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Co-ordinated Volunteers' Dilemmas

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> "... every one is more negligent of what another is to see to, as well as himself, than of his own private business; as in a family one is often worse served by many servants than by a few." *Aristotle's Politics* Book II, Chapter 3

For many tasks, the efforts of a single individual are sufficient to serve an entire group. One sentinel can alert an entire community. One group member can investigate and report on the honesty of a vendor or the usefulness of a consumer good. Thus we might expect clustering into groups to be advantageous. But as the epigraph from Aristotle suggests, there is a countervailing force. The returns to scale offered by larger groups are often offset by free-rider problems that grow with group size. This tension was explored by Mancur Olson [6], who argued that because of free-rider problems, large groups are less able to act in their common interest than small ones and thus in politics it is possible for a minority with concentrated interests to dominate an opposing majority of individuals, each of whom cares less intensely, even though total willingness-to-pay of the majority is higher than that of the minority. Here we explore the group free-rider problems observed by Aristotle and Olson and the relation between efficiency and group size.

1 A Social Problem

As you drive home on a well-travelled street, you encounter a broken traffic signal or perhaps a pile of traffic-obstructing debris. You wonder whether to take the trouble to phone traffic authorities about this condition. You realize that many other commuters face the same choice. If someone else calls, your effort will be wasted. On the other hand, if everybody believes that someone else will call, the hazard will go unreported.

1.1 The Bystander Effect

This conundrum was explored by social psychologists, John M. Darley and Bibb Latané [3], who were inspired by a newspaper story [5] of a grizzly series of events; the murder of Kitty Genovese on the streets of New York City. The story reports that she screamed for help as her killer stalked her for more than half an hour and stabbed her in three separate attacks. According to the story, thirty-eight neighbors heard the commotion, but none came to Kitty's aid and none called the police.

Many commentators took this story as evidence that urban residents suffered from apathy and alienation. Darley and Latané suggested that a more convincing explanation might be what they call the "bystander effect". Observers were aware that many others saw the same events and believed that almost certainly someone else would call the police. To test this notion, Darley and Latané conducted a laboratory experiment in which subjects were presented with what seemed to be the cries of a victim of an epileptic seizure. In alternative experimental treatments, the subjects were led to believe they were the only observer, one of two observers, and one of five observers. The experimenters recorded the time elapsed before a subject would walk down the hall to report the emergency. All of the subjects who believed they were the only observer, but only 62% of the observers who believed they were one of five, reported the incident within 6 minutes. The authors report that "by any point in time, more subjects from the two-person groups had responded than from the three-person groups, and more from the three-person groups than from the six-person groups." Darley and Latané conclude that

> "When there are several observers present, however, the pressures to intervene do not focus on any one of the observers; instead the responsibility for intervention is shared among all the onlookers and is not unique to any one. As a result, no one helps."

1.2 Diekmann's Model

Another sociologist, Andreas Diekmann analyzed the effect discussed by Darley and Latané. by constructing an n-player symmetric game, which he called the *Volunteer's Dilemma.*[4] In Diekmann's game, every player has the option of taking the action *help*, which is costly. If at least one player chooses *help*, then all players receive benefits b. Those who chose not to help have net benefits b and those who chose to help have net benefits $b - c > 0$. If no player helps, then all get a payoff of 0.

In this game, it is clear that there can not be a Nash equilibrium where everyone helps, nor can there be an equilibrium where no one helps. For a group of size n, there is a unique symmetric equilibrium (an equilibrium where all players use the same strategies) in which each player offers help with some probability p between 0 and 1. Diekmann shows that, not surprisingly, the equilibrium probability that any single player volunteers must diminish as the number of players increases. More remarkably, he found that in symmetric equilibrium, *the probability that nobody volunteers increases* with the number of players.

Proposition 1 (Diekmann)**.** *In symmetric equilibrium for the Volunteer's Dilemma, as the number of players increases, the probability that each player volunteers diminishes, and the probability that no player volunteers increases, asymptotically approaches* $\frac{c}{b}$ *from below.*

2 Co-ordinated Volunteer's Dilemmas

In Diekmann's Volunteer's Dilemma, everyone who offers to help must bear the cost of helping, even though only one player's help is needed. Sometimes the efforts of volunteers can be managed more efficiently. An organization may be able to solicit volunteers for a one-person task and then choose just one volunteer who is asked to perform the task. For example, the *National Marrow Donor Program* (NMDP) maintains a registry of persons who avow their willingness to donate stem cells to a leukemia patient with a matching immune system if the need should occur. For patients with common immunity types, there are likely to be many eligible volunteers in the registry. The NMDP chooses just one of the matching registrants to make the donation.[1]

We model a co-ordinated Volunteer's Dilemma as follows: Each of n players can either *volunteer* to perform a task or *refuse* to do so. If nobody volunteers, the task is not done. If one or more players volunteer, exactly one of them will be randomly selected to perform the task.

Payoffs are assigned as follows. If no player volunteers, the task is not performed and every player gets a payoff of 0. If one or more players volunteer, all players except the selected player get utility payoffs of b, while the selected player performs the task at cost c and hence gets a payoff of $b - c > 0$.

This game has no pure strategy symmetric Nash equilibrium, but it does have a symmetric mixed-strategy Nash equilibrium in which each player volunteers with probability $p = 1 - q$ and refuses with probability q. In equilibrium, the probability that all n players refuse is then q^n and the probability that at least one person volunteers is $1 - q^n$.

The comparative statics of symmetric equilibrium in the Single-payer Volunteer's Dilemma are consistent with Aristotle's remark that "every one is more negligent of what another is to see to." Just as Aristotle suggested, each person is less likely to volunteer as the number of players increases. Stated more formally, we have the following result, which we prove in the Appendix.

Proposition 2. *In the Coordinated Volunteer's dilemma game with $n \geq 2$ identical players, there is a unique symmetric mixed strategy Nash equilibrium. The equilibrium probability q_n that an individual refuses to volunteer is increasing in the ratio of costs to benefits and also increasing in the number of players.*

The second part of our quotation from Aristotle makes a stronger claim: "one is often worse served by many servants than by a few." The Single-payer Volunteer's Dilemma model predicts this effect as well. When there are more players, not only is each player less likely to volunteer, but, despite the additional players, it becomes less likely that *at least one* of the players will volunteer.

Proposition 3. *In the symmetric Nash equilibrium of the Single-payer Volunteer's Dilemma game with identical players, the larger the number of players, the greater is the probability that nobody volunteers.*

It follows from Proposition 3 that as n increases, the equilibrium probability that nobody volunteers is a bounded increasing sequence and hence must approach a limit. The limiting value of this probability can not in general be expressed in terms of standard elementary functions. However this limit can be characterized by a well-studied function that has found applications in physics, engineering, biology, epidemiology, and computer science. This is the Lambert W function [2]. Much is known about the qualitative properties of Lambert's W and routines for its computation are readily available in Maple, Mathematica, and R.

Proposition 4. *In the symmetric Nash equilibrium of the Single-payer Volunteer's Dilemma game with identical players, as $n \to \infty$ the probability that* *no one volunteers converges to* $-\frac{\frac{c}{b}}{W\left(-\frac{c}{b}e^{-\frac{c}{b}}\right)}$ *where* $W()$ *is the lower-branch of the lambert-*W *function.*

Sharing the Task

Sometimes burden of a task can be lightened by sharing the work among volunteers. Suppose that there are n players and a task that must be performed. Players may either *volunteer* or *refuse*. Volunteers work together to perform the task and the cost to any volunteer is $c(k)$ where k is the number of volunteers. If there are $k \geq 1$ volunteers, the task is performed. Those who did not volunteer will get payoffs of b, while those among who volunteered get payoffs of $b - c(k)$.

A simple case for analysis is the case where the work can be divided equally among volunteers with constant returns to scale, so that $c(k) = c/k$. This case is formally isomorphic to the case of the coordinated volunteers dilemma, where one volunteer is selected at random. In this case, the *expected* cost to a volunteer when there are k volunteers is c/k. With equal task sharing, the expected cost is also c/k.

It might be good to have some other examples. Maybe this could be done for small numbers of people, or alternatively we could do a couple of numerical solutions for "interesting" examples of $c(k)$ functions.

Differing Costs and Private Information

The all-pay case

Suppose that the costs c of taking helpful action and the values b of having the action performed differ within the population. We make the following assumptions about the distribution of values and of beliefs.

Assumption 1. *(i) Individuals know their own values of* c *and* b*, but this information is private. (ii) All individuals share the correct belief that the other participants in the game that they play have parameters* c *and* b *such that the ratio* c/b *is an independent random draw from a population with the continuous cumulative distribution* $F(\cdot)$ *with support* $[\ell, u]$ *where* $\ell < 1$*.*

Remark 1. *Given Assumption 1, for any set of* $n \geq 2$ *players, there is a unique threshold* k_n *such that there is a Bayes-Nash equilibrium in which all individuals for whom* $c/b < k_n$ *volunteer and all of those for whom the inequality is reversed will refuse to volunteer.*

An example: The Pareto distribution

Suppose that the ratio of costs to benefits has a Pareto distribution of the form

$$F(x) = 1 - \left(\frac{\ell}{x}\right)^{\alpha} \tag{1}$$

over the interval $[\ell, \infty)$ for some $\alpha > 0$. When there are n players, for a player with the threshold level $c/b = k_n$, it must be that

$$b\,(1 - F(k_n))^{n-1} = b - k_n b \tag{2}$$

From Equations 1 and 2, it follows that

$$\left(\frac{\ell}{k_n}\right)^{\alpha(n-1)} = k_n \tag{3}$$

and hence

$$k_n = \ell^{\frac{\alpha(n-1)}{1+\alpha(n-1)}} \tag{4}$$

With n players, the equilibrium probability that nobody volunteers is

$$(1 - F(k_n))^n = \left(\frac{\ell}{k_n}\right)^{\alpha n} \tag{5}$$

From Equations 4 and 5 it follows that

$$(1 - F(k_n))^n = \ell^{\frac{\alpha n}{\alpha n + (1-\alpha)}} \tag{6}$$

From Equation 6 it can be seen that if $0 < \alpha < 1$, the probability that nobody contributes is decreasing in n and asymptotically approaches ℓ from above. If $\alpha = 1$, this probability is constant with respect to n if $\alpha > 1$ this probabiliy is increasing in n and asymptotically approaches ℓ from below.

Appendix

Proof of Proposition 1, Diekmann's Theorem

Proof. Suppose that each player helps with probability $p \in (0,1)$ and let $q = 1 - p$. The probability that no player volunteers is then q^n and the probability that at least one player volunteers is $1 - q^n$. The expected utility from the strategy *volunteer* is $b - c$ and the expected utility from the strategy *not volunteer* is $b\left(1 - q^{n-1}\right)$. In the symmetric mixed strategy Nash equilibrium for this game, all players are indifferent between choosing the pure strategies *volunteer* and *not volunteer*. Therefore in equilibrium it must be that

$$b - c = b\left(1 - q^{n-1}\right). \tag{7}$$

From Equation 7 it follows that in a symmetric Nash equilibrium,

$$q = \left(\frac{c}{b}\right)^{\frac{1}{n-1}} \tag{8}$$

In equilibrium, the probability that no player helps is

$$q^n = \left(\frac{c}{b}\right)^{\frac{n}{n-1}}. \tag{9}$$

Since, by assumption, $c/b < 1$, it follows from Equation 7 that the probability q that an individual *does not* volunteer increases with n. Furthermore, it follows from Equation 9 that the probability q^n that nobody volunteers also increases with n. From Equation 9, it is also apparent that as n increases, the equilibrium value of q^n approaches $\frac{c}{b}$ asymptotically from below.

□

Proof of Proposition 2

If a player chooses the pure strategy *refuse* while all other players refuse with probability q, then the probability that there is at least one volunteer is $1 - q^{n-1}$. In this case, the expected payoff to an individual who refused to volunteer would be

$$R(q,n) = b(1 - q^{n-1}). \tag{10}$$

In a symmetric mixed strategy Nash equilibrium, it must be that when all other players volunteer with probability p, the payoff to a player who volunteers with certainty is the same as that of a player who refuses to volunteer. In the case of a coordinated volunteer's dilemma, the cost of volunteering is a random variable which depends on the number of other players who volunteer.

Lemma 1. *If there are n players and if the expected cost to each volunteer is $c/(x+1)$ when a total of $x+1$ players volunteer, then if all other players volunteer with probability $p = 1-q > 0$, the expected cost for a player who volunteers with certainty is*

$$c\left(\frac{(1-q^n)}{(1-q)n}\right) = \frac{c}{n}\left(1 + q + q^2 + \dots q^{n-1}\right).$$

Proof. [1] The number of other players who volunteer is a binomial random variable equal to the number of successes in $n-1$ trials with probability p of success on each trial. If there are x volunteers among the other players, then the cost to a player who volunteers with certainty is $1/(1+x)$. Therefore the expected cost of volunteering is the expected value of $c/(x+1)$ where x has the binomial distribution $B(n-1,p)$. We show that this expected value is $c(1-q^n)/pn$.

We have

$$\begin{aligned} 1 &= (p+q)^n, \\ &= q^n + \sum_{k=1}^{n} \binom{n}{k} q^{n-k} p^k, \\ &= q^n + \sum_{k=1}^{n} \frac{n}{k} \binom{n-1}{k-1} q^{n-k} p^k \end{aligned}$$

It follows that

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} q^{n-k} p^k = \frac{1-q^n}{n}$$

[1] An alternative route to this result is as follows: In a symmetric mixed-strategy equilibrium, it must be that for each player, the expected utility of playing the equilibrium mixed strategy is the same as that of playing either of the two pure strategies. If all players use the mixed strategy *refuse with probability* q, then the task will be performed with probability $1-q^n$ and if the task is performed, it is equally likely to be assigned to each player. Therefore the the expected utility of each player when all refuse with probability q is

$$M(q,n) = (1-q^n)\left(b - \frac{c}{n}\right).$$

If we set $R(q_n, n) = M(q_n, n)$, and rearrange terms, we arrive at Equation 11.

Making a change of variables, $x = k - 1$, we can rewrite this expression as

$$\sum_{x=0}^{n} \frac{1}{x+1} \binom{n-1}{x} q^{n-1-x} p^{x+1} = \frac{1-q^n}{n}$$

It then follows that

$$\sum_{x=0}^{n} \frac{1}{x+1} \binom{n-1}{x} q^{n-1-x} p^{x} = \frac{1-q^n}{pn}.$$

Therefore the expected cost of volunteering is

$$c \left(\sum_{x=0}^{n} \frac{1}{x+1} \binom{n-1}{x} q^{n-1-x} p^{x} \right) = c \left(\frac{1-q^n}{pn} \right).$$

We observe that

$$\begin{aligned} c \left(\frac{1-q^n}{pn} \right) &= \frac{c}{n} \left(\frac{1-q^n}{1-q} \right) \\ &= \frac{c}{n} \left(1 + q + \cdots + q^{n-1} \right). \end{aligned}$$

This establishes the claimed result. □

Lemma 2. *Where we define*

$$H(q,n) = \frac{q^{1-n} + q^{2-n} + \cdots + q^{-1} + 1}{n},$$

it must be that in symmetric Nash equilibrium,

$$H(q_n, n) = \frac{b}{c}. \tag{11}$$

Proof. From Lemma 1, it follows that when all other players volunteer with probability $p = 1 - q$, the expected payoff to a player who volunteers with certainty is

$$V(q,n) = b - \frac{c}{n} \left(1 + q + \cdots + q^{n-1} \right). \tag{12}$$

If there are n players and q_n is the symmetric mixed-strategy Nash equilibrium probability q_n that a player refuses, it must be that

$$R(q_n, n) = V(q_n, n). \tag{13}$$

Equations 10, 12, and 13 imply that

$$b - bq_n^{n-1} = b - c\left(\frac{1 + q_n + \cdots + q_n^{n-1}}{n}\right) \tag{14}$$

Rearranging terms of this expression yields

$$H(q_n, n) = \frac{b}{c} \tag{15}$$

□

Comparative statics are aided by the following properties of the function $H(q, n)$.

Lemma 3. *Where*

$$H(q, n) = \frac{q^{1-n} + q^{2-n} + \cdots + q^{-1} + 1}{n}, \tag{16}$$

for all positive integers $n > 1$, *and for all* $q \in (0, 1)$, *it must be that*

(i) $H(q, n)$ *is continuous and strictly decreasing in* q

(ii) $H(q, n + 1) > H(q, n)$

Proof. The derivative of $H(q, n)$ with respect to q is easily seen to be negative for all $q \in (0, 1)$ and hence H is continuous and decreasing in q.

From Equation 16, we see that

$$\begin{aligned} H(q, n+1) &= \frac{q^{-n} + q^{1-n} + q^{2-n} \cdots + q^{-1} + 1}{n+1} \\ &= \left(\frac{1}{n+1}\right) q^{-n} + \left(\frac{n}{n+1}\right) H(q, n) \\ &> H(q, n) \end{aligned} \tag{17}$$

where the final inequality follows from that fact that for $i = 1 \ldots n$ and for all $q \in (0, 1)$, $q^{-n} > q^{i-n}$ and therefore $q^{-n} > H(q_n, n)$. □

We now use these results to complete the proof of Proposition 2

Proof of Proposition 2. In a symmetric Nash equilibrium, each player refuses to volunteer with probability q_n where $H(q_n, n) = b/c$. It is easily verified that $\lim_{q \to 0} H(q, n) = \infty$ and $\lim_{q \to 1} H(q, n) = 1$. Since, by assumption, $b > c$, it must be that $H(q, n) < b/c$ for k close to 1 and $H(q, n) > b/c$

for q close to zero. According to Lemma 3, H is a continuous decreasing function of q. Therefore there must be a unique $q_n \in (0,1)$ such that $H(q_n, n) = b/c$.

To see that q_n decreases as b/c increases, we note that according to Lemma 3, $H(q,n)$ is a decreasing function of q and that $H(q_n, n) = b/c$. Therefore it must be that q_n decreases as b/c increases.

Where q_n and q_{n+1} are the symmetric Nash equilibrium probabilities that an individual will refuse to volunteer when there are n and $n+1$ players, respectively, it must be that $H(q_n, n) = b/c$ and $H(q_{n+1}, n+1) = b/c$. Lemma 3(ii) implies that $H(q_n, n+1) > H(q_n, n) = b/c = H(q_{n+1}, n+1)$. Since $H(q,n)$ is a decreasing function of q, it must be that $q_n < q_{n+1}$. Thus we conclude that the symmetric Nash equilibrium probability that an individual refuses to volunteer increases with the number of players.

□

Proof or Proposition 3

Proof of Proposition 3. The probability that nobody volunteers will be larger for a group of size $n+1$ than for a group of size n if $q_{n+1}^{n+1} > q_n^n$. This will be the case if $q_{n+1} > q_n^{n/(n+1)}$. The equilibrium conditions require that

$$H\left(q_{n+1}, n+1\right) = H\left(q_n, n\right) = \frac{b}{c}. \tag{18}$$

Lemma 4, which is proved below, implies that $H\left(q_n^{n/n+1}, n+1\right) > H(q_n, n)$. According to Lemma 3, $H(q, n+1)$ is a decreasing function of q and since, according to Equation 18, $H(q_{n+1}, n+1) = H(q_n, n)$, it must be that $q_{n+1} > q_n^{n/(n+1)}$ and hence $q_{n+1}^{n+1} > q_n^n$. But this implies that the probability that nobody volunteers increases with the number of players. □

Lemma 4. *For all $q \in (0,1)$ and all integers $n \geq 1$, $H\left(q^{n/n+1}, n+1\right) > H(q,n)$.*

Proof. We note that

$$\begin{aligned} H(q,n) &= \frac{q^{1-n} + q^{2-n} + \cdots + q^{-1} + 1}{n} \\ &= \frac{1}{n}\left(\frac{q}{1-q}\right)\left(\frac{1-q^n}{q^n}\right). \end{aligned} \tag{19}$$

Then

$$H(q^{\frac{n}{n+1}}, n+1) = \frac{1}{n+1}\left(\frac{q^{\frac{n}{n+1}}}{1-q^{\frac{n}{n+1}}}\right)\left(\frac{1-\left(q^{\frac{n}{n+1}}\right)^{n+1}}{\left(q^{\frac{n}{n+1}}\right)^{n+1}}\right)$$
$$= \frac{1}{n+1}\left(\frac{q^{\frac{n}{n+1}}}{1-q^{\frac{n}{n+1}}}\right)\left(\frac{1-q^n}{q^n}\right) \tag{20}$$

It follows from Equations 19 and 20 that

$$H(q^{\frac{n}{n+1}}, n+1) > H(q,n) \tag{21}$$

if and only if

$$\frac{1}{n+1}\left(\frac{q^{\frac{n}{n+1}}}{1-q^{\frac{n}{n+1}}}\right) > \frac{1}{n}\left(\frac{q}{1-q}\right). \tag{22}$$

Rearranging terms, we see that the inequality 22 is equivalent to

$$\frac{n}{n+1}\left(\frac{1-q}{1-q^{\frac{n}{n+1}}}\right) > q^{\frac{1}{n+1}} \tag{23}$$

Define $x = q^{\frac{1}{n+1}}$. Since $q \in (0,1)$, it must be that $0 < x < 1$. Inequality 23 can be written as

$$\frac{1-x^{n+1}}{x(1-x^n)} > \frac{n+1}{n}, \tag{24}$$

which is equivalent to

$$\frac{(1+x+\cdots+x^n)(1-x)}{x(1+x+\cdots+x^{n-1})(1-x)} > \frac{1+n}{n}. \tag{25}$$

Inequality 25 simplifies to

$$\frac{1+x+\cdots+x^n}{x+\ldots x^n} > \frac{1+n}{n}, \tag{26}$$

which is equivalent to

$$x+\cdots+x^n < n. \tag{27}$$

Since, for all $x \in (0,1)$, Inequality 27 must hold, and since this inequality is equivalent to Inequality 21, the lemma is proved. $\square$

Proof of Proposition 4

Proof of Proposition 4. Where q_n is the equilibrium probability that an individual does not volunteer,

$$\frac{c}{b} = \frac{n\left(1 - q_n\right) q_n^{n-1}}{1 - q_n^n} \tag{28}$$

Let $Q_n = q_n^n$, the probability no one volunteers. Then Equation 28 can be written as

$$\frac{c}{b} = \frac{n\left(1 - Q_n^{\frac{1}{n}}\right) Q_n^{\frac{n-1}{n}}}{1 - Q_n} \tag{29}$$

which implies that

$$\left(1 - Q_n\right)\frac{c}{b} = n\left(Q_n^{1-\frac{1}{n}} - Q_n\right) \tag{30}$$

Where $Q = \lim_{n\to\infty} Q_n$, making a change of variables $t = 1/n$ and taking limits, we find that

$$\begin{aligned}\Big(1 - Q\Big)\frac{c}{b} &= \lim_{n\to\infty} n\left(Q_n^{1-\frac{1}{n}} - Q_n\right) \\ &= \lim_{t\to 0}\frac{Q^{1-t} - Q}{t} \end{aligned} \tag{31}$$

If we define the function $g(x) = Q^x$, then we see that

$$\lim_{t\to 0}\frac{Q^{1-t} - Q}{t} = -g'(1) \tag{32}$$

$$= -Q\ln Q \tag{33}$$

Therefore it must be that in the limit as $n \to \infty$, the equilibrium probability Q that nobody volunteers satisfies the equation

$$\left(1 - Q\right)\frac{c}{b} = -Q\ln Q \tag{34}$$

Let us define $\theta = \frac{c}{b}$. Then From Equation 34 it follows that

$$\left(\theta - \frac{\theta}{Q}\right) = \ln Q \tag{35}$$

and hence

$$e^{\theta}e^{-\frac{\theta}{Q}} = Q. \tag{36}$$

From Equation 36, it follows that

$$-\frac{\theta}{Q}e^{-\frac{\theta}{Q}} = -\theta e^{-\theta} \tag{37}$$

Equation 37 enables us to find a unique solution $Q < 1$ expressed in terms of Lambert's W function, where $W(x)$ is defined to be a solution to the equation

$$W(x)e^{W(x)} = x \tag{38}$$

Thus we must have

$$-\frac{\theta}{Q} = W\left(-\theta e^{-\theta}\right) \tag{39}$$

Since, by assumption, $0 < \frac{c}{b} < 1$, it must be that $-\theta e^{-\theta} \in (-e^{-1}, 0)$. It is known that for all $x \in (-e^{-1}, 0)$, there are exactly two solutions to Equation 38.[2] One of these solutions lies on the "upper branch" of W and has a value $y \in (-1, 0)$ one lies on the "lower branch" and has a value $y \in (-\infty, -1)$.

Recalling that $\theta = \frac{c}{b}$, we have now shown that the limit as n gets large of the probability Q that nobody volunteers is

$$Q = -\frac{\frac{c}{b}}{W\left(-\frac{c}{b}e^{-\frac{c}{b}}\right)} \tag{40}$$

where $W\left(-\frac{c}{b}e^{-\frac{c}{b}}\right)$ takes the value on the lower branch solution of Lambert's W function.

[2]Lambert's W is single valued over the range $[0, \infty]$. Equation 38 has no solutions for $x < e^{-1}$.[2]

Table 1: Volunteer Probabilities by Group Size $b/c = 1.1$

n	q	q^n	p	$1 - q^n$
1	0	0	1	1
2	.833	.694	.167	.306
3	.912	.758	.088	.242
4	.934	.761	.066	.239
5	.955	.792	.045	.208
6	.964	.804	.036	.196
7	.969	.805	.031	.195
50	.996	.827	.004	.173
100	.998	.827	.002	.173
200	.999	.835	.001	.165

2.1 Numerical Solutions

I did numerical solutions using MatLab's polynomial solver. For this purpose it is handy to rearrange the equilibrium condition

$$bq^{n-1} = c\frac{1 - q^n}{n(1 - q)} \tag{41}$$

in the form

$$(n\frac{b}{c} - 1)q^n - n\frac{b}{c}q^{n-1} + 1 = 0 \tag{42}$$

I have constructed tables for $b/c = 1.1$, $b/c = 2$, and $b/c = 4$ and shown the symmetric equilibrium probabilities for numbers of players n from 1 to 7 and for 50, 100, and 200. The last column shows the probability that someone helps.

Table 2: Volunteer Probabilities by Group Size $b/c = 2$

n	q	q^n	p	$1 - q^n$
1	0	0	1	1
2	.333	.111	.667	.889
3	.558	.174	.442	.826
4	.672	.203	.328	.797
5	.739	.222	.261	.778
6	.784	.232	.216	.768
7	.816	.240	.184	.760
50	.974	.262	.026	.738
100	.987	.273	.013	.723
200	.994	.283	.006	.717

Table 3: Volunteer Probabilities by Group Size $b/c = 4$

n	q	q^n	p	$1 - q^n$
1	0	0	1	1
2	.143	.020	.857	.980
3	.350	.043	.650	.957
4	.486	.056	.514	..944
5	.577	.064	.423	.936
6	.641	.069	.359	.931
7	.688	.073	.312	.927
50	.954	.094	.046	.906
100	.977	.0946	.023	.9054
200	.988	.0950	.012	.9050

References

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