Human Capital in an Economic Growth Model

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HUMAN CAPITAL IN AN ECONOMIC GROWTH MODEL

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By
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September 1967
I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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INTRODUCTION

Human Capital as a Factor of Production.

Recent attempts ([1], [26], [5]) to attribute growth of the United States economy to increased inputs of capital and labor have run into a common difficulty. More growth is found than can readily be explained. Presumably even the most dour practitioners of the "dismal science" will find satisfaction that the United States economy is performing better than the straightforward application of simple growth models would predict. Yet an economist, no matter how sanguine, is reluctant to accept a beneficent *deux ex machina* (even if it is called *technical progress*) as an adequate explanation of past growth or a sufficiently helpful indicator of potential growth.

It appears to be worthwhile for economists to examine other measurable productive inputs which may play a significant role in economic growth. The conventional measures of aggregate labor inputs do not account for improvements in the quality of the labor force. Investments made in improved laborers are not included as part of the capital aggregate. Some economists, for instance E. F. Denison [5] and Wassily Leontief [14], have suggested that the amount of labor inserted into the production function be adjusted to allow for changes in the quality of labor. Such a procedure, as Denison illustrates, is very helpful in accounting for the sources of economic growth. However,
mere adjustments in the estimates of labor inputs tend to obscure at least two important differences between population growth and improved labor quality. There is little doubt that the supply of education in the labor force responds in quite a different way to economic stimuli than does the supply of children. Furthermore, it seems unlikely that education and people should be perfect substitutes in production. For similar reasons, inclusion of investment in human beings as part of the aggregate amount of physical capital would not be particularly enlightening.

In this paper human capital will be included as a third factor of production in the aggregate production function. Human capital when referred to here will usually mean formal education. There may very well be other significant investments in human beings which behave sufficiently like formal education to justify their inclusion in the aggregate here discussed.

The general form of the aggregate production function to be discussed will be $Y = F(K, H, L)$, where $Y$ is output, $K$ the stock of physical capital as ordinarily defined, $H$ the stock of human capital and $L$ the number of people in the labor force. A special case of this aggregate production function is that in which the function $F$ is separable in the sense that we may write $Y = F(K, g(H,L))$. The function $g$ might be considered a measure in quality units of the amount of human effort exerted by the labor force. Denison [5] makes the implicit assumption that $g(H,L) = \alpha H + L$ where $H$ is the number of days of school attendance represented in the labor force, $L$ is the number of laborers adjusted for age, sex and hours worked, and $\alpha$ is a constant
representing the improvement of labor quality due to an extra day's schooling.

Theodore Schultz [21] has made estimates of the stock of educational capital valued at factor costs in 1956 prices. His estimates indicate that between 1929 and 1957 the amount of education held by the average member of the labor force had risen approximately twice as fast as per capita income. In the period between 1900 and 1929, the per capita stock of educational capital rose at approximately the same rate as per capita income. Estimates which I have made (see Chapter IV of this paper) indicate that in the period from 1890 to 1965, expenditure on education per capita has been rising about twice as rapidly as per capita income. Cross-country studies ([9], [4], [22]) also lend some support to the hypothesis that educational investment increases more than in proportion to income.

There are at least three alternative though not necessarily mutually exclusive explanations for an increase over time in the ratio of the stock of human capital to income. These are: (1) Education is considered a consumption good by those who are educated or by those who pay for their education. The income elasticity of demand for education exceeds unity at sufficiently high levels of income. (2) There are or have been significant imperfections in the capital market for educational investment. These imperfections have been reduced over the years. As a result the return necessary for profitable investment in human capital has fallen relative to the return on physical capital. For this reason the supply of human capital has expanded more rapidly than the supply of physical capital. (3) Changes in productive techniques or in the composition of demand have had non-neutral effects, tending to save on
capital and raw labor and to make education more important in production. Thus a large expansion in the input of human capital has been a response to upward shifts of derived demand.

In this paper the first explanation is given most weight.

I will assume that the supply of new human capital is dependent on per capita income and on the existing stock of human capital. Since in the United States the stock of human capital appears to have been rising much faster than per capita income and the percentage of national income spent on education has increased with per capita income, I assume that at least at some levels of income, education is a luxury good.

One of the implications of such a model is that as per capita income grows, the stock of human capital and the portion of income spent on education grow even faster. If the economy is capable of forever maintaining a positive rate of growth of income per head and the income elasticity of demand for education remains greater than one, the amount spent on education would ultimately reach and then exceed the entire national income. In the very long run if income is growing we would expect the income elasticity of demand to fall and thus avert such a heady (and absurd) state of affairs. It may still be true that over a large range of per capita income, the income elasticity of demand for education remains roughly constant and greater than unity.

I have argued that it is probably useful to separate an aggregate called human capital away from both "labor" and "physical capital." It is then relevant to inquire whether human capital can be defined as a sufficiently well-behaved aggregate to justify its use as an analytical
tool. Closely interwoven with this theoretical problem of aggregation is the practical problem of estimating the stock of human capital and comparing amounts of human capital over time. (The conceptual problems of comparing stocks of human capital at different places at the same time are virtually the same as those of comparing stocks in the same place at different times. The argument below would be little changed if "space" were substituted for "time".)

If there were no quality changes in education over time and if a day in school were equally effective in raising a laborer's output regardless of the level or kind of schooling, we would have a very simple measure of human capital as a factor of production. We would need only to count up the number of days of schooling held by the labor force.

The world in which we live is, of course, not so simple. Even if the quality of education of a given sort had not changed over time, the mix of different kinds of education has changed considerably. The most obvious change for the United States over the past fifty years is the large increase in the amount of advanced education relative to more elementary education. Certainly it would not be legitimate to weight a year of elementary school as equivalent to a year of college. The latter is much more expensive and probably somewhat more productive.

Theodore Schultz [21] estimates the stock of human capital in the United States for various years in the following way. Days of elementary school, high school, and college are weighted by their relative factor costs, (including income foregone by students) in some base year.

*For general theoretical discussions of the problem of aggregation see Hicks [10] and Solow [25].
Then the weighted days of school held by the labor force at a given time are added up. If the quality of education of a particular sort has not changed over time, this method corresponds to Denison's [5] criterion for an index of the stock of physical capital "...the amount it would have cost at base year prices to produce the actual stock of capital goods in a given year."

Let us consider under what conditions the use of base year prices as weights is a good index of the relative productive powers of the various kinds of education.

Suppose that the quality of a given kind of education remains unchanged over time and that all kinds of education are perfect substitutes in production. Then we could write the aggregate production as follows: \( Y = F(K, H, L) \) where \( H = \sum_i \alpha_i H_i \), \( H_i \) is the number of days of the \( i \)th kind of human capital held by the labor force, \( \sum \alpha_i = 1 \) and \( \frac{\partial Y}{\partial H_i}/(\partial Y/\partial H_j) = \alpha_i/\alpha_j \) for all \( i, j \). The weights \( \alpha_i \) would be invariant over time and would correspond to the relative productivities of the various forms of education.

If, in addition, the social rates of return were the same for all forms of education, then the relative base year costs would correspond very closely to relative productivities. Relative costs would equal relative productivities if the length of life of the various sorts of education was the same.

If the internal rates of return on some kinds of education were higher than on other kinds, base year costs would not properly measure relative productivities. Studies by Becker [3] indicated that the rate of return from sending an additional child to elementary school
is much higher than the corresponding rate of return from high school and college education. If this is the case, the relative productivity of elementary school is considerably underestimated by the use of costs as weights. If, for example, a day of elementary education and a day of other education resulted in constant marginal products $MP_{el}$ and $MP_0$ respectively for all time, the formula for the internal rate of return would give us $r_{el} C_{el} = MP_{el}$ and $r_0 C_0 = MP_0$ where $r_{el}$ and $r_0$ represent internal rates of return and $C_{el}$ and $C_0$ costs of production of elementary and other education. If $r_{el} > r_0$ then,

$$\frac{MP_{el}}{MP_0} = \frac{r_{el}}{r_0} \frac{C_{el}}{C_0} > \frac{C_{el}}{C_0}.$$ Since the amount of higher education has increased relative to the amount of elementary education Schultz's figures would, on this score, overestimate the gain in the stock of human capital over time.

It is not necessarily indicative of a non-optimal educational mix that $r_{el} > r_0$. The costs and rates of return referred to above are those rates which would be appropriate if one is considering whether to send an additional child to grammar school, an additional teenager to high school or an additional young adult to college. In the United States in recent years, almost everyone of the ages 7 - 14 has been enrolled in grade school. Practically the only way to expand the amount of elementary education in the United States is to increase the quantity of inputs other than children. There is an absolutely limited supply of this factor which is particularly well suited for production of elementary education. We would have more third graders except that we have run out of eight year-olds! It would not therefore be surprising if the rate of return to elementary education, when
calculated without imputing rental prices to children, should exceed that to higher levels of education even if the social return to a marginal expenditure were the same at all levels.

This difficulty is serious for the practical calculation of weights for elementary and higher forms of education, but does not affect the theoretical legitimacy of human capital as an aggregate factor of production so long as all forms of human capital are perfect substitutes. One need only know their relative marginal products, and using these as weights he can construct a well-behaved aggregate of human capital.

If we assume that some forms of education are partial complements to others, new difficulties are introduced. In particular suppose $Y = F[K, g(H_1, H_2), L]$ where $g$ is smooth and concave. $H_1$ and $H_2$ are the amounts of two sorts of human capital. It may be for example that education in economics and education in history are to some extent complementary. This represents no problem if the market approximately equates rates of return on the two sorts of education and if their relative costs do not change over time. The ratio $H_1/H_2$ would remain unchanged over time since any change in this ratio would change the relative marginal products of $H_1$ and $H_2$. Even an incorrect weighting of $H_1$ relative to $H_2$ would not bias comparison of human capital stocks over time.

In the case of elementary versus higher forms of education, complementarity would present a more serious problem. It is clear that the amount of advanced education has increased relative to the amount of elementary education. The actual estimation of the relative productivities
of elementary and higher forms of education becomes difficult. Furthermore the aggregate $H$ which could be constructed by weighting according to relative marginal productivities would be a very awkward theoretical beast. The contribution of elementary education to output depends on the amount of advanced education and vice versa. Knowledge of the aggregate amount of human capital is not sufficient to give us any information concerning the productivity of additional education at any particular level.

If an aggregate of human capital is used to discuss growth during a period in which virtually everyone receives an elementary education we lose no relevant information if we aggregate elementary education with the exogenously growing labor force rather than with higher education. For time spans which include periods in which elementary education is not universal, it would seem that the legitimacy of aggregating advanced education with elementary education would depend on how closely the levels of education approach perfect substitutability.

Now suppose that the quality of education changes over time. If the change in quality is due to increased inputs per day of school, for example, more (or more intensively trained) teachers per student, base year prices of a day of school would overestimate the value of schooling received in earlier years relative to that of more recent schooling. A day of education in the base year would not be the same thing as a day of schooling in a non-base year. It would not be correct to count a day of school as the same thing no matter when it occurred. A better measure of schooling received in a given year would be the cost
at base year prices of producing a day of schooling of the same quality as that actually received in the given year.

If the quality of education has changed because improved knowledge is available or discoveries have been made concerning teaching methods, base year costs will again underestimate the productive power of recent education relative to that of education received at an earlier date. One could either adjust the weight of education in the stock of human capital according to the time when it was received, (see Solow [24]) or could retain the cost weighting method and attribute residual gains in total product to the advance of the state of the arts (see Denison [5]). The advantage of the latter procedure is that quality changes due to the advance of knowledge are very difficult to measure directly. The advantage of the former method is that it indicates the important role of new human capital as an instrument which brings improved knowledge into use.

The preceding has dealt with the difficulties of constructing an aggregate which is well suited for describing the role of human capital as a factor of production. Since this paper deals with human capital as a direct source of utility as well, it is also important that the aggregate constructed for human capital be a well-behaved consumer's good. The weights which are appropriate for various forms of education when we are considering human capital as a factor of production do not necessarily correspond to the weights appropriate when education is considered as a consumption good. It may be that education in the humanities merits a higher weight relative to technical education on the consumption side than it does on the production side.
Though this question is important and interesting in its own right, it shall be largely ignored in the subsequent discussion.

Discoveries which improve the efficiency of new human capital can be allowed for by counting a day of school as more human capital if received when knowledge is better. The amount of effective education when considered from the consumption side need not change in the same way. A simple alternative assumption would be that days of education remain the relevant measure for education as a consumption good. This question will be further pursued in the discussion of technical progress.
CHAPTER I

A GROWTH MODEL DESCRIBING THE INTERACTION BETWEEN

INCOME AND THE STOCK OF HUMAN CAPITAL

Notation

Let

\( Y(T) \) be output,

\( K(T) \) the stock of physical capital,

\( H(T) \) the stock of human capital,

\( L(T) \) the number of people in the labor force,

all at time \( T \).

\[ y(T) = \frac{Y(T)}{L(T)} \]

\[ k(T) = \frac{K(T)}{L(T)} \]

\[ h(T) = \frac{H(T)}{L(T)} \]

\[ \dot{x}(T) = \frac{dx(T)}{dT} \] for any variable \( x \),

\[ \hat{x}(T) = \frac{\dot{x}(T)}{x(T)} \] for any variable \( x \).

Whenever the meaning is unambiguous variables will be written without the argument \( T \).
Assumptions

(1) All goods in the economy are produced according to a single aggregate production function of the Cobb-Douglas form with constant returns to scale in the three factors K, H, and L. \[ Y = K^\alpha H^\beta L^{1-\alpha-\beta}, \]
\[ 0 < \alpha < 1, \quad 0 < \beta < 1, \quad 0 < 1-\alpha-\beta < 1. \]

(2) The labor force is growing at the constant rate \( n \).

(3) \( \dot{K}/L = s(Y/L)^{\delta} \) where \( s \) and \( \delta \) are positive constants. This is a generalization of the neoclassical savings function. If \( \delta = 1 \), \( \dot{K} = sY \). If \( \delta < 1 \), an increase in per capita income results in a less than proportionate increase in per capita saving and investment.

(4) \( \dot{H}/L = c(Y/L)^{\epsilon} + a(H/L) \) where \( c \) and \( \epsilon \) are positive constants and \( a < n \).

Equation (4) requires some justification. If \( a \) is zero, per capita investment in education has an elasticity of \( \epsilon \) with respect to per capita income. It is possible that the amount of new education per capita also depends on the stock of existing human capital. This effect is represented by the parameter \( a \). One might claim a positive value for \( a \) on the grounds that a more educated populace probably desires to spend more on the education of its youth. On the other hand, as the stock of education increases, the rate of depreciation of the existing stock also increases, thus lowering the rate of net human capital formation. The assumption that \( a < n \) is needed to
insure the existence of a steady growth path.*

Equation (4) assumes that educational expenditure depends largely on per capita income rather than on private or social monetary returns to education. It is hard to argue that in reality these rates of return have no effect on the supply of new education. My hypothesis is that although such effects may exist, the large consumption benefits from education and the highly imperfect capital market for educational investment make rising per capita income the dominant force for the expansion of educational investment.

Rewriting assumptions (1a) - (4a) in per capita form, we have

\[ y = k^\alpha h^\beta \]  \hspace{1cm} (1a)'

*One set of assumptions which would lead to equation (4) is the following. New human capital enters the labor force only when new laborers enter. The level of education of the new laborers depends on per capita income and the amount of existing human capital per man at the time they enter the labor force. Human capital depreciates at the rate \( d_H \). Then we could write

\[ \frac{\dot{H}(T) + d_H h(T)}{L(T)} = c' \left[ \frac{Y(T)}{L(T)} \right] c + a' \left[ \frac{H(T)}{L(T)} \right] . \]

Since \( \dot{L} = nL \), this equation can be rewritten

\[ \frac{\dot{H}}{L} = nc' \left[ \frac{Y}{L} \right] c + [na' - d_H] \left[ \frac{H}{L} \right] \]

\[ = c \left[ \frac{Y}{L} \right] c + a \left[ \frac{H}{L} \right] \]

where \( c = nc' \), \( a = na' - d_H \). \( a < n \) if \( a' < 1 + d_H/n \).
\[ \hat{L} = n \] (2a)'

\[ \hat{k} = \frac{L}{K} \frac{\dot{K}}{L} - n = \frac{\epsilon y^6}{k} - n \] (3a)'

\[ \hat{n} = \frac{L}{H} \frac{\dot{H}}{L} - n = \frac{cy^6}{h} - \mu \quad \text{where} \quad \mu = n-a \] (4a)'

Given assumptions (1a) - (4a), the following propositions hold.

**Proposition I:**

There exists a unique stationary state equilibrium if \( n > 0 \), \( \mu > 0 \), and \( 1 - \beta \varepsilon - 5\alpha \neq 0 \).

**Proposition II:**

If there exists a stationary state equilibrium, the system converges asymptotically to the stationary state if \( 1 - \beta \varepsilon - 5\alpha > 0 \). The system is explosive if \( 1 - \beta \varepsilon - 5\alpha < 0 \).

**Proof of Proposition I:** (existence)

Taking logs of equations (1a)', (3a)', and (4a)' we have

\[ \log y = \alpha \log k + \beta \log h \] (1a)"

\[ k \searrow 0 \quad \text{as} \quad 8 \log y - \log k \searrow \log \frac{n}{s} \] (3a)"
\( \hat{h} \gtrsim 0 \quad \text{as} \quad \epsilon \log y - \log h \gtrsim \log \frac{\mu}{c} \) \hspace{1cm} (4a)''

There exists a unique stationary state equilibrium if and only if there exists a unique simultaneous solution to (la)'', (3a)'', and (4a)'' where the equalities hold in (3a)'' and (4a)''.

Substituting (1a)'' into (3a)'' and (4a)'' and rearranging terms we see that this is equivalent to the existence of a unique solution to the simultaneous equations

\[
\log k = \frac{\epsilon \beta}{1 - \alpha \epsilon} \log h - \frac{1}{1 - \alpha \epsilon} \log \frac{n}{s}
\]

and

\[
\log k = \frac{1 - \beta \epsilon}{\epsilon \alpha} \log h - \frac{1}{\epsilon \alpha} \log \frac{\mu}{c}
\]

Such a solution exists if \( 1 - \beta \epsilon - \beta \alpha \neq 0 \). This can quickly be seen if we graph the two equations which are linear in the logs of \( k \) and \( h \)

![Figure 1](image-url)
The two lines $\hat{k} = 0$ and $\hat{h} = 0$ must intersect somewhere in the log space if their slopes are unequal; that is if $\delta \beta / (1 - 5\alpha) \neq (1 - \beta \varepsilon) / \varepsilon \alpha$ or equivalently $1 - \beta \varepsilon - 5\alpha \neq 0$.

Proof of Proposition II: (stability)

From (1a)'', (3a)'', and (4a)'' we have

$$\hat{k} \searrow 0 \quad \text{as} \quad \left[5\alpha - 1\right] \log k + 5\beta \log h \searrow \log \frac{n}{s}$$

and

$$\hat{h} \searrow 0 \quad \text{as} \quad \varepsilon \alpha \log k + \left[\varepsilon \beta - 1\right] \log h \searrow \log \frac{H}{c}.$$

Since $(5\alpha - 1) < 0$ and $\varepsilon \alpha > 0$, these relations give us

$$\hat{k} \nearrow 0 \quad \text{as} \quad \log k \nearrow \frac{5\beta}{1 - \alpha \delta} \log h - \log \frac{n}{s} \quad (3a)'''$$

and

$$\hat{h} \nearrow 0 \quad \text{as} \quad \log k \nearrow \frac{1 - \beta \varepsilon}{\varepsilon \alpha} \log h + \log \frac{H}{c} \quad (4a)''''$$

Since there exists a point $(k^*, h^*)$ where $\hat{h} = \hat{k} = 0$, it is legitimate to choose units of $H$ and $K$ in such a way that $k^* = h^* = 1$ and $\log k^* = \log h^* = 0$.

A phase diagram for $(3a)'''$ and $(4a)''''$ will look like either (a) or (b), below.
Case (a):

Case (b):

FIGURE 2

FIGURE 3
Case (a) is stable, case (b) is unstable. If the \( \hat{k} = 0 \) line is steeper than the \( \hat{h} = 0 \) line we have case (a). The condition for this to occur is \( (1-\beta\varepsilon)/\varepsilon \alpha > \delta\beta/(1-\alpha\varepsilon) \). But this is equivalent to \( 1 - \beta\varepsilon - \delta\alpha + \delta\beta\varepsilon \alpha > \delta\varepsilon \alpha \) or \( 1 - \beta\varepsilon - \delta\alpha > 0 \). If \( 1 - \beta\varepsilon - \delta\alpha < 0 \) Case (b) obtains and the model is unstable.

Propositions I and II can be generalized in a straightforward way to the case where there are \( n \) forms of capital. This is demonstrated in Appendix I. A Lyapunov transformation is used to prove stability.

If \( 1 - \beta\varepsilon - \alpha\delta < 0 \), then the interaction of expanding income and even more rapidly expanding expenditure on education is sufficient to cause increasing rates of growth of income. This condition may be compared with the well-known results of Solow [23] and Swan [27] for a one sector, two factor economy with constant returns to scale, no technical change, and a rate of investment proportional to income. Under these conditions, if there are diminishing returns to capital, the economy will approach a balanced growth path with a constant output per laborer. The reason is that as the capital-labor ratio rises, the output capital ratio falls and with net investment proportional to total output, the rate of growth of capital must approach the rate of growth of the labor force. If, however, the income elasticity of supply of some forms of capital exceeds unity, it is possible that the rising percentage of income invested will overwhelm the diminishing marginal products of capital goods which accompany capital deepening. Such would be the case here if \( 1 - \beta\varepsilon - \alpha\delta < 0 \).
The parameters \( \alpha, \beta, \delta, \epsilon \) and in particular \( 1 - \beta \epsilon - \alpha \delta \) assume critical importance for the behavior of this model. Estimates of \( \alpha \), the functional share of physical capital, and \( \delta \), the income elasticity of supply of physical capital, can be found in existing studies of the sources of economic growth. Available evidence [37] seems to indicate that \( \alpha \) is about 0.25 and that the stock of physical capital has been increasing slightly less rapidly than income. A reasonable estimate of \( \delta \) would be in the range of 0.9 [11]. The reader is invited to use his own favorite estimates of \( \alpha \) and \( \delta \).

The functional share of human capital and the income elasticity of supply of human capital have not been systematically investigated in past discussions of economic growth. The functional share of human capital, \( \beta \), which I wish to estimate is that part of the conventionally calculated share of labor which is a return to education. The part of earnings which is not due to education is estimated by the amount which a laborer who is average in all other aspects would earn had he received no education. Using United States census data, I estimated the functional share of human capital for 1939 and 1959. The estimate for each of these years is about 0.3.

To estimate the income elasticity of the supply of human capital I used time series estimates of per capita income, costs of education per student, and school enrollment from 1890 to 1965. The estimate for \( \epsilon \) is about 2. If these estimates are approximately correct, 
\[
1 - \beta \epsilon - \alpha \delta \approx 1 - (2)(.3) - (.9)(.25) = .17.
\] The stable case appears to be the more relevant for the United States in this century.
If the parameters of the United States economy are such that, at least over long periods of time, the interaction of income and education is not sufficient to produce sustained economic growth, we must look elsewhere to explain the observed growth of per capita income. In the next chapter, I will explore the possible interactions between education and the advance of knowledge which may produce a sustained positive rate of growth of per capita income.

It is important to remember, especially when we consider the possibility of the unstable case, that this model with a rigid $\epsilon$ greater than unity cannot be interpreted as valid in the very long run. The income elasticity of demand for education must ultimately fall or at some high level of income we would reach the totally cerebral state. All income (and more) would be spent on education. At very low levels of income, it is also likely that the income elasticity of educational expansion will be small. Additional income may be needed just for sustenance. If $\epsilon$ changes with income, locally stable equilibrium rates of growth may obtain at some levels of income while there are unstable equilibria at other levels. Figure 4 below describes
this situation. There is an unstable equilibrium at \( E_2 \). If the system moves to the right of \( E_2 \), \( h \), \( k \) and \( y \) rise, inducing a fall in \( \epsilon \). The \( \hat{h} = 0 \) line recrosses the \( \hat{k} = 0 \) line from below at some higher level of income to give a stable equilibrium at \( E_3 \). There could also be a stable equilibrium at the low level \( E_1 \).

The implications of an unstable equilibrium are not then so absurd as the model with rigid \( \epsilon \) would indicate. Furthermore, the fact that the equilibrium rate of growth is stable at a given level of income does not necessarily mean that it is globally stable. A sufficient jump in income might permit the economy to break out of the low growth rate "trap" and move toward a new and higher growth rate.
CHAPTER II

INCREASING RETURNS TO SCALE AND THE ADVANCE OF THE ARTS

In the model of Chapter I, I postulated constant returns to scale. This assumption seems better suited to deal with the process of learning previously discovered knowledge than with the discovery of new knowledge. The model assumes that learning is costly, that individuals choose to learn more as their income rises, and that education increases an individual's productivity. But what an individual learns engenders no external economics. When one considers that part of man's intellectual endeavor which results in new knowledge, the assumption of constant returns to scale becomes dubious. A new discovery can be employed by anyone who finds it worth the learning cost to avail himself to it. The larger the economy to which the discovery is applied, the larger the benefit accruing from it.

If a rise in the general level of education contributes to new discoveries, it is appropriate to postulate increasing returns to scale. So long as we wish to maintain the simplifying device of a single production function for all goods, these increasing returns must extend throughout the economy. That is, additional human capital put to use in the production of any good must have the same external effects in the production of all goods.
Suppose that \( Y = K^\alpha h^\beta L^\gamma \) where \( \alpha + \beta + \gamma = 1 + m, m > 0 \). Then \( y(T) = k(T)^\alpha h(T)^\beta L(T)^m \).

If we choose units of labor so that \( L(0) = 1 \), then

\[
y(T) = e^{mnT} k(T)^\alpha h(T)^\beta.
\]

Notice that this production function is formally very similar to the function which describes neutral technical progress at a constant rate \( \rho \), under constant returns to scale. This form is \( y = e^{\rho T} k^\alpha h^\beta \). The only significant difference between the two formulations is that in the case of increasing returns, the rate of growth of efficiency is proportional to the rate of growth of population.

If we retain the assumptions made in Chapter I concerning the growth rates of physical and human capital, we have the following system,

\[
y = e^{mnT} k^\alpha h^\beta \quad (1b)
\]

\[
\hat{L} = n \quad (2b)
\]

\[
\hat{k} = \frac{sv\delta}{k} - n \quad (3b)
\]

\[
\hat{h} = \frac{cv\xi}{n} - \mu \quad (4b)
\]

Consider the conditions under which the growth rates of the two forms of capital both remain constant (but not necessarily equal). I shall call this steady growth.
From (5b),

\[
\frac{d}{dT} \hat{k} = 0 \quad \text{if} \quad \frac{d}{dT} \frac{sy^5}{k} = 0 \quad \text{or} \quad \hat{k} = \delta y.
\]

From (4b),

\[
\frac{d}{dT} \hat{h} = 0 \quad \text{if} \quad \hat{h} = e y.
\]

From (1b),

\[
\hat{y} = mn + \alpha \hat{k} + \beta \hat{h}.
\]

Substituting the steady growth conditions for \( k \) and \( h \) into the last expression we have

\[
\hat{y} = mn + \alpha \delta \hat{y} + \beta \epsilon \hat{y},
\]

or

\[
\hat{y} = \frac{mn}{1 - \alpha \delta - \beta \epsilon}.
\]

Then on a steady growth path,

\[
\hat{h} = \frac{\epsilon mn}{1 - \alpha \delta - \beta \epsilon},
\]

and

\[
\hat{k} = \frac{\delta mn}{1 - \alpha \delta - \beta \epsilon}.
\]
It can now be shown that the same conditions which in the model of Chapter I assured the existence and stability of the stationary state will in this model assure the existence and stability of the steady growth path.

Proposition III:

Given assumptions (1b) - (4b), the steady growth path is asymptotically approached from any initial endowments of $h$ and $k$ if $\lambda - \beta \epsilon - \delta \alpha > 0$. If $\lambda - \beta \epsilon - \delta \alpha < 0$ the system diverges from the steady growth path. Let

$$z = \log k - \left(\frac{\lambda n \delta}{1 - \alpha \delta - \beta \epsilon}\right) T,$$

and

$$x = \log h - \left(\frac{\lambda n \epsilon}{1 - \alpha \delta - \beta \epsilon}\right) T.$$

Then if

$$\dot{z} > 0 \quad \dot{k} > \frac{\lambda n \delta}{1 - \alpha \delta - \beta \epsilon}.$$

If

$$\dot{x} < 0 \quad \dot{h} < \frac{\lambda n \epsilon}{1 - \alpha \delta - \beta \epsilon}.$$

Thus we have
\[ \hat{z} \gtrsim 0 \quad \text{if} \quad \frac{xy}{k} \lesssim \frac{1}{s} \left[ n + \frac{mn\delta}{1 - \alpha \beta - \beta \epsilon} \right] \quad (3b)'
\]

or if

\[ 6 \log y - \log k \lesssim \log \frac{1}{s} \left[ n + \frac{mn\delta}{1 - \alpha \beta - \beta \epsilon} \right], \]

\[ \hat{x} \gtrsim 0 \quad \text{if} \quad \frac{cy}{h} \lesssim \frac{1}{c} \left[ \mu + \frac{mne}{1 - \alpha \beta - \beta \epsilon} \right] \quad (4b)'
\]

or if

\[ \epsilon \log y - \log h \lesssim \log \frac{1}{c} \left[ \mu + \frac{mne}{1 - \alpha \beta - \beta \epsilon} \right]. \]

Substitution for \( \log y \) give us

\[ \hat{z} \gtrsim 0 \quad \text{if} \quad 5mnT + 5\beta \log h + (6\alpha - 1) \log k \lesssim \log \frac{1}{s} \left[ n + \frac{mn\delta}{1 - \alpha \beta - \beta \epsilon} \right] \quad (3b)''
\]

and

\[ \hat{x} \gtrsim 0 \quad \text{if} \quad emnT + (\epsilon\beta - 1) \log h + e\alpha \log k \lesssim \log \frac{1}{c} \left[ \mu + \frac{mne}{1 - \alpha \beta - \beta \epsilon} \right]. \quad (4b)''
\]

Equations (3b)'' and (4b)'' can be rewritten in terms of \( z \) and \( x \) as follows:

\[ \hat{z} \gtrsim 0 \quad \text{if} \quad 5\delta x + (6\alpha - 1) z \lesssim \log \frac{1}{s} \left[ \frac{mn\delta}{1 - \alpha \beta - \beta \epsilon} + n \right] \quad (3b)'''
\]

and
\[
\dot{x} \geq 0 \quad \text{if} \quad (\varepsilon\beta - 1)x + \alpha z \geq \log \frac{1}{c} \left[ \frac{\mu n e}{1 - \alpha \delta - \beta \varepsilon} + \mu \right].
\]

Finally since \((\delta\alpha - 1) < 0\) and \(\beta \varepsilon > 0\) rearrangement of terms gives us

\[
\dot{z} \geq 0 \quad \text{if} \quad z \leq \frac{\beta \varepsilon}{1 - \delta \alpha} x - \frac{1}{1 - \delta \alpha} \log \frac{1}{c} \left[ \frac{\mu n \delta}{1 - \alpha \delta - \beta \varepsilon} + n \right],
\]

\[
\dot{x} \geq 0 \quad \text{if} \quad z \geq \frac{1 - \beta \varepsilon}{\alpha \varepsilon} x + \frac{1}{\alpha \varepsilon} \log \frac{1}{c} \left[ \frac{\mu n e}{1 - \alpha \delta - \beta \varepsilon} + \mu \right].
\]

These equations are formally the same as \((3a)''\) and \((4a)''\) in the model with constant returns. The only difference is that the variables are now \(z\) and \(x\) rather than \(\log k\) and \(\log h\). In the special case of constant returns, \(m = 0\), \(z = \log k\) and \(x = \log h\).

If \(1 - \beta \varepsilon - \alpha \delta > 0\), \(z\) and \(x\) will converge asymptotically to some equilibrium values \(z^*\) and \(x^*\). For the case where \(1 - \beta \varepsilon - \alpha \delta > 0\), a phase diagram is drawn in Figure 5. \(z\) and \(x\) both approach stationary values. This means that \(\hat{k}\) approaches \(mn\delta/(1 - \beta \varepsilon - \alpha \delta)\) and \(\hat{h}\) approaches \(mn/(1 - \beta \varepsilon - \alpha \delta)\). \(\hat{y}\) approaches \(mn/(1 - \beta \varepsilon - \alpha \delta)\). If \(1 - \beta \varepsilon - \delta \alpha < 0\), the \(\dot{z} = 0\) line is steeper than the \(\dot{x} = 0\) line and as in the model of Chapter I, the growth rates diverge from steady growth.

The model with increasing returns to scale is capable of steady growth at a positive rate if the labor force is growing. If \(\delta\) and \(\varepsilon\) are not equal to one, physical and human capital per man
will increase at constant rates but neither rate will be the same as the growth rate of per capita income.

Increasing returns to scale could be reconciled with constant returns to scale for firms if part of the returns to human capital are external to the firm. The private market would underreward human capital so that factor rewards would not exceed the total product.

If we assume that physical capital and labor receive their marginal products, then capital receives $\alpha Y$, labor receives $(1+m-\alpha-\beta)Y$ and human capital receives the remainder, $Y - (1+m-\alpha-\beta)Y - \alpha Y = (\beta-m)Y$.

The payment to a single unit of human capital is $(\beta-m) Y/H$. The marginal product of human capital is $\beta Y/H$. 

29
Improvement in the Quality of Education

So far we have considered only increasing returns among factors contemporaneously employed. It is likely that in addition to this effect, succeeding generations benefit from advances in knowledge made by their forebears. This section presents a model in which the productive power of a day's schooling increases at a constant rate over time. It is assumed that improvement in the knowledge taught in school causes a day's schooling to become more productive the later the year in which that day of schooling was received.

If we assume that, on the demand side, $\hat{H}/L = C(Y/L)^\varepsilon$ where $\hat{H}$ is measured in quality units, the set of equations which describes the system is the same as that for the system previously discussed. Our theoretical results would be unchanged. However, in constructing empirical estimates of $\varepsilon$, it must be remembered that $\hat{H}$ should be measured in quality units rather than in days of schooling.

Alternatively, it could be assumed that on the demand side, people are interested only in the number of days of schooling. The implications of this assumption are discussed here.

I shall assume that there is no depreciation. Depreciation would introduce considerable complexity though I believe for my purposes, little is lost by ignoring it.

Let $\check{H}$ be the amount of human capital measured in efficiency units. Let $\tilde{H}$ be the number of units of human capital measured in days of school (with proper weightings to aggregate the various levels of schooling received in the same year).
Since in this case it is the number of days of schooling which depends on per capita income, we let \( \hat{H}/L = cy^\varepsilon \). Now let \( \dot{H}(T) = \dot{H} e_{H^T}^T \).

This assumption means that new education becomes more productive at a constant rate, \( \rho_H \), as time passes. Education once acquired does not change in productivity. We now have \( \hat{H}/L = e_{H^T}^T, \dot{H}/L = e_{H^T}^T cy^\varepsilon \). From this we derive

\[
\dot{h} = \left[ \frac{cy^\varepsilon}{h} \right] e_{H^T}^T - n \tag{4c}
\]

where \( \dot{h} \) is the rate of growth in efficiency units of the per capita stock of human capital.

There is little difficulty introduced by adding the assumption that new physical capital goods improve at a constant rate over time if we assume that the expenditure on new capital goods is unaffected by the change in their productive power. Exactly similar arguments lead us to

\[
\dot{k} = \frac{sy^\delta}{k} e_{K^T}^T - n \tag{3c}
\]

where \( \dot{k} \) is the rate of growth in efficiency units of the per capita stock of physical capital. In addition, let us assume

\[
y = e^{mn^T} k^\alpha h^\beta \quad \text{where } m \geq 0 \tag{1c}
\]

If \( m = 0 \), there are constant returns to scale. If \( m > 0 \), we have increasing returns.
An alternative rationalization of the same set of assumptions would be the following:

Suppose that \( \frac{C}{L_C} = e^{\frac{mnT}{L} \ k_c^\alpha h_c^\beta} \)

\[
\frac{\dot{H}}{L_H} = \exp((mn + \rho_H)T) \ k_H^\alpha h_H^\beta \quad \text{and} \quad \frac{\dot{K}}{L_K} = \exp((mn + \rho_K)T) \ k_K^\alpha h_K^\beta
\]

where \( C \) is the output of consumption goods, \( \dot{H} \) and \( \dot{K} \) are the outputs in productive units of new human capital and physical capital respectively. \( K_i, L_i, H_i \) are the amounts of capital, labor and human capital employed in the \( i \)th sector. \( k_i = K_i/L_i \quad \text{and} \quad h_i = H_i/L_i \quad \sum_{i=C,K,H} K_i = K. \)

\[
\sum_{i=C,K,H} H_i = H, \quad \sum_{i=C,K,H} L_i = L. \quad \text{Total income is} \quad Y \quad \text{where}
\]

\[
Y = C + p_K \dot{K} + p_H \dot{H}
\]

where the consumption good is the numeraire, \( p_K \) is the price of new capital and \( p_H \) is the price of new human capital. Then

\[
\frac{Y}{L} = \frac{L_C}{L} \frac{C}{L_C} + p_K \left[ \frac{L_K}{L} \frac{\dot{K}}{L_K} \right] + p_H \left[ \frac{L_H}{L} \frac{\dot{H}}{L_H} \right]
\]

or

\[
\frac{Y}{L} = e^{\frac{mnT}{L} \left\{ \frac{L_C}{L} k_c^\alpha h_c^\beta + \frac{L_K}{L} p_K e^{\frac{\rho_K T}{L} k_K^\alpha h_K^\beta} + \frac{L_H}{L} k_H^\alpha h_H^\beta p_H e^{\frac{\rho_H T}{L}} \right\}}
\]

If marginal rates of substitution between each pair of factors are equalized in the production of every good, it can easily be shown that \( k(T) = k_c(T) = k_K(T) = k_H(T) \) and that \( h_c(T) = h_k(T) = h_H(T) = h(T) \)
at every moment of time. If this is true, and goods are priced at their marginal rates of transformation, \( P_K(T) = e^{-\rho_K T} \) and \( P_H(T) = e^{-\rho_H T} \). Then

\[
\frac{Y(T)}{L(T)} = e^{mnT} \left( \frac{L_c}{L} \frac{k^\alpha}{h^\beta} + \frac{L_K}{L} \exp(\rho_K T) \frac{k^\alpha}{h^\beta} + \frac{L_H}{L} \exp(\rho_H T) \frac{k^\alpha}{h^\beta} \right)
\]

or

\[
\frac{Y(T)}{L(T)} = e^{mnT} \frac{k^\alpha}{h^\beta}
\]

since \( L_c + L_K + L_H = L \).

Now if the price elasticity of demand for education is unity and the income elasticity of demand is \( \varepsilon \) we have

\[
\frac{P_H H}{L} = C(Y_L) \varepsilon \quad \text{or} \quad e^{-\rho_H T} \frac{H}{L} = C(Y_L) \varepsilon
\]

or

\[
\frac{H}{L} = C(Y_L) \varepsilon e^{\rho_H T}
\]

A unitary price elasticity of demand for capital goods and an income elasticity of \( \delta \) will give us

\[
\frac{\dot{K}}{L} = S(Y_L) \delta e^{\rho_K T}
\]

Thus we have the same expressions for the growth rates of physical and human capital as derived in (3c) and (4c) above.
Consider the conditions for steady growth in the system (1c), (3c), (4c). Since \( \hat{h} = cy^e/h \ e^{\hat{H}^T} - n \), \( d/dT \hat{h} = 0 \) when \( \delta y + \rho_H = \hat{h} \).

Since \( \hat{k} = sy^b/k \ e^{\hat{K}^T} - n \), \( d/dT \hat{k} = 0 \) when \( \delta y + \alpha_K = \hat{k} \). Now

\[
\hat{y} = \alpha \hat{k} + \beta \hat{h} + mn.
\]

On the steady growth path, \( \hat{y} = \alpha \delta \hat{y} + \alpha \rho_K + \beta \delta \hat{y} + \beta \rho_H + mn \)

\[= (\alpha \rho_K + \beta \rho_H + mn)/(1 - \alpha \delta - \beta \delta) = \bar{g}_y \]

where \( \bar{g}_y \) is a constant,

\[
\hat{h} = \epsilon \bar{g}_y + \rho_H \quad \text{and} \quad \hat{k} = \delta \bar{g}_y + \alpha_K.
\]

The condition for stability of the steady growth path remains as in the first two models, \( 1 - \beta \epsilon - \beta \alpha > 0 \).

The proof is virtually the same as the proof for the model in part A of this chapter.

If there is steady improvement in the quality of new human capital, the growth rate of human capital measured in effective units is higher than that measured in days of school. Along the steady growth path, \( \hat{H}/H = \epsilon \hat{y} \) while \( \hat{H}/H = \epsilon \hat{y} + \rho_H \). The stock of human capital, if measured in days of school, would be underestimated for recent years relative to earlier years. The rates of return on human capital would overestimate the marginal product of an effective unit in recent years since the cost of effective units of human capital has been falling.

Nevertheless, the Cobb-Douglas form of the production function would require that the functional share of human capital remain constant over time.
Non-Cobb-Douglas Production Functions

If the aggregate production function is not of the Cobb-Douglas form, there will be changes in the functional shares of the various factors as the ratios in which they are employed change. The amount of human capital relative to the amounts of other factors appears to have increased markedly in the United States over the last fifty years. The income share of human capital appears also to have increased. It would seem interesting to examine the implications of a non-Cobb-Douglas aggregate production function for economic growth.

I shall deal with a function of the form \( Y = F[K,H,L,T] \) where \( F \) is linear homogeneous in \( K, L, \) and \( H. \)

The model considered in this section is as follows:

\[
y = f(k, h, T) \tag{1e}
\]

\[
\hat{L} = n \tag{2e}
\]

\[
\hat{k} = \frac{sv^5}{k} - n \tag{3e}
\]

\[
\hat{h} = \frac{sv^\epsilon}{h} - \mu \tag{4e}
\]

If \( \partial f/\partial T \neq 0 \), if \( f \) is not Cobb-Douglas in \( k \) and \( h \), and if assumptions (3e) and (4e) describe the accumulation of physical and human capital, there does not in general exist a steady growth path.
If \( \frac{dk}{dT} = 0 \) and \( \frac{dh}{dT} = 0 \), \( \dot{k} = \delta \dot{y} \) and \( \dot{h} = \epsilon \dot{y} \) and
\[
\frac{d}{dT} \dot{y} = \frac{d}{dT} (\frac{1}{\delta} \dot{k}) = 0.
\]
Then
\[
\dot{y} = \theta_K(T) \delta \dot{k} + \theta_H(T) \dot{h} + \frac{1}{y} \frac{\partial r}{\partial T} = \theta_K(T) \delta \dot{y} + \theta_H(T) \epsilon \dot{y} + \frac{1}{y} \frac{\partial r}{\partial T} = \frac{\frac{1}{y} \frac{\partial r}{\partial T}}{1 - \theta_K(T) \delta - \theta_H(T) \epsilon}.
\]
Now \( \frac{d}{dT} \dot{y} = 0 \) only if
\[
\frac{\frac{1}{y} \frac{\partial r}{\partial T}}{1 - \theta_K(T) \delta - \theta_H(T) \epsilon}
\]
is constant. Since the denominator of this term is in general changing as \( h/k \) changes, the numerator would have to change in exactly the same proportion, in order that a steady growth equilibrium exist.

I can think of no a priori reasons why technical progress should proceed at a rate proportional to \( 1 - \theta_K(T) \delta - \theta_H(T) \epsilon \). There is an exactly analogous difficulty of nonexistent balanced growth paths in a two-factor neoclassical growth model with a constant savings rate. The two-factor, one-sector growthmen circumvent this difficulty by the assumption of Harrod neutral technical progress [7], [28].

The typical model of this sort is of this form \( y = f(k,T) \) and \( \dot{K}/K = sY/K \). If \( \frac{d}{dT} \dot{k} = 0, \dot{k} = \dot{y} \). On the balanced growth path,
\[
\dot{y} = \theta_K(T) k + \frac{\partial r}{\partial T} \frac{1}{y} = (\frac{\partial r}{\partial T} \frac{1}{y})/(1 - \theta_K(T)).
\]
Thus \( \frac{d}{dT} \dot{y} = 0 \) if and only if
\[ \frac{\partial f}{\partial T} \frac{1}{y} \left( \frac{1}{1 - \theta_K(T)} \right) \text{ is constant.} \]

Suppose there are constant returns to scale and technical progress is Harrod neutral. It can easily be shown that for such a model there exists a stable balanced growth path such that along this path, \( Y/K = (n+\lambda)/s \) where \( n \) is the constant rate of growth of population, \( \lambda \) the constant rate of Harrod neutral progress, and \( s \) the constant rate of saving, if the production function satisfies the following conditions: \( f_K > 0, f_{KK} < 0, \lim_{k \to 0} y/k = \infty, \lim_{k \to \infty} y/k = 0. \)

Since

\[ Y = F[K, L e^{\lambda T}] \]

\[ \hat{Y} = \theta_K(T) \hat{k} + (1 - \theta_K(T)) \hat{L} + [1 - \theta_K(T)] \lambda \]

and

\[ \hat{y} = \theta_K(T) \hat{k} + [1 - \theta_K(T)] \lambda . \]

Thus

\[ \frac{\partial f}{\partial T} \frac{1}{y} = [1 - \theta_K(T)] \lambda \]

and

\[ \frac{\partial f}{\partial T} \frac{1}{y} \left( \frac{1}{1 - \theta_K(T)} \right) = \lambda \]

which by the assumption of Harrod neutral technical progress is constant. Once again we glimpse the great economic truth that judicious assumptions lead to convenient results.
It is not too surprising that we have to stack our cards carefully in order to have a model in which a steady growth equilibrium is possible. Any very detailed description of the real world is not going to find everything growing at constant rates. Our justification for staking the deck has to be that we can get a good approximation to reality within the context of a manageable model if we simplify our assumptions about the world.

It may be that changes in the functional shares of human capital and/or physical capital are sufficiently important that we find it necessary to abandon the discussion of steady growth paths. My suspicion, however, is that changes in $\theta_K$ and $\theta_H$ are slow enough that steady growth is a good approximation to reality.
CHAPTER III

ENDOGENOUS TECHNICAL PROGRESS

A. Dead Philosophers and Economic Growth

In the previous section, improvement in the quality of education depended only on the passage of time. It would be worthwhile to examine models in which the improvement of knowledge is endogenous. Arrow has presented a model in which the efficiency of new physical capital depends on the number of machines built in the past. In Arrow's model, the construction of a machine contributes to growth in two ways. The machine increases the capacity of the economy for current production and the experience gained in its production makes future machines more efficient.

One might suppose alternatively that the productive efficiency of the economy depends on the cumulative amount of past human capital. Recall the words of Keynes [12],

"...the words of economists and political philosophers both when they are right and when they are wrong, are more powerful than is commonly understood...Practical men who believe themselves quite exempt from any intellectual influences are usually the slaves of some defunct economist."

In the construction of my model I shall credit dead plumbers and bank clerks with a role nearly as lofty as that of the defunct economist.

In particular I assume that
\[ Y(T) = A(T)^a K^\alpha H^\beta L^\gamma, \quad \alpha + \beta + \gamma = 1 + m, \quad m \geq 0, \quad 0 < a < 1, \]

and

\[ A(T) = \int_{-\infty}^{T} H(\tau) \exp[-\lambda(T-\tau)] \, d\tau \quad \text{where} \quad \lambda > 0, \quad a > 0. \]

The parameter \( \lambda \) is a forgetting factor: Knowledge acquired long ago has less effect on current production than more recent knowledge. For \( a < 1 \), there are diminishing returns to the stock of accumulated knowledge.

I also assume that \( K/L = \sigma(Y/L)^5 \). This assumption means that the public has a desired relation between per capita income and per capita holdings of physical capital and maintains this relation at all times. Along the steady growth path the assumption that \( K/L = \sigma(Y/L)^5 \), like the assumption \( \hat{k} = \sigma \hat{y} / k - n \), implies that \( \hat{k} = \sigma \hat{y} \).

The equations for this model are, in per capita terms:

\[ y = A(T)^a k^\alpha h^\beta e^{nT} \quad (1d) \]

where

\[ A(T) = \int_{-\infty}^{T} H(\tau) \exp[-\lambda(T-\tau)] \, d\tau, \quad m \geq 0 \quad \text{and} \quad 0 < a < 1. \]

\[ L(T) = e^{nT} \quad (2d) \]

\[ k = \sigma \hat{y}^5 \quad (3d) \]

\[ \hat{h} = \frac{\sigma \hat{y}^c}{h} - \mu \quad (4d) \]
A(T) can be written as

\[ A(T) = e^{-\lambda T} \int_{-\infty}^{T} h(\tau) e^{(\delta + \lambda) \tau} \, d\tau \]

\[ \dot{A} = -\lambda A + h(T) e^{\delta T} \]

\[ \dot{\hat{A}} = -\lambda \hat{A} + \frac{h(T)}{A} e^{\delta T} \]

Also, \( \dot{k} = \beta \dot{y} \) at all times, and \( \dot{y} = a\dot{A} + \alpha \dot{k} + \beta \dot{h} + n. \) Along the steady growth path,

\[ \frac{d}{dt} \dot{y} = \frac{d}{dt} \dot{k} = \frac{d}{dt} \dot{h} = \frac{d}{dt} \dot{A} = 0 \]

\[ \frac{d}{dt} \dot{h} = 0 \quad \text{implies} \quad \dot{h} = \epsilon \dot{y} \]

\[ \frac{d}{dt} \dot{A} = 0 \quad \text{implies} \quad \dot{A} = \dot{h} + n. \]

Then

\[ \ddot{y} = a \epsilon \dot{y} + an + \alpha \beta \dot{y} + \beta \epsilon \dot{y} + mn \]

\[ = \frac{n(a + m)}{1 - \alpha \delta - (\beta + a) \epsilon} \]

The system will asymptotically approach the balanced growth path if \( 1 - \alpha \delta - (\beta + a) \epsilon > 0. \) If \( 1 - \alpha \delta - (\beta + a) \epsilon < 0 \) the system will diverge from the steady growth path.
To prove this we make the transformations

\[ x = \hat{\alpha} - \varepsilon \overline{g}_y \]

where

\[ \overline{g}_y = \frac{n[m+a]}{1 - \alpha \delta - (\beta + \sigma) \varepsilon} \]

\[ z = \hat{\Lambda} - (\varepsilon \overline{g}_y + n) \]

\[ \dot{x} \lesssim 0 \quad \text{as} \quad \frac{d}{dt} \hat{\alpha} \lesssim 0 \quad \text{or} \quad \varepsilon \hat{y} - \hat{\alpha} \lesssim 0 \]

\[ \dot{z} \lesssim 0 \quad \text{as} \quad \frac{d}{dt} \hat{\Lambda} \lesssim 0 \quad \text{or} \quad \hat{\alpha} + n - \hat{\Lambda} \lesssim 0. \]

If we substitute \( x \) and \( z \) into the above expressions we get

\[ \dot{x} \lesssim 0 \quad \text{as} \quad x \lesssim \frac{\varepsilon a}{1 - \varepsilon \beta - \alpha \delta} z \]

\[ \dot{z} \lesssim 0 \quad \text{as} \quad x \lesssim z. \]

The phase diagram would look like Figure 6 if \( \varepsilon a/(1 - \varepsilon \beta - \alpha \delta) < 1 \). In this case the model is stable. \( x \to 0, z \to 0, \hat{\alpha} \to \varepsilon \overline{g}_y, \hat{\Lambda} \to \varepsilon \overline{g}_y + n \) and \( \hat{y} \to \overline{g}_y \).

If \( \varepsilon a/(1 - \varepsilon \beta - \alpha \delta) > 1 \), the \( \dot{x} = 0 \) line is steeper than the \( \dot{z} = 0 \) line and the system is unstable. Thus the necessary and sufficient condition for stability is \( 1 - \alpha \delta - \beta \varepsilon - a \varepsilon > 0 \) and the steady growth rate of per capita income is \( \overline{g}_y = n[m+a]/(1-\alpha \delta - (\beta + \sigma) \varepsilon) \).

This result is strikingly similar to that which Arrow [20] derives when he incorporates a constant gross savings rate, a constant
rate of growth of population, and full employment of labor into his model. Arrow derives a balanced growth rate of \( n/(1-a') \) where \( n \) is the rate of growth of population and \( a' \) is a sort of elasticity of efficiency with respect to cumulative gross investment. (This is my notation. In Arrow's notation, the same term is represented by \( c/(1-n) \).) Levhari [15] has shown that under these assumptions Arrow's balanced growth path is stable where \( a' < 1 \).

In Arrow's "Learning by Doing" model as well as in my "Dead Philosophers" model, the long term rate of balanced growth is proportional to the rate of growth of population. This is not too
surprising since both models depend in an essential way on increasing returns to scale. If the model is stable, the only thing that can cause growth, once the long term equilibrium is reached, is an exogenous force. The only exogenous force in either model is population growth.

If we can increase the rate of steady growth by increasing the rate of population growth, should we regard accelerated population growth as a good thing? Not necessarily. For one thing, I suspect that members of very crowded populations exert strong external diseconomies on each other which are not taken into account by ordinary measures of per capita income. The consumption of privacy, quiet forests, clean air, and unpolluted lakes is not fully subject to the price system in our economy. I believe that the welfare importance of these is systematically underestimated to a significant degree by conventional measures of national product.

Furthermore, although the balanced growth rate is higher for higher rates of growth of population, the level of per capita income at the point on the balanced growth path appropriate to the current level of technology depends inversely on the current rate of growth of population. If we were to choose today between two balanced growth paths, each characterized by a different rate of growth of population we would find that with increasing returns to scale the lower rate of population growth would mean higher incomes at first. The high population growth economy would start at lower per capita incomes but would ultimately reach and surpass the incomes in the slow population growth economy. The preferred path would depend on the economy's rate of time preference. I will discuss this question in somewhat more detail in Appendix II.
B. Human Capital as a Lubricant of Economic Growth

Some writers ([6], [15], [16]) have suggested that the level of the stock of human capital directly affects the rate of growth of income. The argument usually runs to the effect that more educated workers are better equipped to invent and adopt new procedures. This effect may differ from that of ordinary external economies since the stock of human capital has a specifically innovative function.

It seems to me that models of this form face one common difficulty. The assumption that the rate of growth of per capita income depends on the stock of human capital per laborer together with the fact that the stock of human capital per capita has risen greatly in developed countries would predict accelerating rates of growth in developed countries. There may have been a modest increase in the rate of growth of per capita income in the United States. The large changes in the stock of human capital may have been responsible for this. But the effect of the stock of human capital on growth rates would have to be very small (unless counteracted by other forces) to explain the coincidence of greatly increased stocks of human capital and very slightly increased growth rates.

A model where the growth rate depends on the per capita stock of human capital would generally take the form \( \dot{y} = \theta_K \dot{k} + \theta_H \dot{h} + f(h) \) where \( f'(h) > 0 \). The problem of "too much growthiness" can be quickly observed if we suppose that \( \theta_K \) and \( \theta_H \) are constant and that \( \frac{d}{dT} \dot{h} = \frac{d}{dT} \dot{k} = 0 \). Then \( \frac{d}{dt} \dot{y} = f'(h) \dot{h} > 0 \) if \( \dot{h} > 0 \). If \( \dot{h} \) has been observed positive then the growth rate is accelerating. Only
if \( f'(h) \) is small, could such an explanation be consistent with observed data.

Phelps and Nelson [18] in a recent article formulate a "lubricant" model with a slightly different tack. They assume that there is a steady exogenous improvement in known productive technique. They then argue that the time lag between the discovery of new techniques and their adoption depends negatively on the amount of human capital per worker. In its simplest form, the Phelps-Nelson model has a production function of the form \( Y(t) = F[K(t), T_o \exp(\lambda [t-w(h)])] L(t) \) where \( w'(h) < 0 \). As Phelps and Nelson point out, in this model the importance of the stock of human capital for growth depends on the rate of exogenous technical progress. The source of the exogenous improvement \( \lambda \) remains somewhat mysterious. \( \lambda \) could be treated as a control parameter determined by the government's decision to devote resources specifically to research. At any rate the implicit assumption must be that \( \lambda \) is unrelated to the variables \( K \) and \( h \).

The implications of the Phelps-Nelson production function for the growth path in a closed growth model depend crucially on the nature of the function \( w(h) \).

I will generalize their model slightly by including human capital as an ordinary productive factor as well as a lubricant of growth. I will assume that the function \( F \) is of the Cobb-Douglas form with constant returns to scale and that \( \hat{k} = sY^\delta /k - n \) and \( \hat{h} = cY^\epsilon /h - \mu \).

If this is done, we have \( y = \exp(\lambda [t-w(h)]) \) \( k^\alpha \) \( h^\beta \). Suppose that \( w(h) = -b \log h \) for some constant \( b \). Then we can rewrite the
production function as  \( y = e^{\lambda t} k^\alpha h^{\beta+\lambda b} \). If \( \lambda \) is constant, this model has a steady growth path where  \( \hat{y} = \lambda / (1 - \alpha \delta - (\beta + \lambda b) \epsilon) = \hat{y}_y \), a constant,  \( \hat{h} = \epsilon \hat{y}_y \) and  \( \hat{k} = \delta \hat{y}_y \).

The system asymptotically approaches the steady growth path if  \( 1 - \alpha \delta - (\beta + \lambda b) \epsilon > 0 \) and diverges from this path if  \( 1 - \alpha \delta - (\beta + \lambda b) \epsilon < 0 \). Since this particular form of the Phelps-Nelson production function is formally the same as my increasing returns model in Chapter II, the stability proof is virtually the same as that above.

This model differs from the increasing returns model in two important respects. The parameter \( \lambda \) rather than the rate of growth of population plays a crucial role in determining the rate of growth of income along the steady growth path. Also the functional share of human capital here depends on \( \lambda \) as well as the ordinary production parameter \( \beta \). In fact the stability of the steady growth path itself depends on \( \lambda \). If \( \lambda \) were large enough, the functional share of human capital would increase to the point where increasing rates of growth could occur (at least so long as the other parameters remain constant, see Chapter I). If \( \lambda \) could be influenced by policy, benefits might be reaped not only from an increase in the steady growth rate but from a movement from one steady growth path to another originating at a higher level of income.

Phelps and Nelson did not specify the form of \( w(h) \). The implications of their production function would be radically different if for example  \( w(h) = -bh \) where \( b \) is a positive constant. If such were the case we would have  \( y = \exp[\lambda (t + bh)] k^\alpha h^{\beta} \).
\[ \hat{y} = \lambda + \alpha \hat{k} + \beta \hat{h} + \lambda \hat{bh} \]

or

\[ \hat{y} = \lambda + \alpha \hat{k} + [\beta + \lambda \hat{bh}] \hat{h}. \]

If there is a steady growth path, \( \frac{d}{dt} \hat{h} = \frac{d}{dt} \hat{k} = \frac{d}{dt} \hat{y} = 0, \)
\( \hat{h} = \hat{cy} \) and \( \hat{k} = \hat{by}. \) Then

\[ \hat{y} = \frac{\lambda}{1 - \alpha \hat{e} - (\beta + \lambda \hat{bh(t)} \hat{e})} \neq 0. \]

Since \( h \) enters the denominator and \( \hat{h} = h \hat{cy} \neq 0, \frac{d}{dt} \hat{y} \neq 0. \) Thus there is no steady growth path. This is the same difficulty which I discussed earlier. When the rate of growth of income depends on the stock of human capital (rather than just its rate of change) and when the stock of human capital grows with income it is not surprising that the result is accelerating growth of income.
CHAPTER 4

SOME EMPIRICAL ESTIMATES

The behavior of the growth model which I have introduced depends significantly on two parameters which have not been systematically investigated in past discussions of economic growth. One of these is the functional share of human capital. The other is the income elasticity of demand for education. It appears to be possible in principle to estimate both of these magnitudes. The values of both parameters will depend on how human capital is defined and on what income is designated as a return to human capital. I will attempt to fit some of the available data into the theoretical categories suggested by my model. This is not intended to be an exhaustive empirical analysis of the relation between income and economic growth. Rather I hope to present rough estimates of the magnitudes involved and to suggest possible ways to link available and potentially available data on income and education with a specific growth model.

Estimation of the Functional Share of Human Capital.

The concept of human capital which I will employ is formal education at all levels from elementary school through college which is held by the labor force. If all factors are paid their marginal products, the functional share of human capital is that part of the

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conventionally calculated share of labor which the average laborer earns beyond what he would earn if he had no education.

United States census data for 1940 and for 1960 give mean total earnings by level of education. If the average person with no education in a given year were an average member of the labor force in all other attributes but his education, one could directly employ the actual mean earnings of persons who have no education as an estimate of what the average laborer would earn if he had no education. However, the uneducated laborer is more likely to be a Negro, a southerner, and a farmer than the average laborer. Members of each of these groups tend to earn less than the average laborer even when they have the same amount of education. Furthermore, the average uneducated laborer is probably somewhat less intelligent and healthy than the average laborer.

I estimate the earnings which the average laborer would receive if he had no formal education as follows. I separate the labor force into eight categories by race (white, nonwhite), region (North, South), and occupation (farm, nonfarm). I find the average earnings of persons in each category (e.g., Negro, South, nonfarm) who have effectively no formal education. I then multiply this figure by the percentage of the total labor force in this category. The sum of these products is an estimate of the expected earnings of the average laborer had he received no education.

I do not make an explicit allowance for intelligence or health. The adjustment which I do make is to use the earnings, as calculated above, of an average laborer with 0.4 years of education as a surrogate
for the earnings of persons with no education. The sample of persons with no education at all is very small and, one would suppose that this is a very freakish group in many respects. I prefer to deal with persons in the 0-4 year group and assume that any gain which they receive from such a small amount of education is balanced by their subnormality in other respects.

Once we have an estimate of the earnings which the average laborer would receive if he had no education, it is not difficult to estimate the functional share of human capital. The only additional information we need is the average wage and salary earnings of all laborers and the functional share of physical capital.

Let $E$ represent total wage and salary income, $w$ the wage of the average laborer if he had no education, $L$ the number of people in the labor force, $q_H$ the rental price of human capital, and $H$ the amount of human capital. By definition, $E = wL + q_HH$. Dividing both sides of the identity by $E$, we have

$$\frac{1}{E} = \frac{w}{E} \left( \frac{F}{L} \right) + \frac{q_HH}{E}.$$  

If $Y$ is total national income and $\theta_k$ the functional share of physical capital, we can rewrite the identity as

$$\frac{1}{E} = \frac{w}{E} \left( \frac{F}{L} \right) + \frac{q_HH}{Y} \frac{1}{1-\theta_k}.$$  

But $q_HH/Y = \theta_H$ is the functional share of human capital. Thus,
\[ l = \frac{w}{(E/L)} + \theta_H \frac{1}{1-\theta_k} \]

or

\[ \theta_H = \left[ 1 - \frac{w}{(E/L)} \right] \left[ 1 - \theta_k \right] \]

where \( w/(E/L) \) is the ratio of \( w \) to the average earnings of all members of the labor force.

I use the ratio \( w/(E/L) \) for males between 25 and 64 as an estimate of \( w/(E/L) \) for the entire economy. This procedure may overestimate \( w/(E/L) \) since the average woman in the labor force is slightly more educated than the average male laborer and may receive a larger portion of her income as a return to human capital. On the other hand, the very young, and the very old are also excluded. Members of these groups probably receive a somewhat smaller share of their income as a return to human capital.

I have used this procedure to make estimates of the functional share of human capital for 1939 and 1959. For 1959 I also made an estimate of \( \theta_H \) using data collected by Giora Hanoch in his Ph.D. thesis, Personal Earnings and Investment in Schooling, [8]. Mr. Hanoch estimated the expected earnings of persons with 0-4 and 0-7 years of education in the following way. He separated the male population into categories by race (white-nonwhite), region (North-South), and age (5 year intervals). Using the 1960 United States census tape, he standardized the incomes and persons in each category for such variables as family size, city size, place of birth, and industry of employment.
In 1939, the ratio \( w/(E/L) \) was about 0.57 if the wages of persons with 0-4 years of education (adjusted as discussed above) are used to estimate the wages of uneducated labor.

The 1960 census special reports list mean earnings of persons with 0-7 years of education but do not separately list earnings of persons with 0-4 years. When the earnings of persons with 0-7 years of education are estimated according to the above procedure, the ratio \( w_{0-7}/(E/L) \) is about 0.70. If the ratio \( w_{0-4}/w_{0-7} \) which is arrived at by Mr. Hanoch is used to adjust this ratio, \( w_{0-4}/(E/L) \) is calculated to be about 0.61. \( w_{0-4}/(E/L) \) when estimated directly from Mr. Hanoch's data is 0.60.

If the functional share of physical capital is assumed to have been 0.25 in both 1939 and 1959, the estimates of the functional share of human capital are 0.32 and 0.30 for these years respectively.

These figures suggest that the functional share of human capital has changed very little over the period from 1939 to 1959. Some support, therefore, lent to the hypothesis that the aggregate production function is of the form \( y = K^\alpha L^\beta \). We will use \( \theta_H \), the functional share of human capital, as an estimate for the parameter \( \beta \).

It should be pointed out, however, that the 1939 figures and the 1959 figures are not strictly comparable. The figures for 1959 give the average earnings of all persons who received any earnings from wage and salary or self-employment. The figures for 1939 give wage and salary earnings only for those persons who received no income other than wages or salaries. The population considered in 1939 thus excludes farmers with income from sale of crops, non-farm proprietors...
with self-employment income, and persons with income from capital in addition to their wage and salary income. The exclusion of farmers tends to bias the estimate of the functional share of human capital upward since farmers are less educated than the average laborer and receive a smaller portion of their income as a return to human capital. The exclusion of the other three groups tends to bias the estimate downward for similar reasons. Finally 1939 was a year of high unemployment, especially among the uneducated. For this reason, the actual wage of those uneducated persons who were employed may not accurately represent the marginal product of uneducated persons.

The 1950 census does not give earnings by level of education. There are, however, figures for total income by education for 1949. The ratio of the median income of persons with 1-4 years of education to that of all persons with income in 1949 is about 0.46. In 1959 the corresponding ratio is 0.41. Both figures would be raised considerably if the incomes of persons with 1-4 years of education were standardized for race, region, and occupation. If these figures are used as estimates for the ratio $w/(E/L)$, it would appear that the functional share of human capital increased somewhat between 1949 and 1959.

While the functional share of human capital appears to have been fairly stable over time, it is interesting to note that the functional share of human capital is probably smaller in the North than in the South. Yet, the North has considerably more human capital per man than the South. The ratio $w/(E/L)$ for 1959 is 0.62 for the North and 0.55 for the South. If the functional share of physical capital were $1/4$ for both regions, the functional share of human capital in the South would be 0.34. In the North it would be 0.29.
This result would be consistent with the hypothesis that the North and South were isolated economies with slightly different Cobb-Douglas production functions. But since there is relatively unrestricted trade between the North and South, it is somewhat surprising that the forces suggested by the Ohlin-Heckscher trade model have not resulted in approximate equalization of the relative (and absolute) rental prices of factors. If relative factor rentals were equalized, the ratio of the wages of uneducated labor to total earnings would be higher in the South than in the North because the South has less human capital per laborer. In fact, we observe this ratio to be higher in the North.

Estimation of the Income Elasticity of Demand for Education.

Estimates of the income elasticity of demand for human capital will depend on the way in which the change in the stock of human capital is measured. Some of the increase over time in real per capita expenditures has been due to increased real expenditure per student and some to the larger fraction of the population which is enrolled in school. Any estimate of the annual increase of the stock of education will be influenced by the extent to which additional expenditures per student reflect more real inputs rather than higher costs per efficiency unit of education.

If it is assumed that actual expenditures on education measured in constant dollars represent the amount of human capital produced in any given year, we get a relatively high estimate of $e$. If it is
assumed that a year's schooling at any particular grade level is the same in quality, no matter when it was received, a much smaller estimate of $\epsilon$ is obtained. An intermediate estimate is made if we assume that changes in direct outlays on education (measured in constant dollars) per year of school represent changes in the quality of a year's schooling. Changes in annual income foregone by students due to rising real wages of labor are assumed, however, to have no effect on the quality of a year's education.

One set of estimates of $\epsilon$ is made on the hypothesis that

$$\frac{\dot{H}}{L} = c(Y/L)^\epsilon$$

where $\dot{H}$ represents the annual amount of schooling received by persons in the United States when schooling is measured in quality units, $Y/L$ is per capita GNP and $L$ is the population. Since there appear to be strong reasons for persons to receive their educations while they are young, it is probably more reasonable to use the hypothesis

$$\frac{\dot{H}}{L_{5-25}} = c(Y/L)^\epsilon$$

where $L_{5-25}$ is the population aged 5-25. If the population had been growing at a constant rate over time, the age distribution would remain roughly constant and the $\epsilon$ estimated would be about the same whether $\dot{H}/L = c(Y/L)^\epsilon$ or $\dot{H}/L_{5-25} = c(Y/L)^\epsilon$ were used as the supply function of education (see page 14 above). There have been some spurts and slowdowns in the birth rate in the United States since 1890. Particularly
striking is the relatively low birth rate of the 1930's and the correspondingly low percentage of the population which was of school age in the 1940's and early 1950's. Estimates of $\epsilon$ are, therefore, presented for both $\dot{H}/L = c(Y/L)^\epsilon$ and $\dot{H}/L_{5-25} = c(Y/L)^\epsilon$. These estimates are quite similar, but the concept $\dot{H}/L_{5-25}$ gives slightly higher estimates of $\epsilon$ with slightly better fits.

Estimates of $\epsilon$ are made from a regression on time series observations for a series of years from 1890 to 1965.

The first estimate is calculated assuming that actual expenditures on education (measured in 1929 dollars) represent the annual increment in the stock of education. To make this estimate, I used Professor Schultz's estimates [21] of the expenditure on education (including foregone earnings) for the years 1900, 1910, 1920, 1930, 1940, 1950, and 1956. I deflated these estimates which were measured in current prices by using the overall price index for the appropriate years. The relevant figures for these years are listed in Table 7 below. The estimate for $\epsilon$ which is derived from this procedure is 2.45.

A second estimate is made on the assumption that there has been no change in the quality of a year's school attendance for any particular grade level. An estimate of $\epsilon$ can be made if weights are attached to years of elementary, high school, and college attendance. This I do by using Schultz's estimates of the relative factor costs of elementary, high school, and college education in 1956 to measure the relative importance of a year of schooling at the different levels.
Since in 1956 one year of college cost 11.8 times as much as a year of elementary school, a year of college is considered to be equivalent to 11.8 years of elementary school. One year of high school cost 5.07 times as much as a year of elementary school and is considered to be equivalent to 5.07 years of elementary school. The data used to estimate \( \varepsilon \) by this procedure are summarized in Table 8. \( \varepsilon \), when estimated in this way, is about 1.1.

The third estimate of \( \varepsilon \) is calculated in the following manner. Actual educational expenditures of educational institutions measured in 1929 dollars are estimated for the years 1890, 1900, 1910, 1920, 1930, 1940, 1950, 1956, 1960, 1962, and 1965. Foregone earnings for each of these years are calculated as the value in 1929 dollars of income foregone per student in 1956 for each level of education multiplied by the number of students enrolled at that level of education for the appropriate year. The estimates of direct costs and foregone earnings which are made in this way are then added to obtain an estimate of the amount of education which took place in each of these years. When \( \hat{H} \) is estimated in this way, the increase in real expenditures on education which is due to rising real wages is excluded from the estimate. The data used to estimate \( \hat{H} \) by this procedure are presented in Tables 5 and 6. The corresponding value for \( \varepsilon \) is about 2.

Table A below lists the estimates of \( \varepsilon \) which are derived from the various procedures described above.
TABLE A

<table>
<thead>
<tr>
<th>Concept of $\epsilon$</th>
<th>$\frac{\hat{H}}{L} = c(x_L)^\epsilon$</th>
<th>$\frac{\hat{H}}{L_{5-25}} = c(x_L)^\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Procedure I:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Observed real costs represent $\hat{H}$)</td>
<td>$2.14$</td>
<td>$0.97$</td>
</tr>
<tr>
<td>Procedure II:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Quality of a year of a given level of schooling is unchanged over time)</td>
<td>$0.95$</td>
<td>$0.94$</td>
</tr>
<tr>
<td>Procedure III:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Observed real direct costs plus constant real foregone earnings represent $\hat{H}$)</td>
<td>$1.83$</td>
<td>$0.96$</td>
</tr>
</tbody>
</table>

These results give some idea of the reasonable range for the value of $\epsilon$. Better estimates could, of course, be obtained if we had a direct estimate of the change over time in the quality of education at any given level.

Since $\beta > 0$, the condition $1-\beta(1-\delta) > 0$ may be written $(1-\delta)/(1-\delta) > \epsilon$. If $\beta$ (the functional share of human capital) is 0.3, $\alpha$ (the functional share of physical capital) is 0.25, and $\delta$ (the income elasticity of the supply of physical capital) is 0.9, then $(1-\delta)/(1-\delta) \approx 2.58$. Under these circumstances, all of my estimates of

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\( \epsilon \) are less than \((1-\alpha)/\beta\) and \(1-\beta \epsilon - \alpha \delta > 0\) for any of these estimates of \( \epsilon \). Thus, it appears likely that the parameters relevant to the United States economy for the twentieth century are such that the model which I have discussed in this paper would be stable.
TABLE 1A
THE EXPECTED EARNINGS IN 1940 OF PERSONS WITH 0-4 YEARS OF EDUCATION

| Mean earnings in 1940 of persons with income only from wages and salaries and having 0-4 years of education |
| % of total white labor force age 25-64 |

<table>
<thead>
<tr>
<th>Whites</th>
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</thead>
<tbody>
<tr>
<td>Residence Age 25-34:</td>
<td></td>
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</tr>
<tr>
<td>South Rural Farm (S.R.F.)</td>
<td>342</td>
<td>3.14</td>
</tr>
<tr>
<td>South Nonfarm (S.N.F.)</td>
<td>584</td>
<td>6.78</td>
</tr>
<tr>
<td>North Rural Farm (N.R.F.)</td>
<td>396</td>
<td>3.60</td>
</tr>
<tr>
<td>North Nonfarm (N.N.F.)</td>
<td>867</td>
<td>21.63</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Age 35-44:</th>
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</thead>
<tbody>
<tr>
<td>S.R.F.</td>
<td>397</td>
<td>2.53</td>
</tr>
<tr>
<td>S.N.F.</td>
<td>713</td>
<td>5.26</td>
</tr>
<tr>
<td>N.R.F.</td>
<td>500</td>
<td>3.13</td>
</tr>
<tr>
<td>N.N.F.</td>
<td>967</td>
<td>17.08</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Age 45-54:</th>
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</thead>
<tbody>
<tr>
<td>S.R.F.</td>
<td>425</td>
<td>2.26</td>
</tr>
<tr>
<td>S.N.F.</td>
<td>783</td>
<td>3.76</td>
</tr>
<tr>
<td>N.R.F.</td>
<td>504</td>
<td>3.03</td>
</tr>
<tr>
<td>N.N.F.</td>
<td>1015</td>
<td>13.09</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Age 55-64:</th>
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</thead>
<tbody>
<tr>
<td>S.R.F.</td>
<td>387</td>
<td>1.74</td>
</tr>
<tr>
<td>S.N.F.</td>
<td>696</td>
<td>2.38</td>
</tr>
<tr>
<td>N.R.F.</td>
<td>485</td>
<td>2.30</td>
</tr>
<tr>
<td>N.N.F.</td>
<td>975</td>
<td>8.26</td>
</tr>
</tbody>
</table>

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TABLE 1B

THE EXPECTED EARNINGS IN 1940 OF PERSONS WITH 0-4 YEARS OF EDUCATION

<table>
<thead>
<tr>
<th></th>
<th>Estimated mean earnings in 1940 of persons with 0-4 years of education</th>
<th>% of total labor force</th>
<th>Average earnings all laborers</th>
<th>( \frac{W_{0-4}}{F_L} )</th>
<th>( \sigma_H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whites</td>
<td>$783</td>
<td>89.75</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nonwhites</td>
<td>$480</td>
<td>10.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>$752</td>
<td>100.00</td>
<td>$1320</td>
<td>.57</td>
<td>.32</td>
</tr>
</tbody>
</table>

Sources: Bureau of the Census Special Report. Population, Education 16th Census 1940; 3a, Table 29 for whites; Table 31 for nonwhites.
### TABLE 2A

THE EXPECTED EARNINGS OF MALES WITH 0-4 AND 0-7 YEARS OF EDUCATION CALCULATED USING HANOC'H'S DATA FOR 1959

<table>
<thead>
<tr>
<th>Age</th>
<th># Males (in Thousands)</th>
<th>% of Total # of Males</th>
<th>Expected Earnings Persons with 0-4 Years of Education</th>
<th>Expected Earnings Persons with 0-7 Years of Education</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>North/white</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25-29</td>
<td>3345</td>
<td>9.13</td>
<td>2570</td>
<td>2989</td>
</tr>
<tr>
<td>30-34</td>
<td>3000</td>
<td>8.19</td>
<td>3299</td>
<td>3810</td>
</tr>
<tr>
<td>35-39</td>
<td>4141</td>
<td>11.30</td>
<td>3672</td>
<td>4183</td>
</tr>
<tr>
<td>40-44</td>
<td>3038</td>
<td>8.29</td>
<td>3738</td>
<td>4340</td>
</tr>
<tr>
<td>45-49</td>
<td>3354</td>
<td>9.78</td>
<td>3717</td>
<td>4342</td>
</tr>
<tr>
<td>50-54</td>
<td>2648</td>
<td>7.23</td>
<td>3655</td>
<td>4242</td>
</tr>
<tr>
<td>55-59</td>
<td>2751</td>
<td>7.45</td>
<td>3404</td>
<td>3914</td>
</tr>
<tr>
<td>60-64</td>
<td>1788</td>
<td>4.88</td>
<td>2779</td>
<td>3872</td>
</tr>
<tr>
<td><strong>North/nonwhite</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25-29</td>
<td>286</td>
<td>.78</td>
<td>3021</td>
<td>2820</td>
</tr>
<tr>
<td>30-34</td>
<td>250</td>
<td>.68</td>
<td>3370</td>
<td>3038</td>
</tr>
<tr>
<td>35-39</td>
<td>317</td>
<td>.86</td>
<td>2879</td>
<td>3171</td>
</tr>
<tr>
<td>40-44</td>
<td>240</td>
<td>.65</td>
<td>2747</td>
<td>3257</td>
</tr>
<tr>
<td>45-49</td>
<td>249</td>
<td>.68</td>
<td>2898</td>
<td>3189</td>
</tr>
<tr>
<td>50-54</td>
<td>192</td>
<td>.52</td>
<td>2872</td>
<td>2987</td>
</tr>
<tr>
<td>55-59</td>
<td>168</td>
<td>.46</td>
<td>2503</td>
<td>2658</td>
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<td>60-64</td>
<td>92</td>
<td>.25</td>
<td>1799</td>
<td>1891</td>
</tr>
<tr>
<td><strong>South/white</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25-29</td>
<td>1254</td>
<td>3.42</td>
<td>1767</td>
<td>2415</td>
</tr>
<tr>
<td>30-34</td>
<td>1128</td>
<td>3.08</td>
<td>2455</td>
<td>3075</td>
</tr>
<tr>
<td>35-39</td>
<td>1474</td>
<td>4.02</td>
<td>2860</td>
<td>3430</td>
</tr>
<tr>
<td>40-44</td>
<td>1138</td>
<td>3.11</td>
<td>2867</td>
<td>3467</td>
</tr>
<tr>
<td>45-49</td>
<td>1262</td>
<td>3.44</td>
<td>2775</td>
<td>3360</td>
</tr>
<tr>
<td>50-54</td>
<td>899</td>
<td>2.45</td>
<td>2468</td>
<td>3038</td>
</tr>
<tr>
<td>55-59</td>
<td>957</td>
<td>2.61</td>
<td>2365</td>
<td>2805</td>
</tr>
<tr>
<td>60-64</td>
<td>622</td>
<td>1.70</td>
<td>1934</td>
<td>2272</td>
</tr>
<tr>
<td><strong>South/nonwhite</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25-29</td>
<td>299</td>
<td>.82</td>
<td>1459</td>
<td>1598</td>
</tr>
<tr>
<td>30-34</td>
<td>212</td>
<td>.58</td>
<td>1768</td>
<td>1844</td>
</tr>
<tr>
<td>35-39</td>
<td>323</td>
<td>.88</td>
<td>1924</td>
<td>1899</td>
</tr>
<tr>
<td>40-44</td>
<td>210</td>
<td>.57</td>
<td>1922</td>
<td>1895</td>
</tr>
<tr>
<td>45-49</td>
<td>287</td>
<td>.78</td>
<td>1785</td>
<td>1875</td>
</tr>
<tr>
<td>50-54</td>
<td>191</td>
<td>.52</td>
<td>1694</td>
<td>1807</td>
</tr>
<tr>
<td>55-59</td>
<td>214</td>
<td>.58</td>
<td>1495</td>
<td>1554</td>
</tr>
<tr>
<td>60-64</td>
<td>111</td>
<td>.30</td>
<td>1047</td>
<td>1066</td>
</tr>
</tbody>
</table>

**Sources:**
1. Column 1, Number of people in 5 year intervals kindly supplied to me by Mr. Hanoch.
2. Column 2, Calculated from Column 1.
TABLE 2B
ESTIMATION OF THE FUNCTIONAL SHARE OF HUMAN CAPITAL IN 1959 FROM TABLE 2

<table>
<thead>
<tr>
<th></th>
<th>NORTH AND WEST</th>
<th>SOUTH</th>
<th>UNITED STATES</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Estimated mean earnings, males age 25-64 with 0-4 years education, $w_{0.4}$</td>
<td>3355</td>
<td>2353</td>
<td>3066</td>
</tr>
<tr>
<td>(2) Estimated mean earnings, males age 25-64 with 0-7 years education, $w_{0-7}$</td>
<td>3902</td>
<td>2814</td>
<td>3588</td>
</tr>
<tr>
<td>(3) Mean earnings all males age 25-64</td>
<td>5521</td>
<td>4259</td>
<td>5127</td>
</tr>
<tr>
<td>(4) Estimated $w_{0.4}/(E/L)$</td>
<td>.608</td>
<td>.552</td>
<td>.598</td>
</tr>
<tr>
<td>(5) Estimated $w_{0.7}/(E/L)$</td>
<td>.707</td>
<td>.561</td>
<td>.696</td>
</tr>
<tr>
<td>(6) Estimated $\theta_{H0.4}$</td>
<td></td>
<td></td>
<td>.302</td>
</tr>
<tr>
<td>(7) Estimated $\theta_{H0.7}$</td>
<td></td>
<td></td>
<td>.228</td>
</tr>
</tbody>
</table>

Sources:
Row 1. The elements of Column 3, Table 2, are multiplied by the corresponding elements of Column 2, Table 2, and summed.

Row 2. Similar procedure except that Column 4, Table 2, is used instead of Column 3.

Row 3. Hanoch: Table 4.

Row 4. Row 1 ÷ Row 3.

Row 5. Row 2 ÷ Row 3.

Row 6. It is assumed that the functional share of physical capital is $1/4$. Then by formula developed in the text of Chapter IV

$$\theta_H = \frac{3}{4} \left( \frac{1-w}{E} \right)^{\frac{1}{4}}.$$
TABLE 3A

MEAN EARNINGS OF MALES AGE 25-64 WITH 0-7 YEARS OF EDUCATION AND SOME EARNINGS IN 1959 BY AGE, RACE, REGION, AND OCCUPATION

<table>
<thead>
<tr>
<th>Age</th>
<th>North/Whites</th>
<th>South/Whites</th>
<th>South/Nonwhites</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>% of Total Earnings</td>
<td>Mean</td>
</tr>
<tr>
<td></td>
<td>Force</td>
<td></td>
<td>Force</td>
</tr>
<tr>
<td>25-34</td>
<td>Farmers (F)</td>
<td>3305 .57</td>
<td>1901 .22</td>
</tr>
<tr>
<td></td>
<td>Farm Laborers (FL)</td>
<td>1867 .27</td>
<td>1413 .13</td>
</tr>
<tr>
<td></td>
<td>Nonfarm (NF)</td>
<td>4201 17.35</td>
<td>3329 6.31</td>
</tr>
<tr>
<td>35-44</td>
<td>F</td>
<td>3373 .86</td>
<td>2127 .18</td>
</tr>
<tr>
<td></td>
<td>FL</td>
<td>2054 .21</td>
<td>1502 .12</td>
</tr>
<tr>
<td></td>
<td>NF</td>
<td>4711 18.95</td>
<td>3740 6.56</td>
</tr>
<tr>
<td>45-54</td>
<td>F</td>
<td>3294 .92</td>
<td>2165 .51</td>
</tr>
<tr>
<td></td>
<td>FL</td>
<td>2222 .20</td>
<td>1420 .13</td>
</tr>
<tr>
<td></td>
<td>NF</td>
<td>4763 16.16</td>
<td>3757 5.32</td>
</tr>
<tr>
<td>55-64</td>
<td>F</td>
<td>2939 .81</td>
<td>1896 .42</td>
</tr>
<tr>
<td></td>
<td>FL</td>
<td>1890 .16</td>
<td>1275 .09</td>
</tr>
<tr>
<td></td>
<td>NF</td>
<td>4671 10.95</td>
<td>3679 3.11</td>
</tr>
</tbody>
</table>

North/Nonwhites

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>25-34</td>
<td>3290 1.42</td>
<td></td>
</tr>
<tr>
<td>35-44</td>
<td>3708 1.42</td>
<td></td>
</tr>
<tr>
<td>45-54</td>
<td>3687 1.05</td>
<td></td>
</tr>
<tr>
<td>55-64</td>
<td>3528 .66</td>
<td></td>
</tr>
</tbody>
</table>

Sources: United States Census 1960; Subject Report P.C. (2) 7B. Occupation by Earnings and Education, Tables 2 and 3.
### TABLE 3B

**ESTIMATION OF THE FUNCTIONAL SHARE OF HUMAN CAPITAL IN 1959 FROM TABLE 3**

<table>
<thead>
<tr>
<th></th>
<th>North and West</th>
<th>South</th>
<th>United States</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$w_{0-7}$ Estimated mean earnings in 1959 of males 25-64 with 0-7 years of education</td>
<td>4461</td>
<td>3238</td>
</tr>
<tr>
<td>(2)</td>
<td>$E/L$ Mean earnings all males age 25-64 with earnings</td>
<td>6213</td>
<td>4905</td>
</tr>
<tr>
<td>(3)</td>
<td>Estimated $w_{0-7}/(E/L)$</td>
<td>.718</td>
<td>.660</td>
</tr>
<tr>
<td>(4)</td>
<td>Estimated $w_{0-4}/w_{0-7}$</td>
<td>.859</td>
<td>.835</td>
</tr>
<tr>
<td>(5)</td>
<td>Estimated $w_{0-4}/(E/L)$</td>
<td>.617</td>
<td>.551</td>
</tr>
<tr>
<td>(6)</td>
<td>Estimated $\theta_H$</td>
<td>.287</td>
<td>.337</td>
</tr>
</tbody>
</table>

**Sources:**

Row 1: The mean earnings of persons with 0-7 years of education in each age, region, race, occupation category are multiplied by the percent of the total labor force in that category and summed.

Row 2: United States Census 1960: Subject report PC (2) 7B.

Row 3: Row 1 ÷ Row 2.

Row 4: The ratio $w_{0-4}/w_{0-7}$ is assumed to be the same as that which is derived from Hanoch's data.

Row 5: Row 3 x Row 4.

Row 6: Calculated by the formula developed in the text of this appendix

$$\theta_H = \frac{3}{4} \left[ \frac{1-x}{E/L} \right].$$
<table>
<thead>
<tr>
<th></th>
<th>South $\frac{w_{0-4}}{E_L}$</th>
<th>North $\frac{w_{0-4}}{E_L}$</th>
<th>United States $\frac{w}{E_L}$</th>
<th>$\theta_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1939</td>
<td>--</td>
<td>--</td>
<td>.57</td>
<td>.32</td>
</tr>
<tr>
<td>1959</td>
<td>.55</td>
<td>.61</td>
<td>.60</td>
<td>.30</td>
</tr>
<tr>
<td>1959 (Estimate based on Hanoch data)</td>
<td>.55</td>
<td>.62</td>
<td>.61</td>
<td>.30</td>
</tr>
<tr>
<td>1959 (Estimate based on Table 3)</td>
<td>.55</td>
<td>.62</td>
<td>.61</td>
<td>.30</td>
</tr>
</tbody>
</table>

TABLE 4
SUMMARY OF ESTIMATES MADE IN TABLES 1 - 3
TABLE 5
RESOURCE COSTS OF EDUCATIONAL SERVICES RENDERED BY EDUCATIONAL INSTITUTIONS

<table>
<thead>
<tr>
<th>Total Costs of Elementary and High Schools in Current Dollars (millions)</th>
<th>Total Costs of Colleges in Current Dollars (millions)</th>
<th>Total Costs; all Levels of Schooling in Current Dollars (millions)</th>
<th>Price Deflator for Academic Year (1/Price level)</th>
<th>Total Costs of Educational Services Rendered in 1929 Dollars (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1890</td>
<td>160</td>
<td>21.5</td>
<td>181.5</td>
<td>1.98</td>
</tr>
<tr>
<td>1900</td>
<td>252</td>
<td>40</td>
<td>292</td>
<td>2.09</td>
</tr>
<tr>
<td>1910</td>
<td>500</td>
<td>81</td>
<td>581</td>
<td>1.72</td>
</tr>
<tr>
<td>1920</td>
<td>1182</td>
<td>184</td>
<td>1366</td>
<td>.88</td>
</tr>
<tr>
<td>1930</td>
<td>2568</td>
<td>535</td>
<td>3222</td>
<td>1.01</td>
</tr>
<tr>
<td>1940</td>
<td>2945</td>
<td>742</td>
<td>3697</td>
<td>1.17</td>
</tr>
<tr>
<td>1948</td>
<td>5195</td>
<td>1710</td>
<td>6905</td>
<td>.66</td>
</tr>
<tr>
<td>1950</td>
<td>6505</td>
<td>2128</td>
<td>8633</td>
<td>.64</td>
</tr>
<tr>
<td>1956</td>
<td>11884</td>
<td>3500</td>
<td>15384</td>
<td>.55</td>
</tr>
<tr>
<td>1960</td>
<td>17030</td>
<td>5630</td>
<td>22660</td>
<td>.51</td>
</tr>
<tr>
<td>1962</td>
<td>20500</td>
<td>7140</td>
<td>27646</td>
<td>.50</td>
</tr>
<tr>
<td>1965</td>
<td>27030</td>
<td>9720</td>
<td>36750</td>
<td>.48</td>
</tr>
</tbody>
</table>

Sources:

Column 3: Column 1 and Column 2.

Column 4: Price level estimates for academic year is mean of price level in year listed and that in preceding year. Figures from 1890 to 1960 come from Kendrick Productivity Trends in the U.S., NBER, 1961. Table AIIa Column 11 = Table AIIb Column 11. Price indexes for more recent years come from Economic Report of the President, 1965. These figures are multiplied by 1.036 to make them comparable with Kendrick's series.

Column 5: (Column 3) × (Column 4).
<table>
<thead>
<tr>
<th>Academic Year</th>
<th>High School Enrollment (Millions)</th>
<th>Total College Income Foregone by High School Students (Millions of 1929 Dollars)</th>
<th>Total College Income Foregone by all College Students (Millions of 1929 Dollars)</th>
<th>Total Costs of Education (Millions)</th>
<th>Population Aged 5-25 (Millions)</th>
<th>Population Aged (N)</th>
<th>N</th>
<th>N^5-25</th>
<th>GNP per Capita (1929 Dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1890</td>
<td>.25</td>
<td>126</td>
<td>.157</td>
<td>171</td>
<td>291</td>
<td>560</td>
<td>62.9</td>
<td>25.8</td>
<td>10.3</td>
</tr>
<tr>
<td>1900</td>
<td>.70</td>
<td>336</td>
<td>.238</td>
<td>259</td>
<td>595</td>
<td>1205</td>
<td>76.1</td>
<td>30.0</td>
<td>15.8</td>
</tr>
<tr>
<td>1910</td>
<td>1.12</td>
<td>538</td>
<td>.355</td>
<td>387</td>
<td>925</td>
<td>1924</td>
<td>92.4</td>
<td>35.1</td>
<td>20.8</td>
</tr>
<tr>
<td>1920</td>
<td>2.50</td>
<td>1200</td>
<td>.598</td>
<td>652</td>
<td>1852</td>
<td>3054</td>
<td>106.4</td>
<td>38.6</td>
<td>28.7</td>
</tr>
<tr>
<td>1930</td>
<td>4.81</td>
<td>2308</td>
<td>1.10</td>
<td>1199</td>
<td>3507</td>
<td>6762</td>
<td>123.1</td>
<td>44.6</td>
<td>54.9</td>
</tr>
<tr>
<td>1940</td>
<td>7.13</td>
<td>3422</td>
<td>1.49</td>
<td>1624</td>
<td>5046</td>
<td>9371</td>
<td>132.1</td>
<td>44.3</td>
<td>70.9</td>
</tr>
<tr>
<td>1948</td>
<td>6.36</td>
<td>3024</td>
<td>2.62</td>
<td>2856</td>
<td>5880</td>
<td>10437</td>
<td>146.6</td>
<td>43.1</td>
<td>71.2</td>
</tr>
<tr>
<td>1950</td>
<td>6.45</td>
<td>3096</td>
<td>2.88</td>
<td>2899</td>
<td>5995</td>
<td>11220</td>
<td>152.3</td>
<td>44.2</td>
<td>75.6</td>
</tr>
<tr>
<td>1956</td>
<td>7.78</td>
<td>3734</td>
<td>3.00</td>
<td>3276</td>
<td>7004</td>
<td>15465</td>
<td>168.9</td>
<td>49.9</td>
<td>91.6</td>
</tr>
<tr>
<td>1960</td>
<td>9.60</td>
<td>4608</td>
<td>3.22</td>
<td>3510</td>
<td>8118</td>
<td>19678</td>
<td>180.7</td>
<td>56.6</td>
<td>108.9</td>
</tr>
<tr>
<td>1962</td>
<td>10.77</td>
<td>5170</td>
<td>3.73</td>
<td>4066</td>
<td>9236</td>
<td>22956</td>
<td>186.7</td>
<td>60.4</td>
<td>122.9</td>
</tr>
<tr>
<td>1965</td>
<td>12.9</td>
<td>6192</td>
<td>5.4</td>
<td>5886</td>
<td>12078</td>
<td>29718</td>
<td>194.6</td>
<td>66.1</td>
<td>152.7</td>
</tr>
</tbody>
</table>

See next page for sources.
Sources for Table 6:

Columns 1 and 3. Digest of Educational Statistics, 1965. Tables 91
93; Biennial Survey of Education, 1956. Chapter 4:1, Table III,
Table 45.

Column 2. (Column 1) \times 480. 480 is the amount of income foregone
per high school student in 1956 as estimated by Schultz op. cit.,
but deflated into 1929 dollars.

Column 4. (Column 1) \times 1090. 1090 is the equivalent in 1929 dollars
of Schultz's estimate of income foregone per college student in
1956.

Column 5. Column 4 + Column 2.

Column 6. Column 5, Table 6, + Column 5, Table 5.


Column 10. Column 6 ÷ Column 8.

Column 11. Real GNP figures from Kendrick op. cit. Table AIIa for
1890-1950; from Economic Report of the President 1965 for more
recent years. GNP is divided by population (Column 7 above) to
find per capita GNP. (GNP figure used here is the average for
the year listed and the previous year.)
TABLE 7
TOTAL COSTS OF EDUCATION MEASURED IN CURRENT PRICES

<table>
<thead>
<tr>
<th>Academic Year</th>
<th>(1) Annual Earnings</th>
<th>(2) Annual Costs of Services</th>
<th>(3) Total Costs</th>
<th>(4) 5-25</th>
<th>(5) GNP per Capita in 1929 Dollars for Academic Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>2.38</td>
<td>6.10</td>
<td>8.48</td>
<td>11.1</td>
<td>28.3</td>
</tr>
<tr>
<td>1910</td>
<td>3.51</td>
<td>6.99</td>
<td>13.50</td>
<td>14.6</td>
<td>38.5</td>
</tr>
<tr>
<td>1920</td>
<td>9.32</td>
<td>12.02</td>
<td>21.34</td>
<td>20.1</td>
<td>55.3</td>
</tr>
<tr>
<td>1930</td>
<td>16.54</td>
<td>32.55</td>
<td>49.09</td>
<td>39.9</td>
<td>109.4</td>
</tr>
<tr>
<td>1940</td>
<td>24.05</td>
<td>43.25</td>
<td>67.30</td>
<td>51.0</td>
<td>151.9</td>
</tr>
<tr>
<td>1950</td>
<td>49.45</td>
<td>55.25</td>
<td>104.70</td>
<td>68.7</td>
<td>236.9</td>
</tr>
<tr>
<td>1956</td>
<td>62.39</td>
<td>84.61</td>
<td>147.00</td>
<td>87.0</td>
<td>294.6</td>
</tr>
</tbody>
</table>

Sources:

Column 1. Calculated from basic data in Schultze "Education and Economic Growth", op. cit., Table A, B, and C.

Column 2. Table 5 this Appendix; Column 5.

Column 3. Column 1 and Column 2.

Column 4. Column 3 divided by total populations. Population figures listed in Table 6, Column 7, this appendix.

Column 5. Column 3 divided by population between ages 5 and 25. Population figures listed in Table 6, Column 8, this Appendix.

Column 6. Table 6, Column 11.
<table>
<thead>
<tr>
<th>Academic Year</th>
<th>Elementary School Enrollment (Millions)</th>
<th>Secondary School Enrollment (Millions)</th>
<th>Units of Secondary School Education (Millions)</th>
<th>College Enrollment (Millions)</th>
<th>Elementary School Equivalents (Millions)</th>
<th>Total Years of Schooling in Elementary School Equivalents (Millions)</th>
<th>Years Per Capita</th>
<th>Years Person Aged 5 - 25</th>
<th>GNP Per Capita</th>
<th>Dollars for Academic Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1890</td>
<td>10.6</td>
<td>.25</td>
<td>1.3</td>
<td>.16</td>
<td>1.8</td>
<td>13.7</td>
<td>.22</td>
<td>.53</td>
<td>402</td>
<td>1929</td>
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<tr>
<td>1900</td>
<td>13.5</td>
<td>.70</td>
<td>2.9</td>
<td>.24</td>
<td>2.9</td>
<td>19.9</td>
<td>.26</td>
<td>.66</td>
<td>495</td>
<td>1929</td>
</tr>
<tr>
<td>1910</td>
<td>16.3</td>
<td>1.12</td>
<td>4.1</td>
<td>.36</td>
<td>4.1</td>
<td>26.0</td>
<td>.28</td>
<td>.74</td>
<td>608</td>
<td>1929</td>
</tr>
<tr>
<td>1920</td>
<td>18.5</td>
<td>2.50</td>
<td>7.1</td>
<td>.60</td>
<td>7.1</td>
<td>38.2</td>
<td>.36</td>
<td>.99</td>
<td>694</td>
<td>1929</td>
</tr>
<tr>
<td>1930</td>
<td>21.0</td>
<td>4.81</td>
<td>13.0</td>
<td>1.10</td>
<td>13.0</td>
<td>58.3</td>
<td>.47</td>
<td>1.31</td>
<td>810</td>
<td>1929</td>
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<tr>
<td>1940</td>
<td>23.7</td>
<td>7.13</td>
<td>17.7</td>
<td>1.49</td>
<td>17.7</td>
<td>77.7</td>
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<td>1.75</td>
<td>878</td>
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<td>1948</td>
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<td>2.62</td>
<td>31.0</td>
<td>83.7</td>
<td>.57</td>
<td>1.94</td>
<td>1150</td>
<td>1929</td>
</tr>
<tr>
<td>1950</td>
<td>22.2</td>
<td>6.45</td>
<td>31.2</td>
<td>2.66</td>
<td>31.2</td>
<td>86.1</td>
<td>.57</td>
<td>1.95</td>
<td>1175</td>
<td>1929</td>
</tr>
<tr>
<td>1956</td>
<td>28.3</td>
<td>7.78</td>
<td>35.4</td>
<td>3.00</td>
<td>35.4</td>
<td>103.2</td>
<td>.61</td>
<td>2.07</td>
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<td>1929</td>
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<tr>
<td>1960</td>
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<td>38.0</td>
<td>3.22</td>
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<tr>
<td>1965</td>
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<td>12.9</td>
<td>59.5</td>
<td>5.4</td>
<td>59.5</td>
<td>160.8</td>
<td>.83</td>
<td>2.43</td>
<td>1653</td>
<td>1929</td>
</tr>
</tbody>
</table>

See next page for sources.
Sources for Table 8:

Columns 1, 2 and 4. Same references as cited for Columns 1 and 3, Table 6.

Column 3. Column 2 $\times$ 5.07. The figure 5.07 was arrived at as the ratio in 1956 of the factor costs of a year of secondary school to those of a year of elementary school.

Column 5. Column 5 $\times$ 11.8. The figure 11.8 was arrived at as the ratio in 1956 of the factor costs of a year of college to those of a year of elementary school.

Column 6. Column 1 + Column 3 + Column 5.

Column 7. Column 6 divided by population. Population figures taken from Table 6, Column 7.

Column 8. Column 6 divided by population aged 5-25. Population figures taken from Table 6, Column 8.

Column 9. Column 11, Table 6.
APPENDIX I

Many of the results in the main body of this paper can be generalized to the case in which instead of just two forms of capital, physical and human, there are \( n \) forms of capital. It might, for instance, as I mentioned in the discussion of problems of aggregation, be appropriate to separate elementary, high school and college education and treat them as different factors with complementary roles in production and different supply functions.

Suppose there are \( n \) sorts of capital \( k_i, i = 1, \ldots, n \), and that the aggregate production function is of the form

\[
y(T) = e^{\sum_{i=1}^{n} k_i \theta_i} \text{ where } \theta_i \text{ is the (positive, constant) functional share of the } i\text{th form of capital. Suppose also that } \hat{\lambda} = n \text{ and that } \hat{k}_i = (s_i y \epsilon_i)/k_i - \mu_i \text{ where } s_i, \epsilon_i \text{ and } \mu_i \text{ are positive constants. (To make sense of this model, we must also recall the proviso made above that in the relevant range to time, not all income be spent on the accumulation of capital.)}
\]

Proposition I(A):

If
is a nonsingular matrix, then there exists a steady growth path such that \( d/dT \hat{k}_i = 0 \) for \( i = 1, 2, \ldots, n \) and \( d/dT \hat{y} = 0 \). Along this path, \( \hat{y} = mn/(1-\sum \theta_i \epsilon_i) = \bar{y}_y \), a constant, and \( \hat{k}_i = \epsilon_i \bar{y}_y \).

**Proposition II(A):**

If \( 1 - \sum_{i=1}^{n} \theta_i \epsilon_i > 0 \), the steady growth path is asymptotically stable. If \( 1 - \sum_{i=1}^{n} \theta_i \epsilon_i < 0 \), the steady growth path is unstable in the sense of Lyapunov.

**Proof of Proposition I(A).**

Since \( \hat{k}_i = (s_i y \epsilon_i)/k_i - \mu_i \), \( d/dT \hat{k}_i = 0 \) implies that \( \hat{k}_i = \epsilon_i \hat{y}_i \). Now \( \hat{y} = \sum_{i=1}^{n} \theta_i \hat{k}_i + mn = mn/(1-\sum \theta_i \epsilon_i) = \bar{y}_y \), a constant, and \( \hat{k}_i = \epsilon_i \bar{y}_y \) along the steady growth path.

Let
\[
X = \begin{bmatrix}
X_1 \\
\vdots \\
X_n
\end{bmatrix} = \begin{bmatrix}
\frac{\log k_1}{\varepsilon_1} - \bar{g}_y^T \\
\vdots \\
\frac{\log k_n}{\varepsilon_n} - \bar{g}_y^T
\end{bmatrix}
\]

\[
\lambda_1 \geq 0 \text{ if } \lambda_1 \geq \varepsilon_1 \bar{g}_y. \text{ Now } \lambda_1 \geq \varepsilon_1 \bar{g}_y \text{ as } \varepsilon_1 \log y = \log k_1 \]

\[
\geq \log (\mu_1 + \varepsilon_1 \bar{g}_y)/s_i. \text{ Since } \log y = \text{mnT} + \sum_{j=1}^n \theta_j \log k_j, \lambda_1 \geq 0
\]
as the ith element of the column vector

\[
\begin{bmatrix}
\text{mnT} \\
\vdots \\
\text{mnT}
\end{bmatrix} + \begin{bmatrix}
\varepsilon_1 \theta_1 - 1 \\
\vdots \\
\varepsilon_1 \theta_n
\end{bmatrix} \begin{bmatrix}
\log k_1 \\
\vdots \\
\log k_n
\end{bmatrix} = \begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix} \gtrless 0
\]

where \( c_i = \log(\mu_1 + \varepsilon_1 \bar{g}_y)/s_i. \)

If we substitute \( \varepsilon_i X_1 + \varepsilon_i \bar{g}_y^T \) for \( \log k_i \), this can be expressed as follows: \( \lambda_1 \gtrless 0 \) as the ith element of the column vector

\[
\begin{bmatrix}
\varepsilon_1 (\theta_1 \varepsilon_1 - 1) \\
\varepsilon_1 \varepsilon_j \theta_j \\
\vdots \\
\varepsilon_1 \varepsilon_j \theta_j \\
\varepsilon_n (\theta_n \varepsilon_n - 1)
\end{bmatrix} \begin{bmatrix}
X_1 \\
\vdots \\
X_n
\end{bmatrix} = \begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix} \gtrless 0
\]
or as the \( i \)th element of

\[
AX > \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}
\]

In particular if for some \( X \) the equalities hold for all \( i \), there exists a steady growth path. This is guaranteed if \( A \) has an inverse.

Proposition II(A) is proved by means of Lyapunov's direct method [13]. The idea is to find a positive definite scalar-valued function \( V(X) \) such that \( d/dT \ V(X) \) is negative definite. Lyapunov's theorem states that if this can be done, then \( X(T) \) asymptotically approaches the equilibrium value \( X = 0 \).

If there exists a steady growth path, there is no loss of generality is choosing units of capital so that \( \log \left( \mu_1 + \epsilon_1 \bar{z}_y / s_1 \right) = c_1 = 0 \) for all \( i \). If this is done, then \( \dot{X}_1 \geq 0 \) as the \( i \)th element of \( AX \leq 0 \).

Let \( A^* = \tilde{\theta}A \) where

\[
\tilde{\theta} = \begin{bmatrix} \theta_1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & \theta_n \end{bmatrix}
\]
A* is a symmetric matrix which can be written explicitly as

\[
\begin{bmatrix}
\epsilon_1 \theta_1 \left[ \epsilon_1 \theta_1^{-1} \right] & & & \\
& \epsilon_1 \epsilon_j \theta_1 \theta_j & & \\
& & \ddots & \\
& & & \epsilon_n \theta_n \left[ \epsilon_n \theta_n^{-1} \right]
\end{bmatrix}
\]

If \( 1 - \sum \epsilon_j \theta_j > 0 \), then A* is a dominant diagonal matrix [17] with negative diagonal elements. Therefore A* is negative definite. Consider the function \( V(X) = -X^T A^* X \). \( V(X) > 0 \) for \( X \neq 0 \) since A* is a negative definite matrix. \( V(0) = 0 \), \( \frac{dV(X)}{dT} = -2 \dot{X}^T A^* X \) since A* is symmetric. \( \frac{dV(X)}{dT} \) may be rewritten as an inner product \(-2 (\dot{X}^T \vec{\theta}) (AX)\). Every element of the diagonal matrix \( \vec{\theta} \) is positive. Hence the sign of the ith element of the row vector \((\dot{X}^T \vec{\theta})\) is the same as the sign of the ith element of \( \dot{X}^T \). We have already shown that \( \dot{X}_1 < 0 \) as the ith element of \( AX \nless 0 \). Therefore, the sign of the ith element of \( AX \) is the same as the sign of the ith element of \( \dot{X}^T \vec{\theta} \). The inner product \((\dot{X}^T \vec{\theta}) (AX)\) is positive for \( X \neq 0 \) and 0 for \( X = 0 \). \( \frac{dV(X)}{dT} = -2 \dot{X}^T \vec{\theta} AX \) is thus negative definite. By Lyapunov's theorem, the steady growth path must be asymptotically stable.

To prove that if \( 1 - \sum \epsilon_j \theta_j < 0 \), the steady growth path is unstable, we employ Lyapunov's First Theorem of Instability [13].
According to this theorem, if we can find a decrescent function \( V \) such that \( V(X) < 0 \) for points arbitrarily close to \( X = 0 \) and if \( dV(X)/dT \) is a negative definite function, then the equilibrium \( X = 0 \) is unstable.

The function \( V = XA^*X \) satisfies all these conditions if

\[
1 - \sum_j \theta_j \epsilon_j < 0.
\]

Consider the vector

\[
\Delta = \begin{bmatrix}
\delta \\
\delta \\
\vdots \\
\delta
\end{bmatrix}
\]

where \( \delta \) can be made arbitrarily small

\[
V(\Delta) = \Delta A \Delta = -\delta^2 \sum \epsilon_i (\sum \epsilon_j \theta_j - 1) < 0 \quad \text{if} \quad 1 - \sum \epsilon_j \theta_j < 0.
\]

This is sufficient to prove instability since \( dV/dT \) has been proved positive definite.
APPENDIX II

In an increasing returns to scale economy, the steady state rate of growth in general depends on the rate of growth of population. If such an economy increases its rate of growth of population for all future time, it will ultimately experience higher levels of per capita income. But in the short run, if the savings function is independent of the rate of growth of population, the amount of capital per man will grow less rapidly the more rapidly population grows. There will be a period after a once and for all increase in the rate of growth of population during which the lower rate of per capita capital accumulation due to higher rates of population growth overwhelms the increasing returns to scale effect. During this period, per capita income will actually be lowered by an increased rate of population growth.

To illustrate the nature and importance of this effect, I will use a one-sector, two factor model of economic growth similar to that employed in Solow's "Contribution to Economic Growth," [23]. The analysis of this problem where human capital is explicitly included as a factor of production would be considerably more complicated, but the results should be at least qualitatively similar. The method employed here is similar to that used by Ryozo Sato [19] in his article, "Fiscal Policy in a Neo-classical Growth Model."
Let

\[ y = k^\alpha L^\beta, \quad 0 < \alpha < 1, \ 0 < \beta < 1 \]

\[ L(T) = e^{nT}, \quad \alpha + \beta = 1 + m, \ m > 0 \]

\[ \dot{K} = sY, \quad n \text{ a positive constant} \]

\[ r = \frac{Y}{K}, \quad 0 < s < 1. \]

In per capita terms,

\[ y(T) = k(T)^\alpha e^{nmT}, \quad (1) \]

\[ \dot{k}(T) = s \frac{y(T)}{k(T)} - n = sr(T) - n, \quad (2) \]

\[ \dot{y}(T) = \alpha \dot{k}(T) + mn, \quad (3) \]

\[ \dot{r} = \dot{y} - \dot{k}. \quad (4) \]

Suppose that at time zero, this economy is on the steady growth path appropriate to \( n = n_0 \). I shall solve for the effect on per capita income at time \( T \) of an infinitesimal increase in \( n \) at time zero.

Since at time zero, the economy is on the steady growth path appropriate to \( n = n_0 \), we have
sr(0) = n_0 = \hat{k}(0) = \hat{y}(0) = \frac{mn_0}{1-\alpha},

r(0) = \left[\frac{mn_0}{1-\alpha} + n_0 \right] \frac{1}{s} = \left[\frac{\beta n_0}{1-\alpha} \right] \frac{1}{s}.

If the economy retained a population growth rate of \( n_0 \), \( r(T, n_0) \) would remain constant at \( \left[\frac{\beta n_0}{1-\alpha}\right] \frac{1}{s} \).

If the population growth rate increased to \( n' \), we would have (from (3) and (4))

\[ \hat{r}(T, n') = [\alpha-1] \hat{k} + mn' . \]

From (2)

\[ \hat{r}(T, n') = [\alpha-1] [sr(T,n') - n'] + mn', \]

\[ \hat{r}(T, n') = [\alpha-1] sr(T,n')^2 + n'[m+1-\alpha] r(T,n') \]

\[ = [\alpha-1] sr(T,n')^2 + n' \beta r(T,n') . \]

This is a Bernoulli equation which can readily be solved to yield

\[ r(T,n') = \frac{n'\beta}{s[(1-\alpha) + B e^{-n'\beta T}]} \]

where \( B \) is a constant. Since

\[ r(0,n') = r(0,n) - \left[\frac{\beta n_0}{1-\alpha} \right] \frac{1}{s} , \]
we can solve explicitly for $B$.

$$r(0, n') = \frac{n' \beta}{s[1-\alpha + B]} = \frac{\beta n_0}{1-\alpha} \frac{1}{s}$$

or

$$[n' - n_0][1-\alpha] = n_0 B$$

or

$$B = \frac{\Delta n [1-\alpha]}{n_0}.$$

Thus

$$r(T, n') = \frac{n' \beta}{s[(1-\alpha) + \Delta n \frac{n_0}{n_0} (1-\alpha) e^{-n' \beta T}]}$$

$$= \frac{n' \beta n_0}{s[(1-\alpha) n_0 + \Delta n (1-\alpha) e^{-n' \beta T}]}.$$

Now

$$\frac{r(T, n') - r(T, n_0)}{\Delta n} = \frac{1}{\Delta n} \left[ r(T, n') - \frac{\beta n_0}{1-\alpha} \frac{1}{s} \right]$$

$$= \frac{1}{\Delta n} \frac{1}{s} \left[ \frac{\beta n_0}{1-\alpha} \left[ \frac{n'}{n_0 + \Delta n e^{-n' \beta T}} - 1 \right] \right]$$

$$= \frac{1}{\Delta n} \frac{1}{s} \left[ \frac{\beta n_0}{1-\alpha} \right] \frac{n' - n_0 - \Delta n e^{-n' \beta T}}{n_0 + \Delta n e^{-n' \beta T}}$$

$$= \frac{1}{s} \frac{\beta n_0 \Delta n}{1-\alpha} \frac{1 - e^{-n' \beta T}}{n_0 + \Delta n e^{-n' \beta T}}$$
\[ \lim_{\Delta n \to 0} \frac{r(T,n') - r(T,n_0)}{\Delta n} = \frac{1}{s} \frac{\beta}{1-\alpha} \left[ 1 - e^{-r_0 s T} \right] \]

\[ \frac{dr(T,n_0)}{dn} = \left( \frac{1}{r(T,n_0)} \right) \left[ 1 - e^{-r_0 s T} \right] \]

where \((dr(T,n_0)/dn)/r(T,n_0))\) represents the percentage change in \(r\) at time \(T\) due to an infinitesimal increase in \(n\) at time \(0\).

Since

\[ y(T,n_0) = k(T,n_0)^\alpha e^{mn_0 T} \]

\[ = y(T,n_0)^\alpha r(T,n_0)^{-\alpha} e^{mn_0 T} \]

\[ y(T,n_0)^{1-\alpha} = r(T,n_0)^{-\alpha} e^{mn_0 T} \]

Therefore,

\[ (1-\alpha) \frac{dy(T,n_0)}{dn} = -\alpha \frac{dr(T,n_0)}{dn} + mT \]

and

\[ \frac{dy(T,n_0)}{dn} \gg 0 \quad \text{as} \quad \alpha \left[ 1 - e^{-r_0 s T} \right] \frac{1}{n_0} + mT \gg 0 \]
or equivalently as

$$\frac{m_0}{\alpha} T + e^{-n_0 m T} - 1 > 0.$$ 

Let

$$g(T) = \frac{m_0}{\alpha} T + e^{-n_0 m T} - 1.$$ 

Then

$$\text{sign } g(T) = \text{sign } \frac{dy(T, n_0)}{dn}.$$ 

For

$$T = 0, \quad g(T) = \frac{dy(T, n_0)}{dn} = 0.$$ 

For the case in which increasing returns are so large that \( m > \alpha \), it can be shown that \( g(T) \) and \( \frac{dy(T, n_0)}{dn} \) are positive for all positive \( T \).

This is true since \( g(0) = 0 \) and

$$\frac{dg}{dT} = \frac{n_0}{\alpha} \left[ m - \alpha \beta e^{-n_0 m T} \right]$$

$$\geq \frac{n_0}{\alpha} [m - \alpha \beta] \quad \text{for } T \geq 0$$

$$= \frac{n_0}{\alpha} [m - \alpha(1 + m - \alpha)]$$

$$= \frac{n_0}{\alpha} [1 - \alpha] [m - \alpha] > 0 \quad \text{for } m > \alpha.$$
For \( m > \alpha \), which would seem the more normal case, \( \frac{dg(T)}{dT} \) and \( g(T) \) must be negative for sufficiently small positive values of \( T \). In fact, it is easy to show that if \( m < \alpha \), \( g(T) \) has the shape illustrated in Figure 1 below. For \( T \) less than some positive \( T^* \), \( \frac{dy(T,n)}{dn} < 0 \).

![Figure 1](image)

**FIGURE 1**

Table A below presents a few reasonable sounding values for \( m, n_0, \alpha \) and \( \beta \) along with estimates of \( T^* \), the number of years which must elapse before a once and for all increase in the growth rate of population will begin to result in higher per capita incomes.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
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<td>( m )</td>
<td>0.05</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.25</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.80</td>
<td>0.8</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>( n_0 )</td>
<td>0.015</td>
<td>0.015</td>
<td>0.015</td>
<td>0.015</td>
</tr>
<tr>
<td>( T^* )</td>
<td>325</td>
<td>180</td>
<td>85</td>
<td>45</td>
</tr>
</tbody>
</table>
For the parameters listed in column (1), an increase in the rate of growth of population would result in lowered per capita incomes for a period of 325 years! Even in the case of the parameters listed in column (4) where \( m \) is very large relative to \( \alpha \), 45 years would elapse before per capita incomes would reach and begin to surpass the levels which they would have attained had the rate of growth of population not increased.
REFERENCES


