Economies

3.1 Mirror-symmetric Quadratic Utilities

There is an easily calculated closed-form solution for the equilibria of quadratic mirror-symmetric Shapley-Shubik economies. The nonlinear portions of the utility functions in such economies can be expressed in the canonical form, $f(z) = az - z^2/2$.[4] Then the utility functions are:

$$\begin{aligned} U_1(x, y) &= x + ay - \frac{1}{2}y^2 \\ U_2(x, y) &= ax - \frac{1}{2}x^2 + y. \end{aligned} \tag{13}$$

3.1.1 Solving for equilibria

With the utility functions of Equation 13, the inverse marginal utilities are given by $\phi(p) = f'^{-1}(p) = a - p$. If demands are interior at price p, excess demand is given by

$$E(p) = \phi(p) - \frac{1}{p}\phi(1/p) = a - p - a\frac{1}{p} + \frac{1}{p^2} \tag{14}$$

From Equation 14 it follows that $E(1) = 0$ and that

$$E'(1) = \phi(1) + 2\phi'(1) = a - 3. \tag{15}$$

Therefore $E'(1) > 0$ if and only if $a > 3$. The interiority conditions of Definition 2 are satisfied at $p = 1$ when $\bar{x} > \phi(1) = a - 1$, and $\bar{y} > \phi(1) = a - 1$. These facts together with Theorem 2 allow us to conclude that:

[4] This discussion can be extended to allow for free disposibility by making the function f non-decreasing, by letting $f(z) = a^2/2$ for all $z \geq a$. None of the results of this section are altered by this change, since in the economies considered, both consumers have strictly positive marginal utilities at all equilibrium prices.

Remark 6. *In a quadratic mirror-symmetric Shapley-Shubik economy, where* $f(z) = az - z^2/2$*, where* $a > 3$*, and where* $\bar{x} > a - 1$ *and* $\bar{y} > a - 1$*, there is an unstable interior competitive equilibrium at* $p = 1$*, and there are two additional distinct competitive equilibrium prices,* $\bar{p} > 1$ *and* $1/\bar{p} < 1$*, both of which are stable.*

If p is an interior competitive equilibrium price, then $E(p) = 0$ and hence $pE(p) = 0$. Applying Equation 14 and factoring, we find that

$$pE(p) = pa - a - p^2 + \frac{1}{p} = (p-1)\left(a - (1 + p + \frac{1}{p})\right) \tag{16}$$

The interiority conditions are satisfied at p and

$$0 = (p-1)\left(a - (1 + p + \frac{1}{p})\right) \tag{17}$$

Equation 17 is satisfied at p if and only if $p = 1$ or $a = 1 + p + 1/p$. The latter equation is equivalent to:

$$\bar{p}^2 + (1-a)\bar{p} + 1 = 0 \tag{18}$$

Equation 18 has two distinct real roots if and only if $a > 3$. These roots, which can be found by applying the quadratic formula are reciprocals, $\bar{p}$ and $1/\bar{p}$.

3.1.2 Constructing economies with specified equilibrium prices

We can also work backwards, starting with specified prices and finding a parameter value a such that these prices are competitive equilibria in a mirror-symmetric quadratic economy with this parameter. To do this, choose any price $\bar{p} > 1$, and set $a = 1 + \bar{p} + 1/\bar{p}$. In the mirror-symmetric Shapley-Shubik economy with quadratic parameter a, the three solutions of Equation 17 will be $\bar{p}$, 1, and $1/\bar{p}$. It is readily verified that $a = 1 + \bar{p} + 1/\bar{p} > 3$ for all $\bar{p} > 1$. The interiority conditions are satisfied at $p = 1$, $p = \bar{p}$, and $p = 1/\bar{p}$ if $\bar{x} > 1 + \bar{p}$ and $\bar{y} > 1 + \bar{p}$. Therefore:

Remark 7. *For any* $\bar{p} > 1$*, there exists a mirror-symmetric Shapley-Shubik economy in which the non-linear portion of each consumer's utility function is* $f(z) = az - z^2/2$ *with* $a = 1 + \bar{p} + 1/\bar{p}$*. If the initial endowments are* $\bar{x} > 1 + \bar{p}$ *and* $\bar{y} > 1 + \bar{p}$*, then there are exactly three interior competitive equilibria, occurring at prices at* $p = \bar{p}$*,* $p = 1$*, and* $p = 1/\bar{p}$*.*

Example 1. To construct a mirror-symmetric quadratic economy in which the three equilibrium prices are $1/2$, 1, and 2, let $a = 1 + 2 + \frac{1}{2} = \frac{7}{2}$. Then $\phi(p) = 7/2 - p$, and the utility functions are

$$\begin{aligned} U_1(x, y) &= x + \frac{7}{2}y - \frac{1}{2}y^2 \\ U_2(x, y) &= \frac{7}{2}x - \frac{1}{2}x^2 + y \end{aligned} \tag{19}$$

Theorem 3 implies that prices $1/2$, 1, and 2 are interior equilibria if $\bar{x} > 3$ and $\bar{y} > 3$.

Figure 1 graphs the excess demand functions of Consumers 1 and 2, as well as the aggregate excess demand function for Good Y. Consumer 1's demand curve for Y is given by the linear equation $y = 7/2 - p$ for $p \leq 7/2$ and by $y = 0$ for $p > 7/2$. Since Consumer 1 has no initial endowment of good Y, his excess demand for Y is equal to his demand. Consumer 2's demand for Y is given by the equation

$$\bar{y} - \frac{1}{p}\left(\frac{7}{2} - \frac{1}{p}\right) \tag{20}$$

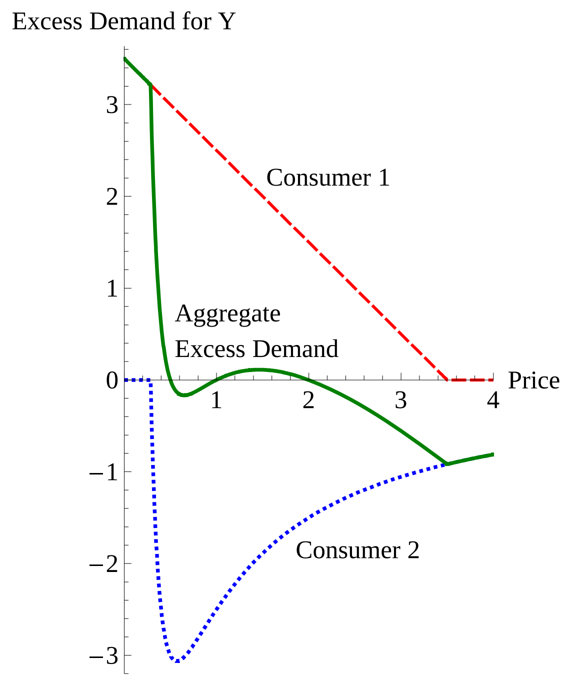
for $p \geq 2/7$ and $y = \bar{y}$ for $p < 2/7$. Consumer 2's excess demand for Y is then

$$-\frac{1}{p}\left(\frac{7}{2} - \frac{1}{p}\right) \tag{21}$$

for $p \geq 2/7$ and 0 for $p < 2/7$. As Figure 1 shows, Consumer 2's excess demand for Y is an increasing function of its price over a range of prices. This is a result of an income effect. As p increases, the value of Consumer 2's endowment increases. Since Y is a "normal good" for Consumer 2 and since her demand for X does not depend on price, the positive income effect of the price increase outweighs the substitution effect and Consumer 2's demand for Y increases with its price. As the figure shows, the aggregate excess demand curve crosses the horizontal axis three times; at $p = 1/2$, $p = 1$, and $p = 2$.

Figure 2 shows the offer curves for this economy in an Edgeworth box with initial endowments $\bar{x} = \bar{y} = 4$. The offer curves intersect three times. At the equilibrium point A, consumption bundles of Consumers 1 and 2, are $(1, 1.5)$ and $(3, 2.5)$ respectively. At equilibrium B, Consumer 1 consumes the bundle $(1.5, 2.5)$ and Consumer 2 consumes $(2.5, 1.5)$. At Equilibrium C, Consumer 1 consumes the bundle $(2.5, 3)$ and Consumer 2 consumes $(1.5, 1)$.

Figure 1: Excess Demand as a Function of Price (Quadratic Utility)



3.2 General Quadratic Utilities

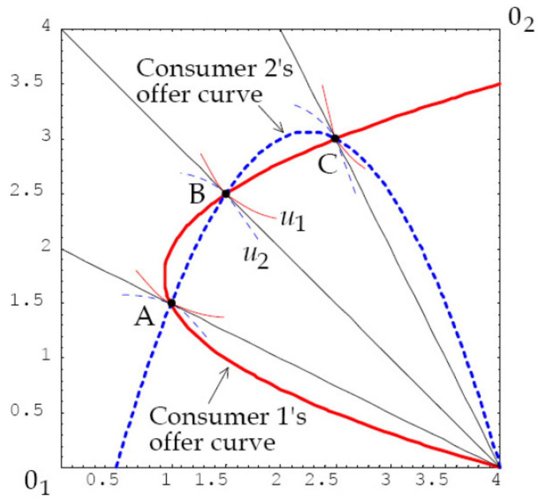
Let us now consider Shapley-Shubik economies in which the nonlinear portions of consumers' utility functions are quadratic but not symmetric. For each consumer i, let

$$f_i(z) = \frac{a_i}{b_i} z - \frac{1}{2b_i} z^2, \tag{22}$$

where $a_i > 0$ and $b_i > 0$. The corresponding demand functions are $\phi_i(p) = a_i - b_i p$ for p in the interval $(0, a_i/b_i]$.

In general, solving for the equilibrium prices of a Shapley-Shubik economy requires solution of a cubic equation. There will be three distinct competitive equilibria if and only if this equation has three distinct roots. In the Appendix, we discuss the somewhat complicated general conditions under which this is the case. On the other hand, it turns out that, given any three distinct positive prices, simple calculations enable us to find parameters of

Figure 2: Offer Curves in an Edgeworth Box (Quadratic Utility)



a (possibly asymmetric) quadratic Shapley-Shubik economy, for which these three prices are the competitive equilibria.

3.2.1 Constructing economies with specified equilibrium prices

For any three distinct positive prices, p_1, p_2, and p_3, if initial endowments are large enough, we can construct a quadratic Shapley-Shubik economy for which each of these prices is a competitive equilibrium, where the utility functions are of the form:

$$\begin{aligned} U_1(x,y) &= x + a_1 y - \frac{1}{2}y^2 \\ U_2(x,y) &= y + a_2 x - \frac{1}{2b}x^2. \end{aligned} \tag{23}$$

We prove this assertion by showing a simple way to choose parameters a_1, a_2, and b to construct such an economy.

With the utility functions in 23, if the interiority conditions are satisfied at price p, excess demand is

$$E(p) = \phi_1(p) - \frac{1}{p}\phi_2(1/p) = a_1 - p - a_2\frac{1}{p} + b\frac{1}{p^2}. \tag{24}$$

At an interior competitive equilibrium, $E(p) = 0$ if and only if

$$p^2 E(p) = -p^3 + a_1 p^2 - a_2 p + b = 0. \tag{25}$$

The cubic equation 25 has three distinct positive roots, p_1, p_2, and p_3 if and only if

$$-p^3 + a_1 p^2 - a_2 p + b = (p_1 - p)(p_2 - p)(p_3 - p). \tag{26}$$

If we carry out the multiplication on the right side of Equation 26, we see that p_1, p_2, and p_3 are equilibrium prices if and only if the following conditions are satisfied.

$$p_1 p_2 p_3 = b \tag{27}$$

$$p_1 + p_2 + p_3 = a_1 \tag{28}$$

$$p_1 p_2 + p_1 p_3 + p_2 p_3 = a_2 \tag{29}$$

From Equations 27 and 29 it follows that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a_2}{b}. \tag{30}$$

For any choice of prices $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$, we choose parameters b, a_1 and a_2 to satisfy Equations 27-29.[5] The quadratic Shapley-Shubik economy with these parameters will have competitive equilibria at p_1, p_2, and p_3 if the interiority conditions are satisfied at each of these prices. From Equation 28 it follows that $\phi_1(p_i) = a_1 - p_i > 0$ for $i = 1, 2, 3$, and from Equation 30 it follows that $\phi_2(p_i) = a_2 - bp_i > 0$ for $i = 1, 2, 3$. The remaining interiority conditions will be satisfied if $\bar{x} > p_i \phi_1(p_i)$ and $p_i \bar{y} > \phi_2(1/p_i)$ for $i = 1, 2, 3$. The equilibria at p_1 and p_3 will be stable, while the equilibrium at p_2 will be unstable.

Example 2. Let us construct a quadratic Shapley-Shubik economy with equilibrium prices $p_1 = 1/2$, $p_2 = 4$, and $p_3 = 5$. Using Equations 27-29, we find $b = 10$, $a_1 = 9.5$, and $a_2 = 24.5$.

Simple calculations show that $p\phi_1(p) < \bar{x}$ for each of these prices if $\bar{x} > 22.5$ and that $\phi_2(1/p) < p\bar{y}$ if $\bar{y} > 9$. Therefore the interiority conditions will be satisfied at all three prices $p_1 = 1/2$, $p_2 = 4$, and $p_3 = 5$ if $\bar{x} > 22.5$ and $\bar{y} > 9$. Therefore there are three competitive equilibria prices, $p_1 = 1/2$, $p_2 = 4$, and $p_3 = 5$ in a quadratic Shapley-Shubik economy with $a_1 = 9.5$, $a_2 = 24.5$, and $b = 10$ in Equations 23 .

[5] This method of choosing parameters for a quadratic equation to generate desired solutions was applied in 1840 by Euler (1972) (pages 253-254) to the study of Diophantine equations.

4 Exponential Utilities

The example presented by Shapley-Shubik has utility functions:

$$\begin{aligned} U_1(x,y) &= x + 100(1 - e^{-y/10}) \\ U_2(x,y) &= y + 110(1 - e^{-x/10}) \end{aligned} \tag{31}$$

This example belongs to a general family of Shapley-Shubik economies that includes all utilities of the form:

$$\begin{aligned} U_1(x,y) &= x - A_1 e^{-b_1 y} \\ U_2(x,y) &= y - A_2 e^{-b_2 x}. \end{aligned} \tag{32}$$

By rescaling units of measurement, as in Section 2.2, we can convert a "four-parameter" economy with utility functions 32 to an equivalent "two-parameter" economy with utilities:

$$\begin{aligned} U_1(x,y) &= x - e^{a_1 - y} \\ U_2(x,y) &= y - e^{a_2 - x}. \end{aligned} \tag{33}$$

where $a_1 = \ln A_1 b_2$ and $a_2 = \ln A_2 b_1$. The inverse marginal utility functions are then $\phi_1(p) = a_1 - \ln p$ and $\phi_2(p) = a_2 - \ln p$.

4.1 Mirror-symmetric Exponential Utilities

For mirror-symmetric Shapley-Shubik economies, the nonlinear portion of utilities take the exponential form $f(z) = -e^{a-z}$. In this case, we have a very simple necessary and sufficient condition for the existence of multiple equilibria.

Remark 8. *In a mirror-symmetric Shapley-Shubik economy with $f(z) = -e^{a-z}$, if $2 < a < \bar{x} = \bar{y}$ there will exactly three equilibrium prices; an unstable equilibrium with $\bar{p} = 1$ and two stable equilibria, $\bar{p} > 1$ and $1/\bar{p} < 1$.*

Proof. Note that $\phi(1) = a$ and $\phi(1) + 2\phi'(1) = a - 2$. Therefore the assumption that $2 < a < \bar{x}$ implies that $\phi(1) = a < \bar{x} = \bar{y}$ and $\phi(1) + 2\phi'(1) = a - 2 > 0$. From Theorem 3 it follows that there are at least three equilibria. From Theorem 4 it follows that there are no more than than three equilibria. $\square$

For an arbitrarily chosen parameter a, there is not a closed form solution for stable equilibrium prices. Solutions must be found by numerical methods.

Working backwards is easier. Starting with an arbitrarily chosen price $\bar{p} > 1$, there is a simple closed form solution for a parameter a such that a mirror-symmetric Shapley-Shubik economy in which $f(z) = -e^{a-z}$ has competitive equilibria at prices, $\bar{p}$, 1, and $1/\bar{p}$.

Remark 9. *For any $\bar{p} > 1$, let*

$$a = \frac{\bar{p}+1}{\bar{p}-1}\ln\bar{p}. \tag{34}$$

In a mirror-symmetric Shapley-Shubik economy where $f(z) = -e^{a-z}$ and where $\bar{x} = \bar{y} > 2\bar{p}\ln\bar{p}/(\bar{p}-1)$, there are interior competitive equilibria at prices at $p = \bar{p}$, $p = 1$, and $p = 1/\bar{p}$.

Proof. There is an interior competitive equilibrium at $\bar{p} > 1$ if and only if the interiority conditions are satisfied at $\bar{p}$ and

$$E(\bar{p}) = a - \ln\bar{p} - \frac{1}{\bar{p}}(a + \ln\bar{p}) = 0 \tag{35}$$

Rearrangement of Equation 35 shows that Equation 35 is satisfied if and only if

$$a = \frac{\bar{p}+1}{\bar{p}-1}\ln\bar{p}. \tag{36}$$

The interiority conditions require that $1/f'(0) < \bar{p} < f'(0)$, $\bar{p}\phi(\bar{p}) < \bar{x}$, and $\phi(1/\bar{p}) < p\bar{x}$. We have

$$f'(0) = e^a = \bar{p}^{\frac{p+1}{p-1}}. \tag{37}$$

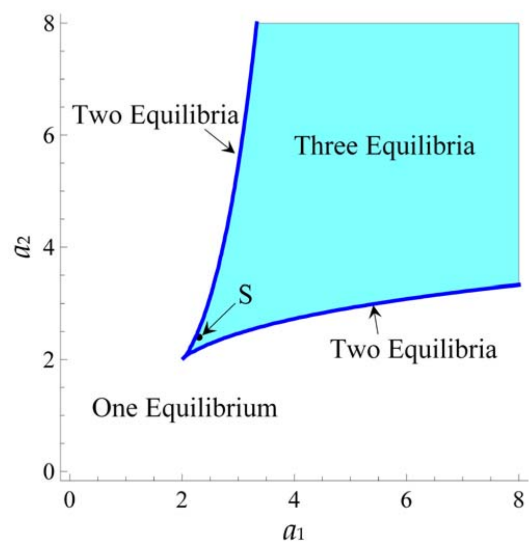
Then for $\bar{p} > 1$, $1/f'(0) = e^{-a} < \bar{p} < f'(0) = e^a$. Calculations show that $\bar{p}\phi(\bar{p}) = \bar{p}(a - \ln\bar{p}) = 2\bar{p}\ln\bar{p}/(1-\bar{p})$ and $\phi(1/\bar{p}) = 2\bar{p}\ln\bar{p}/(1-\bar{p})$. Therefore the interiority conditions $\bar{p}\phi(\bar{p}) < \bar{x}$ and $\phi(1/\bar{p}) < \bar{y}$ will be satisfied if $\bar{x} = \bar{y} > 2\bar{p}\ln\bar{p}/(\bar{p}-1)$. □

Example 3. To construct an economy with exponential utilities and equilibrium prices 2, 1 and 1/2, we set $f(z) = -e^{a-z}$ where $a = ((2+1)/(2-1))\ln 2$. Then $f(x) = -e^{3\ln 2 - x} = -8e^{-x}$ and $\phi(p) = \ln 8 - \ln p$. There will be interior competitive equilibria at $\bar{p} = 2$ and $\bar{p} = 1/2$ if initial endowments are $\bar{x} = \bar{y} > 2\bar{p}\ln(\bar{p}/(\bar{p}-1)) = 4\ln 2$.

4.2 General Exponential Utilities

We have found a closed-form expression, the sign of which determines the number of competitive equilibria for any Shapley-Shubik economy of exponential form. Figure 3, shows parameter regions of utilities for there are one, two, or three competitive equilibria, when utility is written in the canonical form of Equations 33. The point S in Figure 3 marks the parameter values a_1 and a_2 that correspond to the example presented by Shapley and Shubik. In Theorem 4, we characterize the parameter ranges over which there are one, two, or three equilibria.

Figure 3: Parameter ranges for multiple equilibria



In the appendix of this paper, we prove the following result:

Theorem 4. *In a Shapley-Shubik economy where* $f_i(z) = -e^{a_i - z}$ *for* $i = 1, 2$ *and where* $\bar{x} > e^{a_1 - 1}$ *and* $\bar{y} > e^{a_2 - 1}$,

(i) there are no more than three competitive equilibria.
(ii) there is a closed form function $G(a_1, a_2)$ *such that there are one, two, or three distinct competitive equilibria, depending on whether* $G(a_1, a_2)$ *is positive, zero, or negative.*
(iii) there is exactly one competitive equilibrium if $\min\{a_1, a_2\} < 2$.

5 Power Function Utilities

Consider a Shapley-Shubik economy in which the nonlinear portions of utilities are power functions, of the form:

$$f_i(z) = \frac{1}{b_i} z^{b_i} \tag{38}$$

where $b_i < 1$. Then for all $z > 0$, $f_i'(z) = z^{b_i - 1}$ and the inverse marginal utility function is of the constant elasticity form

$$\phi_i(p) = p^{1/(b_i-1)}. \tag{39}$$

We note that for all $b_1 < 1$, $\lim_{z \to 0} = \infty$.[6]

Let us consider Shapley-Shubik utility functions with power function utilities that are also mirror-symmetric. It will be convenient to express these in the form:

$$f(x) = \frac{\epsilon}{1+\epsilon} x^{\frac{\epsilon+1}{\epsilon}} \tag{40}$$

where $\epsilon < 0$. The range of $f'(\cdot)$ is the entire interval $(0, \infty)$. The inverse marginal utility function is $\phi(p) = f'^{-1}(p) = p^{\epsilon}$.

In this economy, if $\bar{x} > 1$ and $\bar{y} > 1$, there will be an interior competitive equilibrium at $p = 1$. At this equilibrium, $E'(1) = \phi(1) + 2\phi'(1) = 1 + 2\epsilon > 0$ if $-1/2 < \epsilon < 0$. Therefore, according to Theorem 3, there must be at least three equilibria if $-1/2 < \epsilon < 0$.

However $p = 1$ is the only positive real solution to the equation

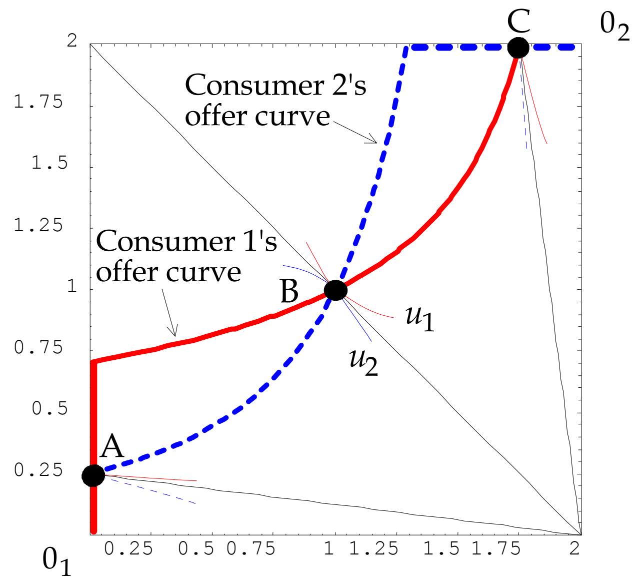
$$0 = \phi(p) - (1/p)\phi(1/p) = p^{\epsilon} - p^{-(1+\epsilon)}. \tag{41}$$

To find the other two equilibrium prices, we must look for "corner solutions." There are two such equilibria. In one of them, Consumer 1 trades away all of his holding of good X and consumes only Y. At the other corner equilibrium, Consumer 2 trades away all of his initial Y and consumes only X.

At the corner equilibrium where Consumer 1 consumes only Y, Consumer 2 must consume $\bar{x}$ units of X and a positive amount of Y. Therefore it must be that $\bar{x} = \phi(1/p) = p^{-\epsilon}$ and hence $p = \bar{x}^{-1/\epsilon}$. At this price, Consumer 1 consumes $y_1 = \bar{x}/p = p^{-(1+\epsilon)}$ units of Y. It is straightforward to verify that with this quantity, Consumer 1's marginal rate of substitution between Y and X exceeds the price p, and hence Consumer 1 is at a corner solution. A similar line of reasoning shows that at the corner solution where Consumer 2 consumes only X, $\bar{y} = p^{\epsilon}$ and hence $p = \bar{y}^{1/\epsilon}$.

[6] In case $b_i < 0$, we must also define $f_i(0) = \lim_{z \to 0} f(0) = -\infty$.

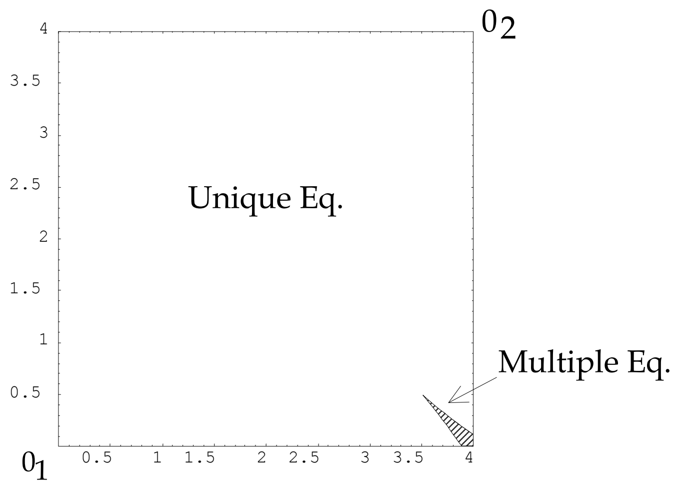
Figure 4: Offer Curves in an Edgeworth Box (Power Function Utility)



Example 4. Consider the mirror-symmetric Shapley-Shubik economy where $f(x) = -(1/2)x^{-2}$ and $\bar{x} = \bar{y} = 2$. In this economy, $\phi(p) = p^{-1/3}$. There is an interior competitive equilibrium at $p = 1$. There is a corner equilibrium where Consumer 1 trades away all of his initial endowment of X and consumes only Y. This occurs at a price $p = 8$. At this price, Consumer 2 will demand $\phi(1/p) = 2$ units of X and $1\frac{3}{4}$ units of Y. Consumer 1 trades all of his X for $1/4$ unit of Y. There is another corner equilibrium where Consumer 2 trades all of his initial endowment of Y and consumes only X. At this equilibrium, $p = 1/8$, Consumer 1 consumes 2 units of Y and $1\frac{3}{4}$ units of X, and Consumer 2 consumes no Y and $1/4$ unit of X.

Figure 4 shows an Edgeworth box and offer curves for this example, where Consumer 1 is endowed with $\bar{x} = 2$ units of x and no y and Consumer 2 is endowed with $\bar{y} = 2$ units of y and no x. The competitive equilibrium A is a corner solution for Consumer 1, who consumes no x. The equilibrium C is a corner solution for Consumer 2, who consumes no y. The equilibrium C is symmetric and interior for both consumers but is unstable.

Figure 5: Endowments Leading to Multiple Equilibria (Quadratic Utility))



6 Interior Endowments and Generalized Shapley-Shubik Economies

Our definition of a Shapley-Shubik economy requires the initial endowment to be at a corner of the Edgeworth box. Each consumer is endowed only with the commodity in which his utility is linear. Many of our results can be extended to *generalized Shapley-Shubik economies* where utility functions are as in a Shapley-Shubik economy, but where one or both consumers have positive initial endowments of both goods.

In general, if a Shapley-Shubik economy has multiple equilibria, the generalized Shapley-Shubik economies constructed by reallocating the same total endowments will continue to have multiple equilibria so long as the initial endowments are "not too distant" from the initial endowments in the corresponding Shapley-Shubik economy. This is illustrated in the Edgeworth box drawn in Figure 5, which applies to the quadratic utility function used in Example 1, where the total initial endowments of X and Y are each 4 units. There are three competitive equilibria if and only if initial endowments are located in the shaded area.[7]

[7] Kumar and Shubik (2003) displayed a similar figure showing the endowments that lead to multiple equilibria in the Shapley-Shubik example.

Let us define a generalized Shapley-Shubik economy as follows. There are two consumers, with utility functions $U_1(x,y) = x + f_1(y)$ and $U_2(x,y) = f_2(x) + y$ where the functions f_i are concave and continuously differentiable, as in an ordinary Shapley-Shubik economy. Each consumer i is endowed with $\bar{x}_i$ units of X and $\bar{y}_i$ units of Y. Then $\bar{x}_1 + \bar{x}_2 = \bar{x}$ and $\bar{y}_1 + \bar{y}_2 = \bar{y}$ are the total supplies of X and Y.

At an interior competitive equilibrium, Consumer 1's consumption of Y is $y_1(p) = \phi_1(p)$ and Consumer 2's consumption of X is $x_2(p) = \phi_2(1/p)$. From Consumer 2's budget constraint, it follows that

$$y_2(p) - \bar{y}_2 = \frac{1}{p}\left(\bar{x}_2 - \phi_2(1/p)\right) \tag{42}$$

At an interior equilibrium, excess demand for good Y is given by

$$E(p) = y_1(p) + y_2(p) - \bar{y}_2 - \bar{y}_1 = \phi_1(p) - \bar{y}_1 - \frac{1}{p}\left(\phi_2(1/p) - \bar{x}_2\right) \tag{43}$$

Let us define $\tilde{\phi}_1(p) = \phi_1(p) - \bar{y}_1$ and $\tilde{\phi}_2(p) = \phi_2(p) - \bar{x}_2$. Then from Equation 43 it follows that at an interior competitive equilibrium,

$$E(p) = \tilde{\phi}_1(p) - (1/p)\tilde{\phi}_2(1/p) = 0. \tag{44}$$

It follows from the two consumers' budget constraints that Consumer 1's consumption of X and Consumer 2's consumption of Y will be positive if and only if

$$p(\phi_1(p) - \bar{y}_1) < \bar{x}_1 \text{ and } \phi_2(1/p) - \bar{x}_2 < p\bar{y}_2. \tag{45}$$

The inequalities in 45 can be written as $p\tilde{\phi}_1(p) < \bar{x}_1$ and $\tilde{\phi}(1/p) < p\bar{y}_1$. Therefore we have the following generalization of Theorem 1.

Lemma 3. *In a generalized Shapley-Shubik economy, p is an interior competitive equilibrium price if and only if $p\tilde{\phi}_1(p) < \bar{x}_1$, $\tilde{\phi}_2(1/p) < p\bar{y}_2$, and $\tilde{\phi}_1(p) = (1/p)\tilde{\phi}_2(1/p)$.*

6.1 Mirror-symmetric Interior Endowments

Within the class of generalized Shapley-Shubik economies, there is an interesting subclass of economies with interior endowments but with sufficient symmetry to allow clear, simple results. Consider a generalized Shapley-Shubik economy with mirror-symmetric utilities in which each person has a positive initial endowment of each good. We will say that there are mirror-symmetric endowments if the two consumers have equal initial endowments of the good in which they have nonlinear utility. When initial endowments are $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$, this means that $\bar{y}_1 = \bar{x}_2 = \bar{z}$ for some for some $\bar{z} > 0$. Where $\phi(p)$ is the inverse marginal utility function, let $\tilde{\phi}(p) = \phi(p) - \bar{z}$. For an interior competitive equilibrium, excess demand for good Y is zero if

$$E(p) = \tilde{\phi}(p) - \frac{1}{p}\tilde{\phi}(1/p) = 0 \tag{46}$$

and the interiority conditions will be satisfied at p if

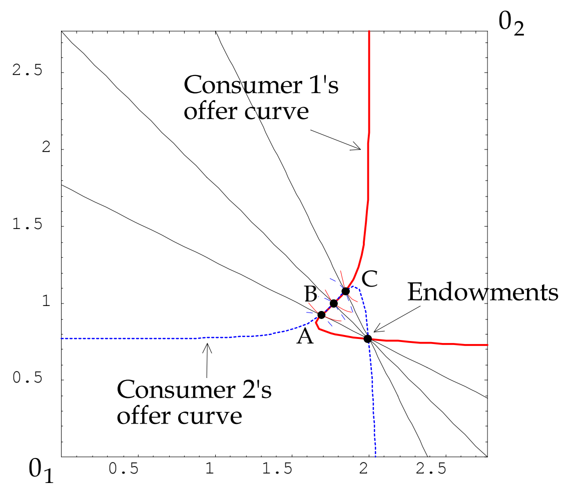
$$p\tilde{\phi}(p) < \bar{x}_1 \text{ and } \tilde{\phi}(1/p) < p\bar{y}_2. \tag{47}$$

From Equation 46 it is immediate that if the interiority conditions are satisfied, there is a competitive equilibrium at price $p = 1$ and that if p is an equilibrium price, then $1/p$ is also an equilibrium price. Under the conditions of Theorem 2, there will be multiple equilibria if $\tilde{\phi}(1) + 2\tilde{\phi}'(1) > 0$. Since, by definition, $\tilde{\phi}(p) = \phi(p) - \bar{z}$, there will be multiple equilibria if $\phi(1) + 2\phi'(1) > \bar{z}$ and if the interiority conditions are satisfied.

Example 5. Consider an economy with the same quadratic utility functions as those in Example 1, but with initial endowments $(\bar{x}_1, \bar{y}_1) = (4, \bar{z})$ and $(\bar{x}_2, \bar{y}_2) = (\bar{z}, 4)$ where $\bar{z} > 0$. There will be three distinct competitive equilibria if $E'(1) > 0$ and if price $p = 1$ satisfies the interiority conditions. For this economy, $\tilde{\phi}(p) = 7/2 - \bar{z} - p$. Therefore $E'(1) = \tilde{\phi}(1) + 2\tilde{\phi}'(1) = 1/2 - \bar{z}$ and hence $E'(1) > 0$ if and only if $z < 1/2$. The interiority conditions will be satisfied at $p = 1$ if $\tilde{\phi}(1) = 5/2 - \bar{z} < \min\{\bar{x}_1, \bar{y}_2\} = 4$, or equivalently if $z < 3/2$. It follows that there are three competitive equilibria if and only if $\bar{z} < 1/2$.

Example 6. Mas-Collel, Whinston, and Green (1995) present an example of an exchange economy with three competitive equilibria in their graduate economic theory textbook (page 521). This example is a generalized Shapley-Shubik economy with mirror-symmetric power function utilities. In Section 5, we showed that if the initial endowment is at a corner of the Edgeworth box then, if there are multiple equilibria, two of these equilibia will be corner solutions. The Mas-Collel, Whinston, Green example is constructed with symmetric interior endowments and this example has three interior competitive equilibria. Figure 6 shows offer curves and the Edgeworth box for the MWG example.

Figure 6: Offer Curves in an Edgeworth Box (The MWG Example)



In this example, the utility functions of Consumers 1 and 2 are

$$\begin{aligned} U_1(x,y) &= x - \frac{1}{8}y^{-8} \\ U_2(x,y) &= -\frac{1}{8}x^{-8} + y \end{aligned} \tag{48}$$

The initial endowments are $(\bar{x}_1, \bar{y}_1) = (2, \bar{z})$ and $(\bar{x}_2, y_2) = (\bar{z}, 2)$. Then $\phi(p) = p^{-1/9}$ and $\tilde{\phi}(p) = p^{-1/9} - \bar{z}$. If p is an interior competitive equilibrium,

$$0 = \tilde{\phi}(p) - \frac{1}{p}\tilde{\phi}(\frac{1}{p}) = p^{-1/9} - p^{-8/9} - \bar{z}(1 - \frac{1}{p}). \tag{49}$$

Price p is a solution to Equation 49 if and only if

$$\bar{z} = \frac{p^{8/9} - p^{1/9}}{p - 1} \tag{50}$$

One can construct an example with any desired pair of equilibrium prices $\bar{p} > 1$ and $1/\bar{p}$ by choosing $\bar{z}$ to satisfy Equation 50. The MWG example is chosen to have equilibria at $p = 2$ and $p = 1/2$. This is the case when $\bar{z} = 2^{8/9} - 2^{1/9}$. To ensure that the solutions $p = 2$ and $p = 1/2$ are both interior solutions, we verify that $2\tilde{\phi}(2) < \bar{x}_1 = 2$ and $1/2\tilde{\phi}(1/2) < \bar{y}_2 = 2$.

7 Conclusion

The family of Shapley-Shubik economies is a class of "special general equilibrium models" that are computationally manageable, yet permit complex outcomes such as multiple equilibria. Shapley-Shubik economies have interesting comparative static properties. For example, if endowments are sufficiently large, equilibrium prices depend only on the demand functions and not on the quantity of goods available.

For mirror-symmetric Shapley-Shubik economies (subject to interiority conditions), we show that a simple calculation of the derivative of excess demand at price 1 determines whether there are multiple equilibria. We find conditions on parameters that determine whether a Shapley-Shubik economy has one, two or three equilibria for the cases where the nonlinear part of utility functions are quadratic, exponential, or power functions. We also show how to "work backwards" so as to choose utility parameters of Shapley-Shubik economies such that any three specified prices are competitive equilibria. Finally, we show that Shapley-Shubik economies can be generalized to include a set of economies in which both consumers have positive endowments of both goods and where there are three distinct competitive equilibria.

A Appendix

A.1 Finding Equilibrium for General Quadratic Utilities

Consider a Shapley-Shubik economy in which the functions f_i take the general quadratic form

$$f_i(z) = a_i z - \frac{1}{2b_i} z^2. \tag{51}$$

The task of finding competitive prices can be simplified by rescaling units of measurement of X and Y so that in the rescaled economy $b_1 = 1$ and $b_2 = 1$. To do so, make the linear transformation of variables $x' = x/k_x$ and $y' = y/k_y$ where

$$k_y = b_1^{2/3} b_2^{1/3} \text{ and } k_x = b_1^{1/3} b_2^{2/3}. \tag{52}$$

In the transformed economy,

$$f_i(z) = A_i z - \frac{1}{2} z^2 \tag{53}$$

for $i = 1, 2$, where $A_1 = a_1 k_y / b_1 k_x$ and $A_2 = a_2 k_x / b_2 k_y$.

The inverse marginal utility functions expressed in the transformed variables are $\phi_i(p) = A_i - p$. In competitive equilibrium,

$$E(p) = A_1 - p + \frac{1}{p}\left(A_2 - \frac{1}{p}\right) = 0. \tag{54}$$

Equation 54 is satisfied at p if and only if p is a root of the cubic equation

$$p^3 - A_1 p^2 + A_2 p - 1 = 0 \tag{55}$$

Closed form solutions for a general cubic function were first published in 1545 by G. Cardan and are known as Cardan's formulas. (A good discussion of solution methods for cubic equations is found in Dickson (1922).) Cardan's formulas can be applied to find the solutions to Equation 55. For general values of A_1 and A_2, these expressions are cumbersome and do not readily yield useful insight. On the other hand, calculating numerical solutions to Equation 55 for specific values of A_1 and A_2 is straightforward.[8]

General necessary and sufficient conditions for the existence of three positive real solutions to Equation 55 are available. As Dickson (1922) explains, a cubic equation will have three distinct real roots if and only if its *discriminant* is strictly positive. The discriminant of Equation 55 turns out to be[9]

$$\Delta = 18A_1A_2 - 4A_1^3 + A_1^2A_2^2 - 4A_2^3 - 27. \tag{56}$$

Equation 55 had three real roots if and only if $\Delta > 0$. These roots will all be positive valued. To see this, note that if $p \leq 0$, then $p^3 - A_1p^2 + A_2p - 1 < 0$ and so p can not be a root of Equation 55.[10] Where $\Delta > 0$, there will be competitive equilibria at each of the three positive roots of Equation 55 so long as the initial endowments $\bar{x}$ and $\bar{y}$ are large enough to satisfy the interiority.

A.2 Proof of Theorem 4

To prove Theorem 4, it is useful to define the function

$$F(p) = \frac{a_1p - a_2}{1 + p} - \ln p. \tag{57}$$

[8]"Cubic equation solvers" which output the solutions for a cubic with any specified real parameters are available on the internet. See for example www.1728.comcubic.htm

[9]The discriminant of any equation is the product of the squares of the differences of its roots. Dickson shows that the discriminant of a cubic equation of the form $x^3 + bx^2 + cx + d$ is $\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2$.

[10]Euler (1972), pages 256-257 provides a similar demonstration that real roots of a cubic with coefficients of alternating signs must be positive.

for all $p \in (0, \infty)$. Our strategy of proof is to show by a series of lemmas that for all $p \in (0, \infty)$, the sign of $F(p)$ is the same as that of aggregate excess demand $E(p)$. Therefore a price p is a competitive equilibrium if and only if $F(p) = 0$. We then study the shape of the graph of $F(p)$ and display a closed form function $G(a_1, a_2)$ such that the sign of $G(a_1, a_2)$ determines the number of solutions to the equation $F(p) = 0$.

Lemma 4. *In a Shapley-Shubik economy with exponential utilities where $a_1 + a_2 > 0$, if $p \in (0, e^{-a_2}]$, then $E(p) > 0$ and $F(p) > 0$. If $p \in [e^{a_1}, \infty)$, then $E(p) < 0$ and $F(p) < 0$.*

Proof. Since $f_i(z) = -e^{a_i - z}$, we have $B_1 = f_1'(0) = e^{a_1}$ and $B_2 = f_2'(0) = e^{a_2}$. From Lemma 1 it follows that $E(p) > 0$ for $p \leq e^{-a_2}$ and that $E(p) < 0$ for $p \geq e^{a_1}$.

To complete the proof we show that $F(p) > 0$ for $p \leq e^{-a_2}$ and $F(p) < 0$ for $p \geq e^{a_1}$. If $p \leq e^{-a_2}$, then $-\ln p \geq a_2$, and therefore

$$F(p) \geq \frac{a_1 p - a_2}{1 + p} + a_2 = \frac{p}{1 + p}(a_1 + a_2) > 0. \tag{58}$$

If $p \geq e^{a_1}$, then $-\ln p \leq -a_1$ and therefore

$$F(p) \leq \frac{a_1 p - a_2}{1 + p} - a_1 = -\frac{a_1 + a_2}{1 + p} < 0. \tag{59}$$

□

Lemma 5. *In a Shapley-Shubik economy with exponential utilities, if $\bar{x} > e^{a_1 - 1}$ and $\bar{y} > e^{a_2 - 1}$, then the interiority conditions of Definition 2 apply for all $p \in (e^{-a_2}, e^{a_1})$.*

Proof. The interiority conditions are satisfied at p if and only if $p \in (e^{-a_2}, e^{a_1})$, $p\phi_1(p) < \bar{x}$, and $\phi_2(1/p) < p\bar{y}$. To show that $p\phi_1(p) < \bar{x}$, we differentiate $p\phi_1(p) = a_1 p - p \ln p$ to find that $p\phi(p)$ is maximized when $a_1 - 1 = \ln p$ and hence when $p = e^{a_1 - 1}$. Therefore for all $p > 0$, $p\phi(p) \leq e^{a_1 - 1}\phi(e^{a_1 - 1}) = e^{a_1 - 1}(a_1 - (a_1 - 1)) = e^{a_1 - 1}$. We have assumed that $\bar{x} > e^{a_1 - 1}$. Therefore $p\phi_1(p) < \bar{x}$ for all $p < 0$. To show that $\phi_2(1/p) < p\bar{y}$ for all $p > 0$, we differentiate $(1/p)\phi_2(1/p)$ and find that $(1/p)\phi_2(1/p)$ is maximized when $p = e^{1 - a_2}$. Therefore, for all $p > 0$, $(1/p)\phi_2(p) \leq e^{a_2 - 1}\phi_2(e^{a_2 - 1}) = e^{a_2 - 1}$. Since we assume that $\bar{y} > e^{a_2 - 1}$, it follows that $1/p\phi_2(p) < \bar{y}$ and hence $\phi_2(p) < p\bar{y}$.

□

Lemma 6. *In a Shapley-Shubik economy with exponential utilities, if $\bar{x} > e^{a_1-1}$ and $\bar{y} > e^{a_2-1}$, then for all $p \in (0,\infty)$, the sign of $F(p)$ is the same as the sign of $E(p)$.*

Proof. According to Lemma 5, the interiority conditions are satisfied at all p in the interval (e^{-a_2}, e^{a_1}). Therefore, for all $p \in (e^{-a_2}, e^{a_1})$, $E(p) = \phi(p) - (1/p)\phi(1/p) = a_1 - \ln p - (1/p)(a_2 + \ln p)$. Rearranging the terms of $F(p)$, we see that

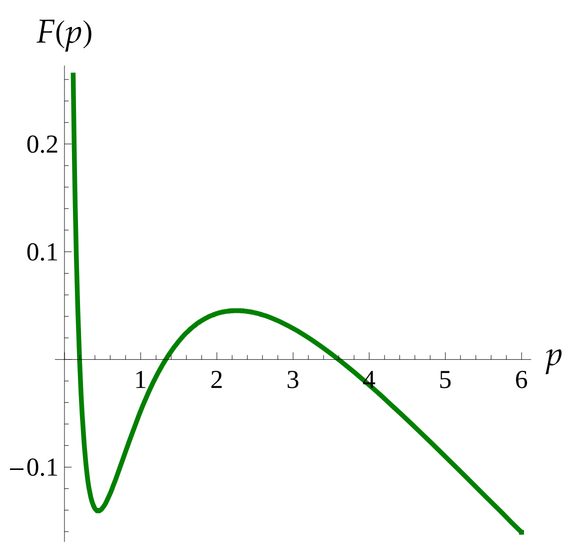
$$F(p) = \frac{p}{1+p}\left(a_1 - \ln p - \frac{1}{p}(a_2 + \ln p)\right) = \frac{p}{1+p}E(p) \tag{60}$$

for all $p \in (e^{-a_2}, e^{a_1})$. Since $p/(1+p) > 0$, the sign of $F(p)$ must be the same as that of $E(p)$ for all $p \in (e^{-a_2}, e^{a_1})$.

Lemma 4 informs us that the sign of $E(p)$ is the same as that of $F(p)$ for all p in the intervals $(0, e^{-a_2}]$ and $[e_1^a, \infty)$. Therefore the sign of $F(p)$ is the same as that of $E(p)$ on the entire interval $(0,\infty)$. □

Figure 7 which graphs the function $F(p)$ with parameters $a_1 = \ln 10 = 2.303$ and $a_2 = \ln 11 = 2.398$ that apply to the Shapley-Shubik example. This graph will be useful in the exposition of the proof of Theorem 4.

Figure 7: The graph of $F(p)$



Proof. of Theorem 4 According to Lemma 6, $E(p) = 0$ if and only if $F(p) = 0$. Calculation shows that

$$F'(p) = \frac{a_1 + a_2}{(1+p)^2} - \frac{1}{p}. \tag{61}$$

Then

$$-p(1+p)^2 F'(p) = p^2 - (a_1 + a_2 - 2)\, p + 1 \tag{62}$$

It follows that the sign of $F'(p)$ is opposite from the sign of the quadratic expression on the right side of Equation 62. This quadratic expression has at most two distinct real roots. Hence the slope of the graph of $F(p)$ can change signs at most twice over the interval (e^{-a_2}, e^{a_1}). It follows that there cannot be more than three solutions to the equation $F(p) = 0$ and hence no more than three distinct competitive equilibrium prices. This proves Assertion (i) of the theorem.

We will prove Assertion (ii) by exhibiting the desired function $G(a_1, a_2)$. To motivate this construction, it is helpful to consider Figure 7. As p ranges from 0 to ∞, the graph of $F(p)$ first slopes downward, then upward, and then downward again. We will show that for any exponential Shapley-Shubik economy with $a_1 + a_2 > 4$, the graph of $F(\cdot)$ changes its direction of slope exactly twice. In the figure, the graph of $F(\cdot)$ crosses the horizontal axis three times, once in the upward-sloping segment and once in each of the downward-sloping segments. In general, the upward-sloping portion of the graph may lie either entirely above or entirely below the horizontal axis, in which case there is only one competitive equilibrium. If the upward-sloping portion of the graph crosses the horizontal axis as in Figure 7, then there will be exactly three competitive equilibria, one stable and two unstable.

If the graph of $F(\cdot)$ has a local minimum and maximum, they must occur at real roots of the quadratic expression in Equation 62. This equation has real roots if and only if $a_1 + a_2 \geq 4$. These roots are distinct if $a_1 + a_2 > 4$ and in this case, both roots must be positive. Let $p_L(a_1, a_2)$ be the smaller of these roots and $p_H(a_1, a_2)$ be the larger. It is readily verified that $p_L(a_1, a_2) p_H(a_1, a_2) = 1$ and hence $p_L(a_1, a_2) < 1 < p_H(a_1, a_2)$. A straightforward calculus argument shows that $p_L(a_1, a_2)$ is a local minimum and $p_H(a_1, a_2)$ is a local maximum of $F(p)$. It is also straightforward to demonstrate that $F'(p) < 0$ if $p < p_L(a_1, a_2)$, that $F'(p) > 0$ if $p_L(a_1, a_2) < p < p_H(a_1, a_2)$ and that $F'(p) < 0$ if $p > p_H(a_1, a_2)$.

If $F(p_L(a_1, a_2))$ and $F(p_H(a_1, a_2))$ are both positive or both negative, the graph of $F(p)$ can cross the horizontal axis at most once. Hence there can be no more than one competitive equilibrium. If $F(p_L(a_1, a_2))$

and $F(p_H(a_1,a_2))$ are of opposite signs, then it must be that as in Figure 7, $F(p_L(a_1,a_2)) < 0$, $F(p_H(a_1,a_2)) > 0$, and there are exactly three distinct solutions to the equation $F(p) = 0$. These include a stable equilibrium in the interval $(e^{-a_2}, p_L(a_1,a_2))$, an unstable equilibrium in the interval $(p_L(a_1,a_2), p_H(a_1,a_2))$, and a stable equilibrium in the interval $(p_H(a_1,a_2), e^{a_1})$. Define

$$G(a_1,a_2) = F(p_L(a_1,a_2))\, F(p_H(a_1,a_2)). \tag{63}$$

Then $G(a_1,a_2)$ will be positive or negative depending on whether $F(p_L(a_1,a_2))$ and $F(p_H(a_1,a_2))$ are of the same or opposite signs. Therefore there are three equilibria if $G(a_1,a_2)$ is negative, and one equilibrium if $G(a_1,a_2)$ is positive. If $G(a_1,a_2) = 0$, then either $F(p_L(a_1,a_2))$ or $F(p_H(a_1,a_2))$ is zero. In this case, the the graph of $F(p)$ is tangent to the horizontal axis either at the local minimum $p_L(a_1,a_2)$ or the local maximum $p_H(a_1,a_2)$, in which case there will be exactly two equilibria, including one at the point of tangency. This completes the proof of Assertion (ii).

From Lemma 1 we see that any competitive equilibrium price must lie in the interval (e^{-a_2}, e^{a_1}) and from Lemma 5 it follows that the interiority conditions are satisfied for all p in this interval. Therefore, a competitive equilibrium must be a point p in the interval (e^{-a_2}, e^{a_1}) such that

$$E(p) = \phi(p) - \frac{1}{p}\phi(1/p) = a_1 - \ln p - \frac{1}{p}(a_2 + \ln p) \tag{64}$$

Differentiating Equation 64, we obtain

$$E'(p) = \frac{1}{p^2}(a_2 - 1 + \ln p - p). \tag{65}$$

With simple calculus, we see that $\ln p - p$ is maximized at $p = 1$ and that $\ln p - p \le -1$ for all $p > 0$. Therefore

$$p^2 E'(p) = a_2 - 1 + \ln p - p \le a_2 - 2 \tag{66}$$

It follows that if $a_2 < 2$, then $E'(p) < 0$ for all $p > 0$. In this case, $E(p)$ is a strictly decreasing function over the entire interval (e^{-a_2}, e^{a_1}). It follows that there can be only one equilibrium when $a_2 < 2$. To show that there is only one equilibrium if $a_1 < 2$, we note that a competitive equilibrium price p must solve the equation

$$0 = pE(p) = p(a_1 - \ln p) - (a_2 + \ln p) \tag{67}$$

Differentiating, we find that

$$\frac{d}{dp}pE(p) = a_1 - 1 - \ln p - \frac{1}{p} \tag{68}$$

With simple calculus, we find that $\ln p + 1/p$ is minimized at $p = 1$ and that $\ln p + 1/p > 1$ for all $p > 0$. Therefore it must be that $\frac{d}{dp} pE(p) \leq a_1 - 2$ for all $p > 0$. Therefore if $a_1 < 2$, then $pE(p)$ is a decreasing function of p and hence there can be only one competitive equilibrium price. This completes the proof of Assertion (iii).

$\square$

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