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Symmetry behavior of the static Taub universe: Effect of curvature anisotropy

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Using the static Taub universe as an example, we study the effect of curvature anisotropy on symmetry breaking of a self-interacting scalar field. The one-loop effective potential of a $\lambda\phi^4$ field with arbitrary coupling (ξ) is computed by ζ-function regularization. It is expressed as a perturbative series in a small anisotropy parameter α measuring the deformation from the spherical Einstein universe with radius of curvature a. This result is used for analyzing the symmetry behavior of such a system as a function of the geometric (a,α) and field (ξ,λ) parameters. The result is also used to address the question of whether and how curvature anisotropy can affect the inflationary scenario, old or new. We find that for a massless scalar field conformally coupled to the background with a prolate configuration (negative scalar curvature) the phase transition is of second order, in which case inflation to the extent necessary for cosmological purposes becomes highly unlikely. For the massless minimally coupled scalar field, first-order phase transitions can occur for a certain range of the radius and deformation parameter. If the curvature radius in the axisymmetric direction is held fixed, increasing deformation can restore the symmetry, whereas if the shape is held constant but the size is allowed to vary, decreasing the radius of the universe can induce symmetry breaking. For the minimally coupled field in a closed universe with high curvature a term linear in the background field in the effective potential appears. The barrier thus generated in the effective potential replaces the broad plateau of the flat-space Coleman-Weinberg potential. The meaning and implication of these results are discussed.

I. INTRODUCTION

In an earlier paper[1] we started a systematic investigation into the symmetry behavior of a self-interacting field in curved spacetime and studied the effect of spacetime curvature and field coupling on phase transitions using the Einstein universe as an example. In this paper, we continue this investigation for homogeneous anisotropic spacetimes, focusing on the effect of curvature anisotropy on symmetry breaking, using a static Taub universe[2] as an example. We also extend our analysis of the Einstein universe and delve somewhat into the effect of topology on symmetry breaking. The Taub metric is an anisotropic generalization of the closed Robertson-Walker metric and a specialization of the mixmaster universe,[3] where two of the three principal radii of curvature are equal.

Our interest in quantum vacuum processes in homogeneous but anisotropic cosmological models is manifold. Foremost is the belief that the universe at very early times may have existed in an anisotropic and inhomogeneous state. Despite the fact that our universe is observed to be highly isotropic today and is believed to have existed in a state of high isotropy since at least the grand-unification (GU) era ($t_{\mathrm{GU}} \sim 10^{-35}$ sec) after a period of inflation,[4,5] our knowledge of the state of the universe before the GU inflationary era is quite lacking. A well-known result[6] in classical general relativity suggests that the universe near the cosmological singularity could be highly anisotropic and inhomogeneous, its generic behavior being described by the (inhomogeneous) mixmaster model.[3] It can be argued that when quantum effects due to particle production are included in our consideration, the isotropic state can be extended as far back to the Planck time ($t_{\mathrm{Pl}} \sim 10^{-43}$ sec), as most of the anisotropies are dissipated in a relatively short duration. However, detailed calculation of these processes have been carried out only for Bianchi type-I universes, which contain anisotropy in the expansion (shear) but no anisotropy in the spatial curvature. From the dynamics of classical anisotropic models and the perturbation analysis of isotropic models, one learns that during the evolution, while the component of shear may decay in time, curvature anisotropy can actually grow. Since these factors are closely coupled via the Einstein equations, it is perhaps reasonable to assume that the strong quantum dissipative effect acting on shear will also suppress curvature anisotropy at the end of the dissi-

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pation epoch. However, it is not entirely implausible that a minute amount of curvature anisotropy left at the Planck era can still grow to a finite amount at the beginning of the grand-unification era. In light of these considerations, one should include the more general class of anisotropic models as permissible initial conditions before the GU era for the consideration of quantum processes like phase transitions.

Granted that a certain degree of anisotropy and inhomogeneity may survive through the Planck epoch and persist to the GU epoch, a natural question to ask is whether they can modify the standard inflationary scenario. This question has been addressed by a number of authors in varied forms. Wald[7] analyzed the late-time behavior of initially expanding homogeneous models with a positive cosmological constant Λ (acting like a vacuum energy density term due to the vacuum expectation value of the Higgs field) and concluded that all Bianchi-type models except type IX (to which the Taub and mixmaster universes belong) evolve exponentially towards the de Sitter solution and the behavior of type-IX universes is similar provided that Λ is sufficiently large compared with spatial curvature terms. These results comply with the "cosmic baldness" conjecture of Bocher, Gibbons, and Hawking[7] that perturbations in a de Sitter universe will in general decay rapidly away. Notice that these statements about the behavior of shear and curvature anisotropy assume that the universe is already in an exponentially expanding stage. If, however, prior to inflation the universe is dominated by shear or negative curvature, then a result due to Barrow and Turner[8] suggests that inflation in the old sense[4] cannot occur. All of the above analyses are based on an examination of the classical Einstein's equations for homogeneous anisotropic universes, no field-theoretical description of the Higgs field is involved. Thereafter, Steigman and Turner[8] considered a Friedmann-Robertson-Walker (FRW) universe perturbed by shear and vorticity. They analyzed their effects on the Hubble expansion rate and on the evolution rate of the Higgs field as governed by a flat-space Coleman-Weinberg (CW) effective potential. They concluded that neither shear nor negative curvature can have significant effect on inflation (in the new sense[5]) during or after the vacuum era. Although an effective potential governing the Higgs field is used, it is not for the same curved-spacetime background which governs the dynamics of the universe. Today many studies of phase transition in the GU epoch assume a curved-spacetime description for cosmology but a flat-space formalism for field theory. The logical consistency and physical soundness of such a hybrid framework is, in our opinion, rather questionable. It is our uneasiness with this state of affairs which prompted us to carry out this series of studies on the effect of spacetime curvature, topology, and field coupling on cosmological phase transitions.[9] Specifically, concerning the role of curvature anisotropy on inflation as is addressed in this paper, we do not think the studies carried out so far[7,8] are conclusive. A good many authors have indeed treated the phase-transition problem in the de Sitter universe in a consistent manner,[10] but the adaptation of results for the de Sitter universe is appropriate only for situations where inflation has already commenced. They cannot answer the question of whether or not inflation can take place in the face of shear and curvature anisotropy, assuming from our earlier discussion that a somewhat chaotic state can exist prior to the GU epoch. For these problems, one should instead use the effective potential in a curved spacetime with shear and anisotropy, the same spacetime one uses for the description of cosmology. As is well-known to practitioners in this field the precise form of the effective potential can dictate rather different outcomes for the behavior of the Higgs field and the dynamics of the universe during the phase transition. A good case in point is the "fine-tuning" problem [i.e., $V''(\hat{\phi}=0)=0$, see Eq. (4.12) and Ref. 5] in the new inflationary scenario. On this point alone, we find that indeed for an important class of curved spacetimes this condition is not satisfied. In a more careful analysis of the Einstein universe (regarded as an instantaneous snapshot of the closed FRW universe), we find that for massless minimally coupled scalar fields a term linear in the background field $\hat{\phi}$ dominates the effective potential near the symmetric state $\hat{\phi}=0$. The existence of such a barrier which increases with curvature violates the CW condition and makes inflation in the new sense difficult to achieve. A phase transition by tunneling can still occur, but that would bring back the well-known difficulties of the old scenario. It is worth noticing that this rather unique behavior of the massless minimal field is generic to spacetimes with topology $R^1\times S^3$ (see Sec. V 2). For conformal fields in the static Taub universe, the symmetric state becomes unstable as the space evolves from an oblate [anisotropic parameter $\alpha>0$, defined below Eq. (2.3a)] to a prolate configuration ($\alpha<0$). Only a second-order phase transition can occur, which renders inflation to the extent necessary for cosmological purposes highly unlikely. These are but a few of the results we obtained from a detailed analysis of the effective potential in a static Taub universe. They should serve to illustrate the relevance of curvature and topology in symmetry-breaking considerations in curved spacetimes. By exemplifying certain conditions in curved space where inflation can fail to occur, we hope to draw the readers' attention to the complexity of issues involved and to sound a note of caution in the ordinary treatment of these problems.

In this work we choose to work with the static Taub universe because it makes a well-defined effective potential possible and allows us to address both the effect of topology and spatial curvature anisotropy. In a more general but perhaps more formal context (than that related to the inflation problem) we also obtained the relation between the critical radius (size) and the deformation parameter (shape) of the geometry. For example, if the curvature radius in the axisymmetric direction is held fixed, an increasing deformation can bring about symmetry restoration. Alternatively, symmetry breaking can occur by scaling down the size of the universe. Our present problem shares some similarity with the problem of finite-size effects on phase transitions,[11] a subject of much recent interest in condensed matter and surface physics. Although as a first step we consider here only static spacetimes, we believe that dynamical effects are probably just as, if not more, important, especially at or near the quantum epoch. Symmetry breaking due to shear is an obvious possibility.[9] The derivation of an effective Lagrangian in general curved spacetime[12] and the study of dynamical effects on phase transitions[13] are currently under investigation. Incorporation of our present result on curvature anisotropy with that obtained elsewhere on the effects due to expansion[13] and shear[14] will enable us to analyze the symmetry behavior of the mixmaster universe, which would constitute an important step towards understanding the quantum nature of the early universe.

This paper is organized as follows: In Sec. II we derive the one-loop effective potential via the ζ-function method, using the eigenmodes of a scalar wave operator in the Taub universe. The geometric quantities of the Taub universe and details of the summation techniques are collected in the Appendices A, B, and C. Using the renormalization condition of Ref. 12, we proceed in Sec. III to renormalize the coupling constants of the effective potential. Our results for the effective potential can be written in terms of geometric invariant quantities, which can provide a more general setting for treating quantum fields in homogeneous but anisotropic spacetimes. This is contained in Appendix D. In Sec. IV we study the symmetry behavior of a massless field and analyze the differences between the minimally coupled and the conformally coupled cases. In Sec. V we discuss the meaning and restrictions of our results and compare our method with other work. We also draw implications of our findings to phase transitions in the inflationary universe and suggest some direction for further investigation.

II. EFFECTIVE POTENTIAL

The metric of a diagonal mixmaster universe is given by[3]

$$ds^2=-dt^2+\sum_{a=1}^{3} l_a^2(\sigma^a)^2 , \tag{2.1}$$

where σ^a are the basis one-forms on the three-sphere satisfying the structure relation $d\,\sigma^a=\frac{1}{2}C^a{}_{bc}\sigma^b\Lambda\sigma^c$. For Bianchi type-IX spaces $C^a{}_{bc}=\epsilon^a{}_{bc}$ is the structure constant of the rotation group SO_3. In the Euler-angle parametrization[3] $(0\le\theta\le\pi, 0\le\phi,\psi\le 2\pi)$, the σ^a are given by

$$\begin{aligned}\sigma^a&=\cos\psi\,d\theta+\sin\psi\sin\theta\,d\phi\ ,\\ \sigma^b&=-\sin\psi\,d\theta+\cos\psi\sin\theta\,d\phi\ ,\\ \sigma^c&=d\psi+\cos\theta\,d\phi\ .\end{aligned} \tag{2.2}$$

The l_a's are the three principal curvature radii of the homogeneous space and are constants for a static universe. The cases when any two of l_a's are equal are the Taub universes. The case when all three l_a's are equal is the closed Friedmann-Robertson-Walker (FRW) universe.

The curvature scalar R is given by

$$R=\frac{4l_1{}^2-l_3{}^2}{2l_1{}^4}=\frac{6(1+\frac{4}{3}\alpha)}{a^2(1+\alpha)} , \tag{2.3a}$$

where $a=2l_1$ and $\alpha=(l_1{}^2/l_3{}^2)-1$ is an anisotropy parameter, with range $-1<\alpha<\infty$. We call the configuration with $\alpha>0$ oblate and that with $\alpha<0$ prolate. In the case of small anisotropy (near-Einstein universe), R can be expanded in powers of α as

$$R=\frac{6}{a^2}\left[1+\frac{\alpha}{3}-\frac{\alpha^2}{3}+O(\alpha^3)\right] . \tag{2.3b}$$

The volume of the Taub universe is given by $\Omega=2\pi^2a^3/\sqrt{1+\alpha}$. (See Appendix A for details.)

Consider a massive (m) scalar field $\tilde\phi$ with quartic self-interaction (λ) coupled to a static Taub universe described by the Lagrangian density

$$L[\tilde\phi,g_{ab}]=-\tfrac{1}{2}\tilde\phi\left[-\Box+m^2+(1-\xi)\frac{R}{6}\right]\tilde\phi-\frac{\lambda}{4!}\tilde\phi^4 , \tag{2.4}$$

where

$$\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right]$$

is the Laplace-Beltrami operator on $R^1\times S^3$ and the coupling constant $\xi=0,1$ denotes conformal and minimal coupling, respectively. This action has a minimum at $\tilde\phi=\hat\phi$, which satisfies the classical equation of motion

$$\left[-\Box+m^2+(1-\xi)\frac{R}{6}+\frac{\lambda}{6}\hat\phi^2\right]\hat\phi=0 . \tag{2.5}$$

Quantum fluctuations $\phi\equiv\tilde\phi-\hat\phi$ around the classical background field $\hat\phi$ satisfy an equation of the form (to lowest order in ϕ)

$$(-\Box+\mathcal{M}^2)\phi(x)=0 , \tag{2.6}$$

where $\mathcal{M}^2=M^2+(1-\xi)R/6$ is an effective mass which depends on the background curvature R, the coupling ξ, and on the background field $\hat\phi$ via $M^2\equiv m^2+\frac{1}{2}\lambda\hat\phi^2$. When contributions from quantum fluctuations are included, the equation satisfied by the background field $\hat\phi$ contains an extra term due to the variance of the fluctuations,

$$\left[-\Box+m^2+(1-\xi)\frac{R}{6}+\frac{\lambda}{6}\hat\phi^2+\frac{\lambda}{2}\langle\phi^2\rangle\right]\hat\phi=0 . \tag{2.7}$$

In the functional-integral perturbative approach, the effective action which is related to the effective Lagrangian $L_{\rm eff}$ by

$$\Gamma[\hat\phi,g_{\mu\nu}]=\int d^4x\sqrt{-g}\,L_{\rm eff} \tag{2.8}$$

is expanded in powers of $\hbar$ as

$$\Gamma[\hat\phi]=S[\hat\phi]+\Gamma^{(1)}+\Gamma' . \tag{2.9}$$

Here $S[\hat\phi]$ is the classical action

$$\begin{aligned}S[\hat\phi]&=\int d^4x\sqrt{-g}\,L_\phi^{(0)} ,\\ L_\phi^{(0)}&=-\tfrac{1}{2}\hat\phi(x)\left[-\Box+m^2+(1-\xi)\frac{R}{6}\right]\hat\phi(x)-\frac{1}{4!}\lambda\hat\phi^4(x) ,\end{aligned} \tag{2.10}$$

$\Gamma^{(1)}$ is the one-loop effective action

$$\Gamma^{(1)}=\frac{i\hbar}{2}\ln{\rm Det}(\mu^{-2}i\hat A)=\int d^4x\sqrt{-g}\,L^{(1)} , \tag{2.11}$$

and Γ' denotes higher-loop contributions. In the above, A is the operator defined by

$$\hat{A} \equiv -\Box + m^2 + (1-\xi)\frac{R}{6} + \frac{\lambda}{2}\hat{\phi}^2 . \tag{2.12}$$

A mass scale μ is introduced to render the measure $d[\phi]$ of the functional integral dimensionless. In a static, homogeneous spacetime, as is the case under study, $\hat{\phi}$ is a constant field. One can then define an effective potential by $V(\hat{\phi}) = -(\text{vol})^{-1}\Gamma(\hat{\phi})$, where vol denotes the spacetime volume.

The one-loop effective potential $V^{(1)}$ is formally divergent. We shall use the ζ-function regularization[15] to render it finite. For operators with a known eigenvalue spectrum on spacetimes admitting Euclidean sections, this method is particularly handy. Denote by λ_N the eigenvalues of the operator A on the Euclideanized metric obtained from (2.1) by a Wick rotation to imaginary time $\tau = it$. A finite-temperature (T) theory is defined by imposing periodic boundary conditions on τ with period $\beta = T^{-1}$. The zero-temperature result is regained by letting β become very large so that the eigenmodes in the imaginary direction become continuous again. One then introduces the generalized ζ-function $\zeta(\nu)$ defined by

$$\zeta(\nu) = \sum_N (\mu^{-2}\lambda_N)^{-\nu} . \tag{2.13}$$

In Hawking's formulation, the one-loop effective potential is given by[16]

$$V^{(1)} = -\frac{\hbar}{2(\text{vol})}\zeta'(0) . \tag{2.14}$$

In the $S^1 \times S^3$ topology of Euclideanized metric, vol$=\Omega\beta$.

We now proceed to calculate $V^{(1)}$ for the static Taub universe with small anisotropy. The final expression (2.32) is given as a series expansion in the deformation parameter α.

The eigenvalues of the operator A are given by

$$\lambda_N = \left[\frac{2\pi n_0}{\beta}\right]^2 + \frac{J(J+1)}{l_1{}^2} + \left[\frac{1}{l_3{}^2} - \frac{1}{l_1{}^2}\right]K^2 + m^2 + (1-\xi)\frac{R}{6} + \frac{\lambda}{2}\hat{\phi}^2 . \tag{2.15}$$

The spatial eigenfunctions and eigenvalues have been derived by one of us before.[17] Here N denotes the collection of temporal and spatial quantum numbers n_0, J, K, M, with ranges $n_0 = 0, \pm 1, \pm 2, \ldots; J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$; and $K, M = -J, -J+1, \ldots, J-1, J$. The quantum number M is totally degenerate. For comparison with the spectra in the deformed space (mixmaster) and the spherical space (Einstein), the relation of eigenfunctions $D^J_{KM}(\theta,\phi,\psi)$, and the hyperspherical harmonics $Y_{nlm}(\chi,\theta,\phi)$ on SO_4-symmetric space can be found in standard texts.[18]

The derivation of the ζ function in (2.13) involves simply carrying out the summations of (2.15) over the appropriate ranges of N. In the low-temperature limit ($\beta \to \infty$), the sum over n can be replaced by $\beta \int dk_0/2\pi$. Integrating over k_0 and summing over M (multiplicity $2J+1$), one gets

$$\zeta(\nu) = \frac{\beta\mu^{2\nu}}{\sqrt{4\pi}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\sum_{n=1}^{\infty}\sum_{K=-J}^{J}\frac{n}{a^{1-2\nu}}\left[n^2+\sigma+\alpha\left[4k^2+\frac{1-\xi}{3}\right]-\left[\frac{1-\xi}{3}\right]\alpha^2\right]^{-\nu+1/2} , \tag{2.16}$$

where we have introduced the notations

$$n = 2J+1, \quad \sigma = -\xi + M^2a^2 . \tag{2.17}$$

The summation over K can be performed if the summand is expanded about $\alpha = 0$ (the Einstein universe) for small α:

$$\sum_{n=1}^{\infty}\Bigg[n^2(n^2+\sigma)^{-\nu+1/2} + (\tfrac{1}{2}-\nu)\frac{\alpha}{3}[n^2(n^2-1)+(1-\xi)n^2](n^2+\sigma)^{-\nu-1/2} - \frac{\alpha^2}{3}(\tfrac{1}{2}-\nu)(1-\xi)n^2(n^2+\sigma)^{-\nu-1/2} + 2(\nu^2-\tfrac{1}{4})\alpha^2(n^2+\sigma)^{-\nu-3/2} \times \left[\frac{n^2}{60}(3n^4-10n^2+7)+\frac{n^2}{18}(n^2-1)(1-\xi)+\frac{(1-\xi)^2}{36}n^2\right] + O(\alpha^3)\Bigg] . \tag{2.18}$$

Define

$$Z(r,\sigma) = \sum_{n=1}^{\infty}\frac{n^2}{(n^2+\sigma)^r} . \tag{2.19}$$

The ζ function can then be expressed as

$$\zeta(\nu) = \frac{\beta}{\sqrt{4\pi}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\frac{(\mu a)^{2\nu}}{a}(E_0 + E_1\alpha + E_2\alpha^2) , \tag{2.20}$$

where

$$E_0 = Z(\nu - \tfrac{1}{2}, \sigma)\ , \tag{2.21a}$$

$$E_1 = \tfrac{1}{3}(\tfrac{1}{2} - \nu)[Z(\nu - \tfrac{1}{2}, \sigma) - (\xi + \sigma) Z(\nu + \tfrac{1}{2}, \sigma)]\ , \tag{2.21b}$$

$$E_2 = -\tfrac{1}{3}(\tfrac{1}{2} - \nu)(1 - \xi) Z(\nu + \tfrac{1}{2}, \sigma) + \tfrac{1}{30}(\nu^2 - \tfrac{1}{4})[\,3Z(\nu - \tfrac{1}{2}, \sigma) - (6\sigma + \tfrac{10}{3}\xi + \tfrac{20}{3}) Z(\nu + \tfrac{1}{2}, \sigma)$$
$$+ (3\sigma^2 + \tfrac{20}{3}\sigma + \tfrac{16}{3} + \tfrac{10}{3}\sigma\xi + \tfrac{5}{3}\xi^2) Z(\nu + \tfrac{3}{2}, \sigma)]\ , \tag{2.21c}$$

The functions $Z(r,\sigma)$ are similar to the generalized ζ functions. They possess poles at $r = \frac{3}{2} - n$, $n = 0,1,2,\ldots$ (the negative integer r terms do not generate divergences). Writing $Z(r,\sigma)$ as

$$Z(r,\sigma) = \frac{Z_{-1}(n,\sigma)}{r - \frac{3}{2} + n} + Z_0(n,\sigma) + O(r - \tfrac{3}{2} + n)\ , \tag{2.22}$$

the E_n's can be separated into a singular part and a regular part:

$$E_0 = \frac{Z_{-1}(2,\sigma)}{\nu} + Z_0(2,\sigma) + O(\nu)\ , \tag{2.23a}$$

$$E_1 = \frac{1}{6\nu}[Z_{-1}(2,\sigma) - Z_{-1}(1,\sigma)(\xi + \sigma)] + \tfrac{1}{6}[Z_0(2,\sigma) - (\xi + \sigma) Z_0(1,\sigma)]$$
$$-\tfrac{1}{3}[Z_{-1}(2,\sigma) - (\xi + \sigma) Z_{-1}(1,\sigma)] - \frac{\nu}{3}[Z_0(2,\sigma) - (\xi + \sigma) Z_0(1,\sigma)] + O(\nu^2)\ , \tag{2.23b}$$

$$E_2 = \frac{1}{\nu}\left[-\tfrac{1}{40} Z_{-1}(2,\sigma) + \left[\frac{\sigma}{20} + \tfrac{7}{36}\xi - \tfrac{1}{9}\right] Z_{-1}(1,\sigma) - \left[\frac{\sigma^2}{40} + \frac{\sigma}{18} + \frac{2}{45} + \frac{\xi^{\sigma}}{36} + \frac{\xi^2}{72}\right] Z_{-1}(0,\sigma)\right]$$
$$+ \left[\frac{1-\xi}{3}\right] Z_{-1}(1,\sigma) - \tfrac{1}{40} Z_0(2,\sigma) + \left[\frac{\sigma}{20} + \tfrac{7}{36}\xi - \tfrac{1}{9}\right] Z_0(1,\sigma)$$
$$- \left[\frac{\sigma^2}{40} + \frac{\sigma}{18} + \frac{2}{45} + \frac{\xi}{36}\sigma + \frac{\xi^2}{72}\right] Z_0(0,\sigma) + \left[\frac{1-\xi}{3}\right] \nu Z_0(1,\sigma)$$
$$+ \frac{\nu}{30}[3Z_{-1}(2,\sigma) - (6\sigma + \tfrac{10}{3}\xi + \tfrac{20}{3}) Z_{-1}(1,\sigma) + (3\sigma^2 + \tfrac{20}{3}\sigma + \tfrac{16}{3} + \tfrac{10}{3}\sigma\xi + \tfrac{5}{3}\xi^2) Z_{-1}(0,\sigma)] + O(\nu^2)\ . \tag{2.23c}$$

The part containing the Γ functions in (2.20) can likewise be expanded as a series in ν, with

$$\lim_{\nu \to 0} \frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} \frac{(a\mu)^{2\nu}}{a} = \frac{-\sqrt{4\pi}}{a}\left[\nu + \left[2 + \ln\frac{a^2\mu^2}{4}\right]\nu^2 + O(\nu^3)\right]\ , \tag{2.24a}$$

$$\lim_{\nu \to 0} \frac{d}{d\nu}\left[\frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} \frac{(a\mu)^{2\nu}}{a}\right] = \lim_{\nu \to 0} \frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} \frac{(a\mu)^{2\nu}}{a}[\psi(\nu - \tfrac{1}{2}) - \psi(0) + \ln(a\mu)^2]$$
$$= -\frac{\sqrt{4\pi}}{a}\left[1 + 2\nu\left[2 + \ln\frac{a^2\mu^2}{4}\right] + O(\nu^2)\right]\ . \tag{2.24b}$$

From (2.20) and (2.24), one obtains

$$\zeta'(0) = \left.\frac{d\zeta(\nu)}{d\nu}\right|_{\nu=0} = \frac{-\beta}{a}\left[(E_0 + E_1\alpha + E_2\alpha^2)_{\text{reg}} + \left[2 + \ln\frac{a^2\mu^2}{4}\right](E_0 + E_1\alpha + E_2\alpha^2)_{\text{pole}}\right]\ . \tag{2.25}$$

The pole terms and the regular terms are readily identified from (2.23) as

$$\text{pole terms} = Z_{-1}(2,\sigma) + \frac{\alpha}{6}[Z_{-1}(2,\sigma) - Z_{-1}(1,\sigma)(\xi + \sigma)]$$
$$-\alpha^2\left[\tfrac{1}{40} Z_{-1}(2,\sigma) - \left[\frac{\sigma}{20} + \tfrac{7}{36}\xi - \tfrac{1}{9}\right] Z_{-1}(1,\sigma) + \left[\frac{\sigma^2}{40} + \frac{\sigma}{18} + \frac{2}{45} + \frac{\xi}{36}\sigma + \frac{\xi^2}{72}\right] Z_{-1}(0,\sigma)\right]\ , \tag{2.26a}$$

$$\text{regular terms} = Z_0(2,\sigma)+\frac{\alpha}{6}[Z_0(2,\sigma)-(\xi+\sigma)Z_0(1,\sigma)]-\frac{\alpha}{3}[Z_{-1}(2,\sigma)-(\xi+\sigma)Z_{-1}(1,\sigma)]$$
$$+\alpha^2\left[\left[\frac{1-\xi}{3}\right]Z_{-1}(1,\sigma)-\tfrac{1}{40}Z_0(2,\sigma)+\left[\frac{\sigma}{20}+\tfrac{7}{36}\xi-\tfrac{1}{9}\right]Z_0(1,\sigma)\right.$$
$$\left.-\left[\frac{\sigma^2}{40}+\frac{\sigma}{18}+\frac{2}{45}+\frac{\xi}{36}\sigma+\frac{\xi^2}{72}\right]Z_0(0,\sigma)\right]. \quad (2.26b)$$

Setting this back to expression (2.14) for the effective potential one gets

$$V^{(1)}=-\frac{\hbar}{2\Omega\beta}\zeta'(0)$$
$$=\frac{\hbar}{2a\Omega}\left[(E_0+E_1\alpha+E_2\alpha^2)_{\text{reg}}\right.$$
$$\left.+\left[2+\ln\frac{a^2\mu^2}{4}\right](E_0+E_1\alpha+E_2\alpha^2)_{\text{pole}}\right]. \quad (2.27)$$

The term containing the volume Ω of the Taub universe when expanded in α becomes

$$\frac{1}{2a\Omega}\simeq\frac{1}{4\pi^2a^4}\left[1+\frac{\alpha}{2}-\frac{\alpha^2}{8}\right]. \quad (2.28)$$

Inside the brackets, the pole part depends on $Z_{-1}(n,\sigma)$, $n=0,1,2$. From (B5)—(B7), we have [see Appendix B and C for the evaluation of $Z(r,\sigma)$]

$$Z_{-1}(0,\sigma)=\tfrac{1}{2}\,,$$
$$Z_{-1}(1,\sigma)=-\frac{\sigma}{4}\,, \quad (2.29)$$
$$Z_{-1}(2,\sigma)=-\frac{\sigma^2}{16}\,.$$

Substituting (2.28) and (2.29) into (2.27) we obtain for the pole part

$$\frac{1}{2a\Omega}(E_0+E_1\alpha+E_2\alpha^2)_{\text{pole}}$$
$$=\frac{1}{4\pi^2a^4}\left[-\frac{\sigma^2}{16}+\frac{\xi}{24}\sigma\alpha\right.$$
$$\left.+\alpha^2\left[-\frac{\sigma}{24}\xi+\frac{1}{360}-\frac{\xi^2}{144}\right]\right]. \quad (2.30)$$

Reexpressing this in terms of ξ by (2.17) and using the geometric expressions in (A7) to (A9), (2.30) becomes

$$\frac{-\xi^2R^2}{2304\pi^2}+\frac{\xi RM^2}{192\pi^2}-\frac{M^4}{64\pi^2}-\frac{C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}}{3840\pi^2}\,. \quad (2.31)$$

Combining this with the regular terms in (2.26b), we obtain finally the one-loop effective potential in small deformation expansion as

$$V^{(1)}=\frac{\hbar}{2a\Omega}(A+B\alpha+C\alpha^2)$$
$$-\hbar\left[2+\ln\frac{a^2\mu^2}{4}\right]\left[\frac{\xi^2R^2}{2304\pi^2}-\frac{\xi RM^2}{192\pi^2}+\frac{M^4}{64\pi^2}\right.$$
$$\left.+\frac{C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}}{3840\pi^2}\right], \quad (2.32)$$

where

$$A=Z_0(2,\sigma)\,, \quad (2.33a)$$
$$B=\tfrac{1}{6}[Z_0(2,\sigma)-M^2a^2Z_0(1,\sigma)]+\tfrac{1}{12}\left[\frac{\sigma^2}{4}-M^2a^2\sigma\right], \quad (2.33b)$$
$$C=-\frac{(1-\xi)}{12}\sigma-\tfrac{1}{40}Z_0(2,\sigma)+\left[\frac{\sigma}{20}-\tfrac{1}{9}+\tfrac{7}{36}\xi\right]Z_0(1,\sigma)$$
$$-\left[\frac{\sigma^2}{40}+\frac{\sigma}{18}+\frac{2}{45}+\frac{\xi}{36}\sigma+\frac{\xi^2}{72}\right]Z_0(0,\sigma)\,. \quad (2.33c)$$

III. RENORMALIZATION

The effective potential V is given as a sum of the quantum (one-loop) effective potential $V^{(1)}$ in (2.32) and the classical potential $V^{(0)}=V_f^{(0)}+V_g^{(0)}$ corresponding to the field $(\hat{\phi})$ and geometry (g):

$$V_f^{(0)}=\tfrac{1}{2}({m_B}^2+\xi_BR)\hat{\phi}^2+\frac{\lambda_B}{4!}\hat{\phi}^4\,,$$
$$V_g^{(0)}=-(\Lambda_B+\kappa_BR+\tfrac{1}{2}\epsilon_{1B}R^2+\tfrac{1}{2}\epsilon_{2B}C^2+\epsilon_3G)\,. \quad (3.1)$$

Here Λ is the cosmological constant, $\kappa=(16\pi G_N)^{-1}$, G_N being Newton's constant, and ϵ_i are the coupling constants for quadratic-curvature terms: $C^2=C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ being the Weyl curvature-squared, and $G=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-4R^{\alpha\beta}R_{\alpha\beta}+R^2$ the Gauss-Bonnet density. (See Ref. 12 for details.) The divergences in V are removed by the renormalization of these parameters which have hitherto been regarded as bare quantities, i.e.,

$$m^2={m_B}^2+\delta m^2\,, \quad (3.2)$$

etc. This is equivalent to the introduction of a counterpotential

$$V^c=V_g^c+V_f^c \quad (3.3)$$

with

$$V_f^c=\tfrac{1}{2}\delta m^2\hat{\phi}^2+\tfrac{1}{2}(\delta\xi)R\hat{\phi}^2+\frac{\delta\lambda}{4!}\hat{\phi}^4$$

and

$$V_g^c=-(\delta\Lambda+\delta\kappa R+\tfrac{1}{2}\delta\epsilon_1 R^2+\tfrac{1}{2}\delta\epsilon_2 C^2)$$

(the pure divergence terms $\Box\phi^2$, $\Box R$, and G do not enter here), which upon addition to $V^{(0)}$ and $V^{(1)}$ renders V finite:

$$V=V_B^{(0)}+V^{(1)}+V^c=V_{\text{ren}}^{(0)}+V_{\text{finite}}^{(1)}\,.$$

The counterterms are determined by imposing suitable renormalization conditions, e.g., $m^2=\partial^2V/\partial\hat{\phi}^2\,|_{\hat{\phi}=0,R=0}$, etc.

This procedure has been discussed at length in our earlier papers (Refs. 1 and 12). In fact, since ultraviolet divergences in the theory occur at the local scale, it suffices to take the small-R (large a) limit of V for the determination of these counterterms. In the near-flat-space limit ($\sigma\gg1$) the coefficients A, B, and C of V in (2.32) are given by

$$\begin{aligned}
A&\simeq\tfrac{1}{16}(\xi^2-2M^2a^2\xi+M^4a^4)\left[\tfrac{1}{2}+\ln\frac{\sigma}{4}\right],\\
B&\simeq\frac{\xi^2}{192}\left[2\ln\frac{\sigma}{4}+5\right]-\frac{\xi M^2a^2}{96}\left[1-2\ln\frac{\sigma}{4}\right]-\frac{M^4a^4}{64}\left[1+2\ln\frac{\sigma}{4}\right],\\
C&\simeq\tfrac{3}{128}M^4a^4\left[\tfrac{1}{2}+\ln\frac{\sigma}{4}\right]+\frac{\xi M^2a^2}{64}\left[\tfrac{5}{2}+\ln\frac{\sigma}{4}\right]-\frac{\xi^2}{1152}\left[\tfrac{85}{2}+37\ln\frac{\sigma}{4}\right]+\tfrac{1}{45}\left[2+\ln\frac{\sigma}{4}\right].
\end{aligned}\tag{3.4}$$

One expects to see the same ultraviolet behavior manifest here as in the Einstein universe[1] or, more generally, in curved spacetime under the small-proper-time Schwinger-DeWitt expansion.[12] The global properties of spacetime which govern the infrared behavior of the fields and their symmetry-breaking patterns will of course be different. One can get V^c by using the expressions for the counterterms in Eq. (4.19) of Ref. 12, which were derived under the renormalization conditions [Eq. (4.18)] therein. Combining V^c with the one-loop potential $V^{(1)}$ in (2.32) and the classical potential $V^{(0)}$ in (3.1), (now with renormalized coupling constants) we obtain the renormalized one-loop effective potential as follows:

$$\begin{aligned}
V=&-\left[\Lambda+\kappa R+\frac{\epsilon_1}{2}R^2+\frac{\epsilon_2}{2}C^2\right]\\
&+\left[\tfrac{1}{2}m^2\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4-\frac{\hbar\lambda^2\hat{\phi}^4}{256\pi^2}\left[\ln\left|\frac{[m^2+(\lambda/2)\hat{\phi}_0{}^2]a^2}{4}\right|+\frac{14}{3}-\frac{8}{3}\frac{m^2[m^2+(\lambda/4)\hat{\phi}_0{}^2]}{[m^2+(\lambda/2)\hat{\phi}_0{}^2]^2}\right]\right]\\
&+\tfrac{1}{12}(1-\xi)R\hat{\phi}^2-\frac{\hbar m^4}{64\pi^2}\left[\tfrac{1}{2}+\ln\left|\frac{m^2a^2}{4}\right|\right]+\frac{\lambda\xi R\hat{\phi}^2\hbar}{384\pi^2}\left[2+\ln\left|\frac{[m^2-(\xi/6)R_0]a^2}{4}\right|\right]+\frac{\hbar\xi m^2R}{192\pi^2}\left[\ln\left|\frac{m^2a^2}{4}\right|+1\right]\\
&-\frac{\hbar\xi^2R^2}{2304\pi^2}\left[2+\ln\left|\frac{[m^2-(\xi/6)R_1]a^2}{4}\right|\right]-\frac{\hbar C^2}{3840\pi^2}\left[2+\ln\left|\frac{[m^2-(\xi/6)R_2]a^2}{4}\right|\right]-\frac{\hbar\lambda m^2\hat{\phi}^2}{64\pi^2}\left[1+\ln\left|\frac{m^2a^2}{4}\right|\right]\\
&-\delta\Lambda(\langle\phi\rangle)+\frac{\hbar}{2a\Omega}(A+B\alpha+C\alpha^2)\,.
\end{aligned}\tag{3.5}$$

The quantum correction terms can be regrouped into a flat-space part, which, together with the classical field terms (in the square brackets) make up the Coleman-Weinberg potential [Eq. (4.23) of Ref. 12], and a curved-space part involving R, R^2, and C^2 terms which has the generic form as in Eq. (4.22) of Ref. 12. Comparing this with the renormalized effective potential for the Einstein universe [Eq. (20) of Ref. 1], we notice that the major difference lies in the terms in $V^{(1)}$ involving the deformation parameter α and, of course (due to the nonconformal flatness of the Taub universe), terms arising from the renormalization of coupling constants associated with C^2, the square of the Weyl tensor.

One may have noticed that in the massless, conformal field case the C^2 term in Eq. (3.5) has an infrared divergence. This is a reflection of the fact that the large-mass limit which was assumed in the derivation of the counterterms is not appropriate. The treatment of this special case is presented in Appendix D.

For the convenience of later use, we record here the values of A, B, and C in the high-curvature limit (small a).

For conformal coupling, $\xi=0$, $|\sigma|\ll1$. From (B14)—(B16) and (2.23) one gets

$$A=\frac{1}{120}-\frac{\sigma}{24}-\frac{\gamma}{8}\sigma^2+O(\sigma^3)\,,\tag{3.6a}$$

$$B=\frac{1}{720}+\frac{\xi}{72}+\sigma\left[\frac{1}{144}+\frac{\gamma}{12}\xi\right]+\frac{\sigma^2}{16}[\gamma+\tfrac{1}{3}-\zeta(3)\xi]+O(\sigma^3)\,,\tag{3.6b}$$

$$C=\tfrac{391}{43200}-\tfrac{2}{45}\gamma-\tfrac{7}{432}\xi-\frac{\xi^2}{72}\gamma+\sigma\left[-\frac{89}{2880}-\frac{\xi}{72}-\frac{\xi\gamma}{8}\right]-\xi\zeta(3)\left[\frac{1}{15}+\frac{\xi^2}{48}\right]$$
$$+\sigma^2[-\tfrac{1}{40}-\tfrac{3}{64}\gamma+(\tfrac{1}{24}+\tfrac{11}{96}\xi)\zeta(3)-(\tfrac{1}{12}+\tfrac{5}{192}\xi^2)\zeta(5)]+O(\sigma^3)\ , \tag{3.6c}$$

where $\gamma=0.5772$ is the Euler constant. For minimal coupling, $\xi=1$, $\sigma\equiv1+y$. From (B18)—(B20), one gets

$$A'=Z_0(2,\sigma)=\sum_{n=0}^{\infty}a_ny^n\ , \tag{3.7a}$$

$$B'=\tfrac{1}{6}[a_0+(1-\xi)b_0]-\tfrac{1}{12}(\tfrac{3}{4}-\xi)+\tfrac{1}{6}[a_1+(1-\xi)b_1+\tfrac{1}{2}(\tfrac{3}{2}-\xi)-b_0]y$$
$$+\tfrac{1}{6}[a_2+(1-\xi)b_2-b_1-\tfrac{3}{8}]y^2+\tfrac{1}{6}\sum_{n=3}^{\infty}[a_n+(1-\xi)b_n-b_{n-1}]y^n\equiv\sum_{n=0}^{\infty}d_ny^n\ , \tag{3.7b}$$

$$C'=\frac{1-\xi}{12}-\frac{a_0}{40}-b_0(-\tfrac{7}{36}\xi+\tfrac{29}{180})-\frac{(1-\xi)^2}{72}c_0$$
$$+y\left[-\frac{1-\xi}{12}-\frac{a_1}{40}+\frac{b_0}{20}-(\tfrac{29}{180}-\tfrac{7}{36}\xi)b_1-\left[\frac{1}{180}+\frac{\xi}{36}\right]c_0+\frac{(1-\xi)^2}{72}c_1\right]$$
$$+y^2\left[-\frac{a_2}{40}+\frac{b_1}{20}-(\tfrac{29}{180}-\tfrac{7}{36}\xi)b_2-\frac{c_0}{40}-\left[\frac{1}{180}+\frac{\xi}{36}\right]c_1-\frac{(1-\xi)^2}{72}c_2\right]$$
$$+\sum_{n=3}^{\infty}\left[-\frac{a_n}{40}+\frac{b_{n-1}}{20}-(\tfrac{29}{180}-\tfrac{7}{36}\xi)b_n-\frac{c_{n-2}}{40}-\left[\frac{1}{180}+\frac{\xi}{36}\right]c_{n-1}-\frac{(1-\xi)^2}{72}c_n\right]y^n\equiv\sum_{n=0}^{\infty}e_ny^n\ . \tag{3.7c}$$

For $n=0,1,2$, the d_n's and e_n's are given below:

$$\begin{aligned}
d_0&=-0.2485+0.2007\xi\ ,\\
d_1&=0.1858-8.545\times10^{-2}\xi\ ,\\
d_2&=-4.219\times10^{-2}-2.190\times10^{-2}\xi\ ,\\
e_0&=0.2075-0.2210\xi+3.531\times10^{-4}\xi^2\ ,\\
e_1&=-0.1044+7.199\times10^{-2}\xi+7.300\times10^{-3}\xi^2\ ,\\
e_2&=-1.950\times10^{-2}+4.502\times10^{-2}\xi-2.438\times10^{-3}\xi^2.
\end{aligned} \tag{3.8}$$

Note that in this case, the $n=1$ terms in the $Z(r,\sigma)$ function are not included, for otherwise an artificial infrared divergence will appear when y approaches 0. Therefore,

$$\frac{\hbar}{2a\Omega}(A+B\alpha+C\alpha^2)$$
$$=\frac{\hbar}{2a\Omega}(a\{m^2+(\lambda/2)\hat{\phi}^2+[(1-\xi)/6]R\}^{1/2}$$
$$+A'+B'\alpha+C'\alpha^2)\ . \tag{3.9}$$

IV. SYMMETRY BEHAVIOR

We now make use of the full expression for the effective potential $V(\hat{\phi})$ to study the spontaneous symmetry breaking of the system. We are particularly interested in how deformation in the form of spatial curvature anisotropy in the background spacetime affects the symmetry of the scalar field system. For simplicity we shall treat the massless field in detail. In this case the curvature effect will be apparent not only on the classical, but also on the quantum level. Generalization to the massive case should be straightforward.

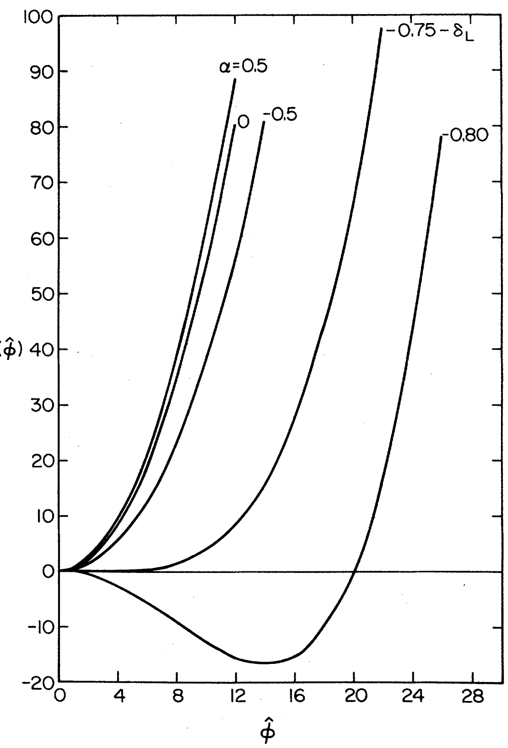

FIG. 1. One-loop effective potential of a massless, conformally coupled scalar field in a static Taub universe with varying deformation parameters (α). The value $\alpha=0$ is the Einstein universe. Symmetry breaking will occur when $\alpha<-0.75-\delta_L$, $\delta_L=6.302\times10^{-6}\lambda$. Here we set $a=1$, $\lambda=10^{-2}$, $\hbar=1$. For all values of $\hat{\phi}_0$ such that $|\hat{\phi}_0a|<4.2\times10^3$ the quantum correction to the ϕ^4 term is negligibly small compared to the classical term $\lambda\phi^4/4!$.

In the small curvature (or large curvature radius) limit corresponding to low-energy or late-time regimes, the effective potential (3.5) reduces to the familiar Coleman-Weinberg form and we recover their result that no symmetry breaking can be induced by one-loop quantum corrections in a $\lambda\phi^4$ theory in flat space. In the large-curvature limit corresponding to conditions prevailing in the early universe under study, the behavior of conformal and minimal fields are quite different. We discuss these cases separately as follows:

(1) Massless Conformal Field. In this case

$$m=0,\quad \xi=0,\quad \sigma=\frac{\lambda}{2}\hat{\phi}^2 a^2 \ll 1 \ . \tag{4.1}$$

Using (3.6) and (2.28), we get

$$\begin{aligned} V(\hat{\phi}) = {} & \frac{R}{12}\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4+\frac{\hbar}{4\pi^2 a^4}\left[\frac{1}{120}+\frac{\alpha}{180}+\alpha^2\left(\tfrac{47}{5400}-\tfrac{2}{45}\gamma\right)\right]+\frac{\lambda\hat{\phi}^2}{8\pi^2 a^2}\hbar\left[-\frac{1}{24}-\frac{\alpha}{72}+\frac{\alpha^2}{15}\left[-\tfrac{1}{3}+\zeta(3)\right]\right] \\ & -\frac{\lambda^2\hat{\phi}^4}{256\pi^2}\left[\ln\frac{\lambda\hat{\phi}_0{}^2 a^2}{8}+\frac{14}{3}-2\gamma+\frac{\alpha}{3}+\alpha^2\left[-\tfrac{7}{30}+\tfrac{2}{3}\zeta(3)-\tfrac{4}{3}\zeta(5)\right]\right]+(\text{gravitational parts}) \ , \end{aligned} \tag{4.2}$$

where "gravitational part" denotes terms proportional to Λ, R, R^2, C^2.

For conformal coupling, the scalar curvature R in the $\frac{1}{12}R\hat{\phi}^2$ term acts like an effective mass which makes symmetry restoration or breaking possible even on the classical level. Recall that

$$R=\frac{6}{a^2}\frac{1+\frac{4}{3}\alpha}{1+\alpha}=6\left[\frac{2\pi^2}{\Omega}\right]^{2/3}\frac{1+\frac{4}{3}\alpha}{(1+\alpha)^{4/3}} \ .$$

For $\alpha > -\frac{3}{4}$, $R>0$, $\hat{\phi}=0$ is the only local minimum. [The form of $V(\phi)$ is sketched in Fig. 1.] Hence no symmetry breaking is possible. Had the system been in the broken-symmetry state, then increasing oblate deformations ($\alpha>0$) can induce symmetry restoration on the classical level. For the spherical Einstein universe ($\alpha=0$), $R=6/a^2$,

$$V(\hat{\phi})=\frac{\hat{\phi}^2}{a^2}\left[\frac{1}{2}-\frac{\lambda\hbar}{192\pi^2}\right]+\frac{\lambda\hat{\phi}^4}{4!}\left[1-\frac{3\lambda\hbar}{32\pi^2}\left[\ln\frac{\lambda a^2\hat{\phi}_0{}^2}{8}+\tfrac{14}{3}-2\gamma\right]\right] \ . \tag{4.3}$$

For the ordinary range of $\lambda \le 10^{-2}$, no symmetry breaking can happen even with quantum corrections. This has been noted in our earlier paper.[1] To determine whether by decreasing $\alpha<0$ in a prolate deformation symmetry can be broken, we look for conditions wherein $V(\hat{\phi})$ is a local minimum at a value of $\hat{\phi}\neq 0$ by computing $V'(\hat{\phi})\equiv dV(\hat{\phi})/d\hat{\phi}$,

$$\begin{aligned} V'(\hat{\phi}) = {} & \frac{\hat{\phi}}{a^2}\left[\frac{1+\frac{4}{3}\alpha}{1+\alpha}-\frac{\lambda\hbar}{4\pi^2}\left[\frac{1}{24}+\frac{\alpha}{72}+\frac{\alpha^2}{15}\left[\tfrac{1}{3}-\zeta(3)\right]\right]\right]+\frac{\lambda\hat{\phi}^3}{3!} \\ & \times\left[1-\frac{3\lambda\hbar}{32\pi^2}\left[\ln\frac{\lambda\hat{\phi}_0{}^2 a^2}{8}+\tfrac{14}{3}-2\gamma+\frac{\alpha}{3}+\alpha^2\left[-\tfrac{7}{30}+\tfrac{2}{3}\zeta(3)-\tfrac{4}{3}\zeta(5)\right]\right]\right] \ . \end{aligned} \tag{4.4}$$

Here $\hat{\phi}=0$ is obviously a solution. A nonzero $\hat{\phi}\neq 0$ solution exists whereby $V'(\hat{\phi})=0$ if and only if

$$-\frac{1}{a^2}\frac{1+\frac{4}{3}\alpha}{1+\alpha}+\frac{\lambda\hbar}{4\pi^2 a^2}\left[\frac{1}{24}+\frac{\alpha}{72}+\frac{\alpha^2}{15}\left[\tfrac{1}{3}-\zeta(3)\right]\right]>0 \ . \tag{4.5}$$

Since $\lambda \ll 1$, an approximate solution can be found

$$\hat{\phi}^2=\frac{6}{\lambda a^2}\left[-\left[\frac{1+\frac{4}{3}\alpha}{1+\alpha}\right]+\frac{\lambda\hbar}{96\pi^2}\left[1+\frac{\alpha}{3}+\alpha^2\left[\tfrac{8}{15}-\tfrac{8}{5}\zeta(3)\right]\right]\right] \ . \tag{4.6}$$

The condition for this to be true is

$$\alpha<\alpha_L=-\tfrac{3}{4}-\delta_L \ ,$$

where (4.7)

$$\delta_L=(6.302\times 10^{-6})\lambda \ .$$

This phase transition (from the $\hat{\phi}=0$ symmetric state to the $\hat{\phi}\neq 0$ broken-symmetry state) is of the second order. This is easily checked by calculating $V''(\hat{\phi})$,

$$V''(\hat{\phi})=\frac{1}{a^2}\left[\frac{1+\frac{4}{3}}{1+\alpha}-\frac{\lambda\hbar}{4\pi}\left[\frac{1}{24}+\frac{\alpha}{72}+\frac{\alpha^2}{15}[\tfrac{1}{3}-\zeta(3)]\right]\right]+\frac{\lambda\hat{\phi}^2}{2}$$
$$\times\left[1-\frac{3\lambda\hbar}{32\pi^2}\left[\ln\frac{\lambda\hat{\phi}_0^2a^2}{8}+\frac{14}{3}-2\gamma+\frac{\alpha}{3}+\alpha^2[-\tfrac{7}{30}+\tfrac{2}{3}\zeta(3)-\tfrac{4}{3}\zeta(5)]\right]\right],$$

and noticing that $V''(0)<0$ when $\alpha<-\frac{3}{4}+\lambda(-6.302\times10^{-6})=\alpha_L$. (See Fig. 1.)

As the value of $\alpha<0$ is decreased (increasingly prolate configuration) the quantum effect will sustain the symmetric state until α_L. For $\alpha\ll\alpha_L$, the classical effect of curvature $R<0$ again dominates the symmetry breaking.

The above result is presented under the assumption of constant-length ($a=2l_1=$const) deformation. For a volume-preserving ($\Omega=$const) deformation, the calculation proceeds in like manner. One should be cautioned that at $\alpha=-\frac{3}{4}$ our small α expansion around $\alpha=0$ may not be valid. However, since in both cases the difference δ from the classical critical point is extremely small it is sufficiently safe to conclude that for conformal massless fields, it is primarily the classical effect of curvature which determines the critical point. Only in a very small range near $R\lesssim0$ (or $\alpha\lesssim-\frac{3}{4}$) will quantum effects of deformation annul the classical effect and suppress symmetry breaking. This can happen at any length scale by exerting only deformation. Our result here agrees with the conclusion of Critchley and Dowker.[19]

(2) Massless Minimal Field. In this case

$$m=0,\quad \xi=1,\quad \sigma=-1+y,\quad y=\frac{\lambda}{2}\hat{\phi}^2a^2\ll1 .$$

From (3.5) and (3.9) the effective potential is

$$V(\hat{\phi})=\frac{\lambda}{4!}\hat{\phi}^4+\frac{\hbar}{4\pi^2a^4}\{(A+D\alpha+E\alpha^2)+[(\lambda/2)(1+\alpha)a^2\hat{\phi}^2]^{1/2}\}+\frac{\lambda\hat{\phi}^2R\hbar}{384\pi^2}\left[\ln\frac{R_0a^2}{24}+2\right]$$
$$-\frac{\lambda^2\hat{\phi}^4\hbar}{256\pi^2}\left[\ln\frac{\lambda\hat{\phi}_0^2a^2}{8}+\frac{14}{3}\right]+(\text{gravitational parts})\,, \tag{4.8}$$

where $D=B'+A'/2$, $E=C'+B'/2-A'/8$.

Due to the absence of the classical curvature effect, the only cause of symmetry breaking is of quantum origin. From (3.7) and (3.8) with $\xi=1$ we can write

$$A+D\alpha+E\alpha^2=\sum_{n=0}^{\infty}(a_n+f_n\alpha+g_n\alpha)(\tfrac{1}{2}\lambda\hat{\phi}^2a^2)^n\,, \tag{4.9}$$

where

$$f_n=d_n+\frac{a_n}{2},\quad g_n=e_n+\frac{d_n}{2}-\frac{a_n}{8}$$

and

$$\begin{aligned}&a_0=-0.4115,\quad a_1=-0.3522,\quad a_2=0.003\,178\,,\\&d_0=-0.0478,\quad d_1=0.1004,\quad d_2=-0.064\,09\,,\\&e_0=-0.013\,15,\quad e_1=-0.025\,19,\quad e_2=0.023\,08\,.\end{aligned} \tag{4.10}$$

Combining this with terms of the same order in $\lambda\hat{\phi}^2$, we get from (4.8)

$$V(\hat{\phi})=\frac{\lambda}{4!}\hat{\phi}^4+\frac{\hbar}{4\pi^2a^3}[(\lambda/2)(1+\alpha)\hat{\phi}^2]^{1/2}+\frac{\hbar}{4\pi^2a^4}(a_0+f_0\alpha+g_0\alpha^2)$$
$$+\frac{\hbar\lambda\hat{\phi}^2}{8\pi^2a^2}\left[(a_1+f_1\alpha+g_1\alpha^2)+\frac{1+\frac{4}{3}\alpha}{8(1+\alpha)}\left[\ln\frac{R_0a^2}{24}+2\right]\right]$$
$$+\frac{\lambda^2\phi^4\hbar}{16\pi^2}\left[a_2+f_2\alpha+g_2\alpha^2-\tfrac{1}{16}\ln\frac{\lambda\hat{\phi}_0^2a^2}{8}-\frac{7}{24}\right]+O((\lambda\hat{\phi}^2)^3)\,. \tag{4.11}$$

The appearance of the term proportional to $(\hat{\phi}^2)^{1/2}$ dictates the rather unusual symmetry behavior of the minimal field. Note that $V(\hat{\phi})$ is nonanalytic at $\hat{\phi}=0$, but is nevertheless a local minimum. Unlike the case of the conformal field with $R>0$, the only admissible phase transition will be of first order. To see if another minimum exists, we look for values of $\hat{\phi}_{\min}\neq0$ at which $V(\hat{\phi}_{\min})$ is a second minimum, i.e.,

$$V'(\hat{\phi}_{\min})=0 , \tag{4.12a}$$

$$V''(\hat{\phi}_{\min})>0 , \tag{4.12b}$$

$$V(\hat{\phi}_{\min})=V(0) , \tag{4.12c}$$

so that $\hat{\phi}_{\min}$ is a critical point. If at the second minima, $V(\hat{\phi}_{\min})\leq V(0)$, the system will undergo symmetry breaking and assume the new state of global minimum as ground state. If $\hat{\phi}_{\min}$ exists, it would provide a natural scale for setting the renormalization parameters $\hat{\phi}_0$. Setting $\hat{\phi}_0=\hat{\phi}_{\min}$ in (4.12c) gives

$$V(\hat{\phi}_0)=\frac{\sqrt{\lambda/2}}{4\pi^2a^4}\left[\hbar\sqrt{1+\alpha}\hat{\phi}_0 a+\left[\frac{\lambda}{2}\right]^{1/2}\hat{\phi}_0^{\,2}a^2X+\left[\frac{\lambda}{2}\right]^{1/2}\frac{\pi^2}{3}\hat{\phi}_0^{\,4}a^4Y\right]=0 , \tag{4.13}$$

where

$$X=\hbar\left[a_1+\tfrac{1}{4}+\alpha(f_1+\tfrac{1}{12})+\alpha^2(g_1-\tfrac{1}{12})+\frac{3+4\alpha}{24(1+\alpha)}\ln\frac{R_0a^2}{24}\right] , \tag{4.13a}$$

$$Y=1+\frac{3\lambda\hbar}{2\pi^2}\left[a_2+f_2\alpha+g_2\alpha^2-\tfrac{7}{24}-\tfrac{1}{16}\ln\frac{\lambda\hat{\phi}_0^{\,2}a^2}{8}\right] , \tag{4.13b}$$

while $V'(\hat{\phi}_0)=0$ gives

$$V'(\hat{\phi}_0)=\frac{\sqrt{\lambda/2}}{4\pi^2a^4\hat{\phi}_0}\left[\hbar\sqrt{1+\alpha}\hat{\phi}_0 a+2\left[\frac{\lambda}{2}\right]^{1/2}\hat{\phi}_0^{\,2}a^2X+\left[\frac{\lambda}{2}\right]^{1/2}\frac{4\pi^2}{3}\hat{\phi}_0^{\,4}a^4Y\right]=0 . \tag{4.14}$$

Solving (4.13) and (4.14) we obtain two equations for $\hat{\phi}_0$ and X:

$$a_c\hat{\phi}_0=\left[\frac{3}{\pi^2Y}\right]^{1/3}\left[\frac{1+\alpha}{2\lambda}\right]^{1/6} , \tag{4.15a}$$

$$X=-\tfrac{3}{2}\left[\frac{4\pi^2}{3}\frac{1+\alpha}{\lambda}Y\right]^{1/3} . \tag{4.15b}$$

For any given R_0 and α (constant-length deformation), these equations determine a_c and $\hat{\phi}_0$, where $V(\hat{\phi}_0)$ has a global minimum. Alternatively, if the values of $\hat{\phi}_0$ and the renormalization point R_0 for the coupling constant ξ are set at the characteristic scale of any realistic model of particle physics like GU, then (4.15) determines the corresponding values of a and α. Using (4.15b) an approximate solution for a_c can be obtained if the term proportional to λ in Y is neglected (to the zeroth order in λ):

$$\ln\frac{R_0a_c^{\,2}}{24}=-\frac{24(1+\alpha)}{3+4\alpha}[a_1+\tfrac{1}{4}+(f_1+\tfrac{1}{12})\alpha+(g_1-\tfrac{1}{12})\alpha^2]-\frac{36(1+\alpha)^{4/3}}{3+4\alpha}\left[\frac{4\pi^2}{3\lambda}\right]^{1/3} . \tag{4.16}$$

For small α, using the small-α expansion for the values of a_1,f_1,g_1 from (4.10) and setting $\lambda\simeq10^{-2}$, one gets approximately

$$\ln\frac{R_0a_c^{\,2}}{24}=-130.7(1+2.522\times10^{-3}\alpha+0.2215\alpha^2) . \tag{4.17}$$

In order for (4.17) to be satisfied, $R_0a_c^{\,2}$ must be a very small number. Note that except for $-1.138\times10^{-2}<\alpha<0$, increasing $|\alpha|$ will reduce a_c even further. This means squashing the Einstein universe tends to restore the symmetry. To see the relation between a_c, R_0, and $\hat{\phi}_0$, let us first consider the case of the spherical Einstein universe. Setting $\alpha=0$ in (4.17) and (4.15a), we have

$$R_0a_c^{02}=4.206\times10^{-56} , \tag{4.18a}$$

$$\hat{\phi}_0a_c^0=1.2905 , \tag{4.18b}$$

where a_c^0 is the critical radius for the undeformed space. Taking $\hat{\phi}_0\simeq10^{15}$ GeV characteristic of the grand unification scale, $a_c\simeq2.58\times10^{-29}$ cm, which is close to the prediction of the inflationary model. Starting with a symmetric state at $\hat{\phi}=0$, by decreasing a beyond the critical radius ($a<a_c^0$) a global minimum will appear, signifying spontaneous symmetry breaking. [See Fig. 2(a)]. However, condition (4.18a) for the existence of a second minimum requires that the coupling constant ξ be renormalized at $R_0\simeq66.4$ cm^{-2}, or a radius of $a=0.3$ cm. If ξ is renormalized at a lower energy or later time (closer to flat space), the critical radius will be larger and the vacuum expectation value of $\hat{\phi}$ will be smaller. This may cause symmetry breaking at a lower-energy scale. One can understand the role of R_0 from the renormalization-group point of view:[20]

$$\xi(\mu)=\xi'(\mu')\left[1+\frac{\lambda}{32\pi^2}\ln\left[\frac{\mu}{\mu'}\right]^2\right] . \tag{4.19}$$

Setting $\xi(\mu')=1$ (minimal) at $\mu'^2=R_0$, the value of ξ at $\mu=1/a_c$ is

$$\xi(a_c)=1+\frac{\lambda}{32\pi^2}\ln\frac{1}{a_c^{\,2}R_0}=1+4.04\times10^{-3} . \tag{4.20}$$

Therefore the classical term $[(1-\xi)/12]R\hat{\phi}^2$ acts as a negative (mass)^2 term which is responsible for the symme-

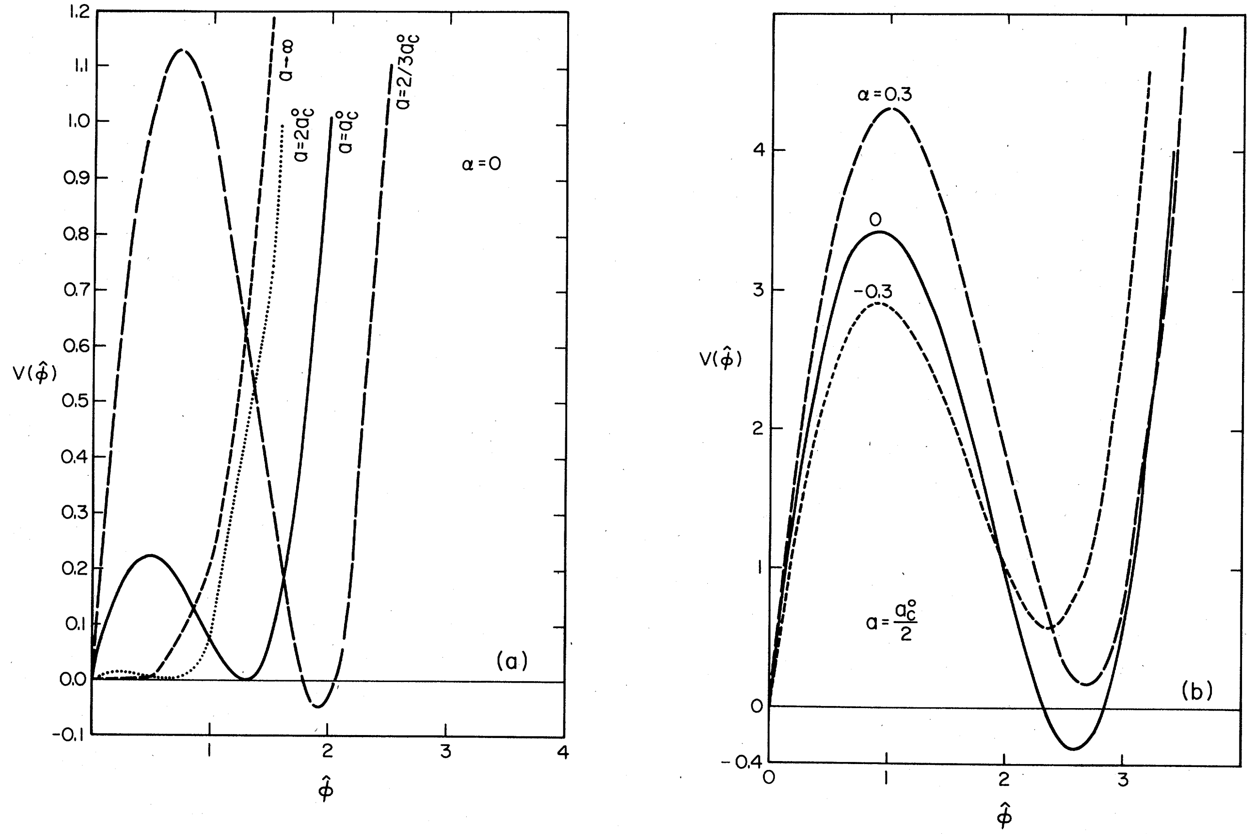

FIG. 2. (a) One-loop effective potential of a massless, minimally coupled scalar field in the Einstein universe with varying curvature radii a. a_c^0 is defined to be the critical radius of the first-order phase transition. When $a < a_c^0$, the broken-symmetry state is energetically preferred. In the limit of $a \to \infty$ the flat-spacetime Coleman-Weinberg $\lambda\phi^4$ potential is recovered. V is plotted in the unit of $\sqrt{\lambda/2}/4\pi^2 a_c^{04}$ with $\lambda=0.01$, $a_c^0=1$, and $\ln(R_0 a_c^{02}/24)=-130.6$ the value of a_c^0 is related to $\hat{\phi}_0$ and R_0 via Eq. (4.18). (b) One-loop effective potential of a massless, minimally coupled scalar field in the Taub universe with varying α. A particular value of $a=a_c^0/2$ is chosen to illustrate the effect of constant-length deformation on the symmetry restoration. All the parameters are as in (a).

try breaking. From this argument one can predict that as α approaches $-\frac{3}{4}$, the curvature scalar vanishes and no spontaneous symmetry breaking can take place. Indeed, Eq. (4.16) gives $a_c=0$ in that limit.

Let us now examine the effect of deformation α on the critical radius a_c. Take $|\alpha|=0.3$ as an example, using (4.17),

$$\text{for } \alpha=0.3 \quad R_0 a_c^{+2}=2.722\times10^{-57}\,, \tag{4.21}$$

$$\text{for } \alpha=-0.3 \quad R_0 a_c^{-2}=3.405\times10^{-57}\,, \tag{4.22}$$

where a_c^+ and a_c^- denote the critical lengths for $\alpha>0$ and $\alpha<0$. Comparing with (4.18a), we see immediately that

$$a_c^+, a_c^- < a_c^0\,, \tag{4.23}$$

which indeed shows the behavior we have stated earlier. Thus, starting with a spherical space in the broken-symmetry state, by deforming it either in the oblate or the prolate direction, one can reduce the critical length and induce symmetry restoration. [See Fig. 2(b).]

These are based on the approximation (4.16) obtained from (4.15) by neglecting the term in Y containing λ. We can estimate the error incurred by using a typical value for $\lambda \simeq 10^{-2}$. Take, for example, the case $\alpha=0$ in (4.18), we get for the contribution of the term in (4.13b)

$$\frac{-3\lambda}{2\pi^2}\left[a_2-\frac{7}{24}-\frac{1}{16}\ln\frac{\lambda\hat{\phi}^2a^2}{8}\right]\simeq-1.48\times10^{-4}\,,$$

which is indeed negligibly small compared to unity.

The above discussion can be repeated in a similar manner for volume-preserving deformations via the relation $\Omega=2\pi^2a^3/\sqrt{1+\alpha}$. A set of solutions exists for $\hat{\phi}'_0$ and R'_0,

$$\hat{\phi}'_0=\left[\frac{6}{Y\Omega}\right]^{1/3}\left[\frac{1}{2\lambda}\right]^{1/6}, \tag{4.24a}$$

$$X'=-3\left[\frac{Y\pi^2}{6\lambda}\right]^{1/3}, \tag{4.24b}$$

where

$$X'=\hbar\left[a_1+\frac{1}{4}-\frac{1}{4}\ln2+(1-\frac{2}{9}\alpha^2)\frac{1}{12}\ln\frac{\Omega}{\Omega_c}\right.$$
$$\left.+(f'_1+\frac{1}{24})\alpha+\alpha^2(g'_1-\frac{5}{48}+\frac{1}{18}\ln2)\right],$$

and

$$f_1' = f_1 - \frac{a_1}{3} \,,$$

$$g_1' = g_1 - \frac{f_1}{3} + \tfrac{2}{9} a_1 \,.$$

For any small α, the critical volume Ω_c can be written as

$$\ln \frac{\Omega_c}{\Omega_0} = (-193.8)(1 + 5.161 \times 10^{-3}\alpha + 0.2191\alpha^2) \,. \quad (4.25)$$

Thus, except for a small range $-2.355 \times 10^{-2} < \alpha < 0$, increasing $|\alpha|$ will again decrease Ω_c and drive the system towards symmetry restoration.

V. DISCUSSIONS

In the above we have presented a rather detailed derivation of the effective potential of a self-interacting scalar field in a homogeneous universe with small spatial curvature anisotropy (Sec. II). We have also discussed at some length symmetry behavior (Sec. IV) of such a system in such a geometry. To complete the discussion we want to compare our methodology and results with some related works and elaborate further on their implication on some current issues.

(1) Comparison with weak-field approximation. Some recent work[21] has attempted to treat symmetry breaking and related problems in curved spacetime by using near-flat-space techniques such as Riemann-normal-coordinate expansion on the metric for the derivation of the effective potential. These approximations, as we have explained here and in our earlier work[1,12] can account only for the local behavior of the theories but not the global. It is thus useful for identifying ultraviolet divergences of the theory (as we have done in Sec. III), but not for investigating the symmetry behavior of the system. One needs to know the infrared behavior of the system. This can be done by analyzing the response of the fundamental mode of the quantum operator of such a system under changes in the order parameter[13] (in our case, e.g., the background field $\hat{\phi}$) or some external parameters (e.g., temperature T). Or, when the exact form is lacking for eigenvalue summation, one should make approximations in energy or length scales corresponding to the infrared limit. The large curvature (or small radius) expansion we used here (Sec. IV) and in earlier work[1] is adopted just for this purpose. For the study of quantum processes in the early universe where the curvature of spacetime is not always small, this usually makes a difference. For example, in Ishikawa's[21] discussion of gravitational effects on symmetry breaking in curved space, he adopted a weak-field approximation for the derivation of the effective potential. His result for the negative-curvature minimal-coupling case does not agree with ours. We suspect that this is related to the inconsistency of his result for high critical curvature to his weak-field assumption. The eigenmode-expansion method used here sees no restriction from the curvature of the background.

(2) Symmetry behavior in the Einstein universe. Symmetry behavior for $\lambda\phi^4$ theory in the Einstein universe, which is a special case of the static Taub universe, has been studied in detail by us[1] and by a number of other authors (see references quoted in Ref. 1), notably, Ford and Toms.[20] In the case of massless minimally coupled fields, Ref. 1 arrives at the critical curvature by examining the condition which leads to the convexity of the effective potential at $\hat{\phi}=0$, thus identifying the critical curvature at which $V''(0)$ changes sign. Reference 20 proceeds by examining the stability of the zero mode of the operator near the classical field $\hat{\phi}=0$. However, these previous results look quite different from our present result, Eq. (4.15) with $\alpha=0$. We will try to address this discrepancy here.

In the limit of $m^2+(1-\xi)R/6 \simeq 0$, Eqs. (30) and (36) of Ref. 1 for the critical curvature based on the criterion $V''(0)=0$ actually has no physical solution, as $V''(0)$ never goes to zero. In fact it has a minimum value greater than zero about $\hat{\phi}=0$. The result we obtained before is not a critical radius, but an approximate solution for the point at which $V''(0)$ takes its minimum value. This means that a second-order phase transition cannot occur. So if the universe is in the region of broken symmetry ($m^2<0$), reducing the size of the universe will gradually bring the asymmetrical ground state closer to $\hat{\phi}=0$ with $V''(0)$ always greater than zero. It is therefore inappropriate to look for a critical point by imposing $V''(0)=0$. Similarly if the universe had started from the symmetric phase [with $m^2<0$ but $m^2+(1-\xi)R/6>0$], as the curvature is reduced, the symmetric phase becomes unstable, but since $V''(0)$ is greater than zero the transition cannot be second order. However, as discussed for the massless case in Sec. IV a global minimum of $V(\hat{\phi})$ does exist at a nonzero value of $\hat{\phi}$ for $a \le a_c$, this will always allow a first-order phase transition to occur.

The physics of this phenomenon is closely related to the occurrence in Eq. (4.11) of the linear term

$$[(\lambda/2)(1+\alpha)\hat{\phi}^2 a^2]^{1/2}/4\pi^2 a^4 \quad (5.1)$$

in Eq. (4.11). Technically speaking, it comes from the first term in the mode sum (2.16),

$$\frac{1}{2\Omega}\{m^2+[(1-\xi)/6]R+(\lambda/2)\hat{\phi}^2\}^{1/2} \quad (5.2)$$

under the condition

$$m^2 + \left[\frac{1-\xi}{6}\right] R \ll \frac{\lambda}{2}\hat{\phi}^2 \,. \quad (5.3)$$

Thus for the massive or nonminimal-coupling case, unless $m^2+[(1-\xi)/6]R \simeq 0$, one can expand (5.2) in powers of $(\lambda/2)\hat{\phi}^2$, and get only the usual $\hat{\phi}^2$ and $\hat{\phi}^4$ terms in the effective potential. The interesting linear dependence in $\hat{\phi}$ does not appear and there is no first-order phase transition. This is why a first-order phase transition occurs in the massless, minimal-coupling case and not the conformal case.

The appearance of this term spoils the analyticity of the effective potential at $\hat{\phi}=0$ and dictates the rather unusual symmetry behavior of the minimal field. Infrared disease at $\hat{\phi}=0$ is commonplace in effective potentials of any massless field. For example, the well-known Coleman-

Weinberg potential is infrared divergent in the fifth derivative with respect to $\hat{\phi}$ (due to the $\hat{\phi}^4\ln\hat{\phi}$ term). The severity of such a problem can be avoided by one derivative order if the renormalization-group equations are used.

The occurrence of (5.2) in the effective potential (4.8) which we believe is generic to universes with topology $R\times S^3$ can be understood more generally by considering the role played by the apparent dimension of the universe. By apparent dimension we mean the dimension obtained when structures of size small compared to the Compton wavelength of the particles or fields defined on the manifold are neglected. Thus to a particle whose Compton wavelength is much smaller than the radius of the three-sphere, the apparent dimension is four, whereas for a particle whose Compton wavelength is larger than the radius of the three-sphere the apparent dimension is one.

If one computes the one-loop correction to the effective potential in one dimension in a fashion similar to Sec. II, one obtains on dividing by the volume of the three-sphere and taking $[(1-\xi)/6]R$ as an effective mass, exactly Eq. (5.2). Thus we see that the origin of the linear term is due to the apparent reduction in the dimension of the spacetime. However, the effective potential is dominated by the term (5.2) rather than terms of $\hat{\phi}^2$ and $\hat{\phi}^4$ orders only when

$$m^2+\frac{1-\xi}{6}R\simeq 0 \text{ and } \hat{\phi}\simeq 0\,. \tag{5.4}$$

It implies that whenever Eq. (5.4) is satisfied, our four-dimensional system can be considered approximately as a one-dimensional one. By the same token the M^3 term in the finite-temperature case of Dolan and Jackiw[22] can be understood as the reduction of $R^3\times S^1$ to R^3, in the infinite-temperature limit.

A theorem of Coleman[23,24] (which is a generalization of an earlier theorem in statistical mechanics by Mermin and Wagner[25] on a lattice to the continuum) states that in dimensions less than or equal to two the infrared divergences are so severe[26] that there is no possibility of spontaneous symmetry breaking for a scalar field; the only vacuum expectation value for $\hat{\phi}$ allowed is zero. This means that as the radius of the universe is decreased, one would naively expect the four-dimensional manifold $(R^1\times S^3)$ is collapsed into a one-dimensional one (R) and Coleman's theorem requires that the symmetry is to be restored. This indeed agrees with our result near $\hat{\phi}=0$. By including the daisy diagram as discussed in Ref. 1, one can obtain a minimum value of $V''(0)$ as $3(\lambda/16\pi^2)^{2/3}$ which is, of course, greater than zero. But when we are away from the region $\hat{\phi}\simeq 0$, the one-dimensional behavior no longer prevails and spontaneous symmetry breaking can take place. This is why we can find the second minimum of the effective potential at $\hat{\phi}_{\min}$. One can check the above reasoning by a careful examination of the length scale which will be used to define the apparent dimension of the universe. The only natural scale in this theory is the mass of the scalar field at $\hat{\phi}_{\min}$ which is governed by the renormalization-group equation. This is given by Eq. (4.14):

$$m^2_{\text{phys}}=\left.\frac{d^2V}{d\hat{\phi}^2}\right|_{\hat{\phi}_{\min}}\simeq -\frac{\lambda X}{2\pi^2a^2}\simeq\frac{\lambda}{96\pi^2}R\ln\frac{R}{R_0} \quad \text{when } a<a_c\,. \tag{5.5}$$

The ratio of the Compton wavelength ($\lambda_{\text{comp}}=2\pi/m_{\text{phys}}$) of the scalar particle to the radius of the universe is

$$\frac{\lambda_{\text{comp}}}{a}=2\pi\left[\frac{2\pi^2}{-\lambda X}\right]^{1/2}\simeq\frac{8\pi^2}{(\lambda\ln R/R_0)^{1/2}},\quad R\gg R_0 \tag{5.6}$$

[where R_0 and a_c are related by Eq. (4.16)], which decreases as the curvature increases. Therefore the apparent size of the universe at the second minimum does increase with increasing curvature. However, we cannot allow R to be too large, otherwise the coupling constant λ which is also governed by the renormalization-group equation will run to a value greater than one at which point our one-loop approximation will break down. At $a=a_c$, from Eq. (4.15) we find $\lambda_{\text{comp}}/a=(4\pi^2/3\lambda)^{1/3}$. Although the ratio is large (at least for sufficiently small λ) the finite-size effect has to be taken into account. Thus the one-dimensional conclusion from Coleman's theorem does not apply in this region and the asymmetrical ground state is not precluded. The complete picture extending from $\hat{\phi}=0$ to $\hat{\phi}=\hat{\phi}_{\min}$ is a combination of one-dimensional and four-dimensional effects. The former generates the barrier while the latter gives the minimum.

In the case of Ref. 20, it was assumed the structure of the potential is dominated by the renormalization-group parts and the effects of finite size contained in the terms $f_{1/2}$ and $f_{-1/2}$ in Eq. (4.3) were dropped which plays an important role in our analysis. Note the minimum at $\hat{\phi}_{\min}$ is, incidentally, not the "fake" minimum discussed by Coleman and Weinberg which can lead to large $\lambda\ln\hat{\phi}/M$ and invalidate the one-loop assumption.[27]

(3) Effect of curvature anisotropy on inflation. We know an inflationary phase can happen if the system stays in a false vacuum for a long time (in the Hubble time scale). There are two ways to achieve this: (a) a barrier between the true vacuum and the false vacuum (old scenario) (b) a slow evolution of the state from the false vacuum to the true vacuum (new scenario). If the phase transition is of the second order neither condition can be satisfied and inflation will not occur. Usually the effective potential used in the new inflationary model is that of the flat-spacetime Coleman-Weinberg form. Now that it has been generalized to curved spacetime, a natural question arises as to whether and how inflation can be affected by spacetime curvature. As we have seen in the case of the conformal field, the symmetric state $\hat{\phi}=0$ will become unstable if the space evolves very slowly (so that the kinetic contribution[12,13] remains insignificant and the effective potential concept can remain useful) from an oblate configuration ($\alpha>0$) to a prolate configuration ($\alpha<0$). A second-order phase transition will occur and inflation to the extent necessary for cosmological purposes becomes highly unlikely. If the field is minimally coupled to the background, and the space is changing its shape and volume in such a way that the two equal axes remain constant (what we called constant-length deformation) then squashing it towards any configuration will restore sym-

metry. If the shape of the universe is kept fixed, but the size is varied, then the collapse of the universe will induce symmetry breaking. As this phase transition is of the first order, it in principle can allow inflation to occur. However, we observe from Fig. 2a that by decreasing the radius of the Einstein universe there arises a high barrier between $\hat{\phi}=0$ and $\hat{\phi}_0$. This effect is, as explained in previous paragraphs, due to the linear term [Eq. (5.1)] which dominates in the effective potential near $\hat{\phi}=0$. This barrier which increases with the curvature will prevent any new inflation[5] to occur (it is assumed that the barrier is unaffected by the dynamics) and bring back the problems of the old scenario.[4] Notice, however, when α is negative (prolate shaped), the height of the barrier can be reduced by a factor $\sqrt{1+\alpha}$. Nonetheless this does not help to generate new inflation. As is discussed below [Eq. (4.20)], if α is less than $-\frac{3}{4}$, we cannot have symmetry breaking at all. The barrier can at best be reduced by half and the broad plateau one needs in new inflation is still lacking. We are currently examining this problem more closely by studying gauge fields and with higher-loop correction.

We have illustrated a few instances where curvature topology and field-coupling effects can alter the flat-space field-theoretical result distinctly. To return to the point brought up in the Introduction, we want to add that the implications on inflation are drawn from the assumption that the Taub metric remains applicable as a faithful description of the universe before the GU epoch. Notice that even in the massless-minimal-field case where the advent of new inflation is ruled out by the presence of the linear term, a phase transition by tunneling is still possible. Once the universe becomes vacuum dominated, it will start expanding exponentially whereby all anisotropy originally present will be wiped out rapidly.[7] Upon entering the inflationary stage one would then need to use the effective potential calculated for the de Sitter universe[10] to discuss the ensuing events. In this work we are interested in discerning the initial conditions which may or may not be conclusive to this transition. We are not equipped to deal with the later stage of development.

Just exactly how the universe evolves from, say, a FRW or Taub or mixmaster spacetime to a de Sitter spacetime is a problem[29] which requires more in-depth analysis in all three aspects—quantum field theoretical, statistical, mechanical, and gravitational (curved spacetime), a problem which we hope to delve into in our future work.

(4) Dynamical and finite-temperature effects. As our calculation is based upon a scalar theory in a static spacetime our discussion of curvature effect on inflation can only be indicative. More realistically, one needs to consider the dynamic (shear) and finite-temperature effects as well. For completeness we shall make a few simple observations on these issues for now and leave the detailed discussion for a later communication.[14] A simple model to see the effect of expansion anisotropy is the Kasner universe, where the shear is defined as Q and where the a_i and a_i' are the principal radii of curvature and their conformal time derivatives. The wave equation for each mode k can be reduced to $\ddot{\chi}+(\Omega^2+Q)\chi=0$, where $\chi=a\phi$ and $\Omega^2=k^2+(m^2+\frac{1}{2}\lambda\hat{\phi}^2)a^2$ is the natural frequency.[30] From this, it is apparent that shear can indeed be viewed as contributing to an effective (mass).[2] Therefore even on the classical level a sufficiently large degree of shear can restore the symmetry. The possibility of symmetry restoration by shear was noticed by us earlier[12] and is discussed recently by Futamase.[31]

As for the finite-temperature correction, one can examine its effect by using the result for the Einstein universe,[30] as it should yield the dominant contribution. The temperature contribution to the effective mass is $(\lambda/48)T^2\hat{\phi}^2$. Adding these factors into our consideration, we observe that even in a highly deformed prolate configuration (with negative R) symmetry will remain unbroken if the shear and/or temperature is sufficiently high. If these additional contributions to the effective mass combine with curvature in such a way that our zero-mass analysis remains valid, we would still expect the behavior of the linear $(\hat{\phi}^2)^{1/2}$ term in $V(\hat{\phi})$ for the minimal case to dominate near the origin and lead to a first-order phase transition. This is a possible candidate for preventing new inflation. These problems are currently under investigation.

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APPENDIX A: GEOMETRICAL QUANTITIES IN A DIAGONAL MIXMASTER UNIVERSE

By defining $\omega^0=dt$, and $\omega^a=l_a\sigma^a$, the diagonal mixmaster metric (2.1) can be written in a Minkowski form $ds^2=\eta_{ab}\omega^a\omega^b$, where $\eta_{ab}=(-1,1,1,1)$. From the defining equations for the connection forms $d\omega^a+\omega^a{}_b\Lambda\omega^b=0$ the connection coefficients $\omega^a{}_b$ are easily calculated to be

$$\omega^0{}_a=k_a\omega^a,\quad \text{where}\quad k_a=\dot{l}_a/l_a\ ,$$
$$\omega^1{}_2=c_3\omega^3,\quad \text{where}\quad c_1=\tfrac{1}{2}(d_2+d_3-d_1)\ , \tag{A1}$$

etc., and

$$d_1=l_1/l_2l_3\ .$$

All other $\omega^a{}_b$ can be obtained by cyclically permuting the indices. From the curvature form

$$\begin{aligned}\mathscr{R}^a{}_b&=d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b\\&=R^a{}_{b|cd|}\omega^c\wedge\omega^d,\quad c<d\end{aligned} \tag{A2}$$

one can obtain the Riemann tensor components in the tetrad basis as

$$\begin{aligned}R^0{}_{101}&=\dot{k}_1+k_1{}^2=\ddot{l}_1/l_1\ ,\\R^0{}_{123}&=k_1d_1-k_2c_3-k_3c_2\ ,\\R^1{}_{203}&=\dot{c}_3+c_3k_3\ ,\\R^1{}_{212}&=k_1k_2+c_3d_3-c_1c_2\ ,\end{aligned} \tag{A3}$$

etc. The Ricci tensor components are

$$R_{00}=-\left[\frac{\ddot{l}_1}{l_1}+\frac{\ddot{l}_2}{l_2}+\frac{\ddot{l}_3}{l_3}\right],$$
$$R_{11}=\frac{\ddot{l}_1}{l_1}+(k_1k_2+c_3d_3-c_1c_2)+(k_1k_3+c_2d_2-c_1c_3)\,, \tag{A4}$$

etc., and the four-dimensional curvature scalar R is

$$R=R^a{}_a=2\left[\frac{\ddot{l}_1}{l_1}+\frac{\ddot{l}_2}{l_2}+\frac{\ddot{l}_3}{l_3}\right]+2(k_1k_2+k_1k_3+k_2k_3+c_1d_1+c_2d_2+c_3d_3-c_1c_2-c_1c_3-c_2c_3)\,. \tag{A5}$$

For the Taub universe, $l_1=l_2\neq l_3$. Define a deformation parameter to be

$$\alpha\equiv\frac{l_1^2}{l_3^2}-1\quad(-1<\alpha<\infty)\,.$$

For the static case under study, all $\dot{l}_a$ and $\ddot{l}_a=0$ in the above equations. From (A3)

$$R^1{}_{212}=\frac{1}{l_1^2}-\frac{3}{4}\frac{l_3^2}{l_1^4}\,,$$
$$R^2{}_{323}=R^3{}_{131}=\frac{1}{4}\frac{l_3^2}{l_1^4}\,. \tag{A6}$$

Thus,

$$\begin{aligned}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}&=4(R_{1212}R^{1212}+R_{2323}R^{2323}+R_{1313}R^{1313})\\&=\frac{3+8\alpha+16\alpha^2}{4l_1^4(1+\alpha)^2}\\&=\frac{4}{l_1^4}\left[\frac{3}{16}+\frac{\alpha}{8}+\frac{9}{16}\alpha^2\right]+O(\alpha^3)\,.\end{aligned} \tag{A7}$$

From (A4)

$$R_{11}=R_{22}=\frac{1}{2l_1^4l_3^2}[l_1^4-(l_1^2-l_3^2)]=\frac{1}{l_1^2}\left[1-\frac{1}{2}\frac{l_3^2}{l_1^2}\right],$$
$$R_{33}=\frac{l_3^2}{2l_1^4}\,. \tag{A8}$$

Thus,

$$R_{\alpha\beta}R^{\alpha\beta}=R_{11}{}^2+R_{22}{}^2+R_{33}{}^2=\frac{3+8\alpha+8\alpha^2}{4l_1^4(1+\alpha)^2}=\frac{1}{l_1^4}\left[\frac{3}{4}+\frac{\alpha}{2}+\frac{\alpha^2}{4}\right]+O(\alpha^3)\,. \tag{A9}$$

From (A5)

$$R=R_{11}+R_{22}+R_{33}=\frac{3+4\alpha}{2l_1^2(1+\alpha)}=\frac{1}{2l_1^2}(3+\alpha-\alpha^2)+O(\alpha^3)\,. \tag{A10}$$

Thus,

$$R^2=\frac{1}{l_1^4}(\tfrac{9}{4}+\tfrac{3}{2}\alpha-\tfrac{5}{4}\alpha^2)+O(\alpha^3)\,. \tag{A11}$$

The Weyl-tensor-squared term C^2 is given by

$$\begin{aligned}C^2\equiv C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}&=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-2R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{3}R^2=\tfrac{2}{3}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-R_{\alpha\beta}R^{\alpha\beta})=2\left[R_{\alpha\beta}R^{\alpha\beta}-\frac{R^2}{3}\right]\\&=\frac{4}{3}\frac{1}{l_1^4}\frac{\alpha^2}{(1+\alpha)^2}=\frac{4\alpha^2}{3l_1^4}+O(\alpha^3)\,.\end{aligned} \tag{A12}$$

One can check that the Gauss-Bonnet quantity vanishes,

$$G=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-4R_{\alpha\beta}R^{\alpha\beta}+R^2=0\,. \tag{A13}$$

Being a topological invariance, this is true for the whole Bianchi type-IX class.

APPENDIX B: DEFINITION AND EVALUATION OF THE Z FUNCTION

Let

$$Z(r,\sigma)=\sum_{n=1}^{\infty}\frac{n^2}{(n^2+\sigma)^r}\,. \tag{B1}$$

This series is similar to the one examined by Toms.[32] Following the same procedure, we find that $Z(r,\sigma)$ have poles at $r=\frac{3}{2}-n$. We denote the singular part by $z_{-1}(r,\sigma)$ and the regular part by $z_0(r,\sigma)$. Employing the Plana sum formula, $z(r,\sigma)$ can be written as

$$\sum_{n=1}^{\infty}\frac{n^2}{(n^2+\sigma)^r}=\int_0^\infty\frac{t^2}{(t^2+\sigma)^r}dt-\int_0^1 dt\frac{t^2}{(t^2+\sigma)^r}+\frac{1}{2(1+\sigma)^r}+I_r(\sigma)\ ,$$

where (B2)

$$I_r(\sigma)=i\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left[\frac{(1+it)^2}{[(1+it)^2+\sigma]^r}-\frac{(1-it)^2}{[(1-it)^2+\sigma]^r}\right].$$

The integral

$$\int_0^\infty\frac{t^2}{(t^2+\sigma)^r}dt=\frac{\sigma^{3/2-r}}{4}\sqrt{\pi}\frac{\Gamma(r-\frac{3}{2})}{\Gamma(r)}$$

vanishes at negative integer r (like the ζ function), but diverges at $r=\frac{3}{2}-n$, where $n=0,1,2,\ldots$. This is the only divergent part on the right-hand side of Eq. (B2). One can expand the singular term in a Laurent series about the pole at $r=\frac{3}{2}-n$ as

$$\int_0^\infty\frac{t^2}{(t^2+\sigma)^r}dt=\lim_{\nu\to 0}\frac{\sqrt{\pi}}{4}\frac{\sigma^n}{\Gamma(3/2-n)}\frac{(-1)^n}{n!}\left[\frac{1}{\nu}+\psi(n+1)-\psi(\tfrac{3}{2}-n)-\ln\sigma\right]+O(\nu)\ ,$$

where (B3)

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)\ .$$

Therefore,

$$Z_{-1}(n,\sigma)=\frac{\sqrt{\pi}}{4}\frac{\sigma^n}{\Gamma(\frac{3}{2}-n)}\frac{(-1)^n}{n!}\ ,$$

$$Z_0(n,\sigma)=\frac{\sqrt{\pi}}{4}\frac{\sigma^n}{\Gamma(\frac{3}{2}-n)}\frac{(-1)^n}{n!}[\psi(n+1)-\psi(\tfrac{3}{2}-n)-\ln\sigma]-\int_0^1\frac{t^2}{(t^2+\sigma)^{3/2-n}}dt+\frac{1}{2(1+\sigma)^{3/2-n}}+I_n(\sigma)\ . \tag{B4}$$

For $n=0,1,2$, we obtain

$$Z_{-1}(0,\sigma)=\tfrac{1}{2}\ , \tag{B5}$$

$$Z_{-1}(1,\sigma)=-\frac{\sigma}{4}\ , \tag{B6}$$

$$Z_{-1}(2,\sigma)=-\frac{\sigma^2}{16}\ . \tag{B7}$$

The expression for the Z_0's are more involved, because of the presence of I_n,

$$Z_0(0,\sigma)=-1-\ln\frac{1+\sqrt{1+\sigma}}{2}+\frac{1}{\sqrt{1+\sigma}}+\frac{1}{2}\frac{1}{(1+\sigma)^{3/2}}+I_2(\sigma)\ , \tag{B8}$$

$$Z_0(1,\sigma)=-\frac{\sigma}{4}+\frac{\sigma}{2}\ln\frac{1+\sqrt{1+\sigma}}{2}-\frac{\sigma}{2\sqrt{1+\sigma}}+I_1(\sigma)\ , \tag{B9}$$

$$Z_0(2,\sigma)=\frac{\sigma^2}{32}+\frac{\sigma^2}{8}\ln\frac{1+\sqrt{1+\sigma}}{2}-\tfrac{1}{8}\sigma\sqrt{1+\sigma}+\tfrac{1}{4}\sqrt{1+\sigma}+I_0(\sigma)\ . \tag{B10}$$

Although an analytic result for $I_r(\sigma)$, is not accessible we may seek the asymptotic expansions for particular ranges of σ. (See Appendix C.) For $\sigma\gg 1$,

$$Z_0(0,\sigma)=-1-\tfrac{1}{2}\ln\frac{\sigma}{4}\ , \tag{B11}$$

$$Z_0(1,\sigma)=-\frac{\sigma}{4}\left[1-\ln\frac{\sigma}{4}\right], \tag{B12}$$

$$Z_0(2,\sigma)=\frac{\sigma^2}{16}\left[\tfrac{1}{2}+\ln\frac{\sigma}{4}\right]. \tag{B13}$$

For $\sigma\simeq 0$,

$$Z_0(0,\sigma)=\gamma-\tfrac{3}{2}\zeta(3)\sigma+\tfrac{15}{8}\zeta(5)\sigma^2+O(\sigma^3)\ , \tag{B14}$$

$$Z_0(1,\sigma)=-\tfrac{1}{12}-\frac{\sigma}{2}(1+\gamma)+\sigma^2\tfrac{3}{8}\zeta(3)+O(\sigma^3)\ , \tag{B15}$$

$$Z_0(2,\sigma)=\tfrac{1}{120}-\frac{\sigma}{24}-\gamma\frac{\sigma^2}{8}+O(\sigma^3)\ . \tag{B16}$$

For σ close to -1, define $y=\sigma+1$. When $y\simeq 0$, we can expand Z' in orders of y (prime means the $n=1$ term is excluded),

$$Z'(r,\sigma)=\sum_{n=2}^{\infty}\frac{n^2}{(n^2-1+y)^r}=D(r)-ryD(r+1)+\frac{r(r+1)}{2}D(r+2)y^2+\cdots\ ,$$

where (B17)

$$D(r)=\sum_{n=2}^{\infty}\frac{n^2}{(n^2-1)^r}\ .$$

The $D(r)$ function can be evaluated similar to the $\sigma\simeq 0$ case (replace r by n through $r=\frac{3}{2}-n$) and the result is

$$Z_0'(2,\sigma)\equiv\sum_{n=0}^{\infty}a_ny^n=-0.4115-0.3522y+3.1781\times10^{-3}y^2+O(y^3)\ , \tag{B18}$$

$$Z_0'(1,\sigma)\equiv\sum_{n=0}^{\infty}b_ny^n=-0.7045+0.01271y+0.1314y^2+O(y^3)\ , \tag{B19}$$

$$Z_0'(0,\sigma)=\sum_{n=0}^{\infty}c_ny^n=-0.02542-0.5256y+0.1755y^2+O(y^3)\ . \tag{B20}$$

The values of $D(-1)$ and $D(-2)$ used here are evaluated by numerical method:

$$D(-1)=0.3504\ , \tag{B21}$$

$$D(-2)=9.361\times10^{-2}\ . \tag{B22}$$

In summary

$$\begin{aligned}&a_0=-0.4115,\ \ a_1=-0.3522,\ \ a_2=3.178\times10^{-3}\ ,\\ &b_0=-0.7045,\ \ b_1=0.01271,\ \ b_2=0.1314\ ,\\ &c_0=-0.02542,\ \ c_1=-0.5256,\ \ c_2=0.1755\ .\end{aligned} \tag{B23}$$

APPENDIX C: EVALUATION OF I_n AND Z_n AND THEIR ASYMPTOTIC EXPANSIONS

$$I_n(\sigma)=i\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}\left[\frac{(1+it)^2}{[(1+it)^2+\sigma]^{3/2-n}}-\frac{(1-it)^2}{[(1-it)^2+\sigma]^{3/2-n}}\right]. \tag{C1}$$

(I) Large-σ limit.

$$[(1+it)^2+\sigma]^{-3/2+n}=\frac{1}{\Gamma(\frac{3}{2}-n)}\int_0^{\infty}e^{-[(1+it)^2+\sigma]x}x^{1/2-n}dx\ , \tag{C2}$$

$$\begin{aligned}(1+it)^2[(1+it)^2+\sigma]^{n-3/2}&=\int_0^{\infty}\frac{(1+it)^2}{\Gamma(\frac{3}{2}-n)}e^{-\sigma x}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(1+it)^{2k}x^{1/2-n+k}dx\\ &=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\frac{3}{2}-n)}\frac{\Gamma(\frac{3}{2}-n+k)}{\sigma^{3/2-n+k}}(1+it)^{2+2k}\ .\end{aligned} \tag{C3}$$

Therefore,

$$I_n(\sigma)=i\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\frac{3}{2}-n)}=-\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{2k+1}{2(2k+3)}\frac{\Gamma(\frac{3}{2}-n+k)}{\Gamma(\frac{3}{2}-n)}\sigma^{n-k-3/2}\ , \tag{C4}$$

where Eq. (3.416.1) of Ref. 33 has been used. The other terms in Z_n can also be expanded in powers of σ^{-1}. Gathering terms together, we get

$$Z_0(2,\sigma)=\frac{\sigma^2}{32}+\frac{\sigma^2}{8}\ln\frac{1+\sqrt{1+\sigma}}{2}-\tfrac{1}{8}\sigma\sqrt{1+\sigma}+\tfrac{1}{4}\sqrt{1+\sigma}+I_2(\sigma)=\frac{\sigma^2}{32}+\frac{\sigma^2}{16}\ln\frac{\sigma}{4}\ , \tag{C5}$$

$$Z_0(1,\sigma)=-\frac{\sigma}{4}+\frac{\sigma}{2}\ln\frac{1+\sqrt{1+\sigma}}{2}-\frac{\sigma}{2\sqrt{1+\sigma}}+I_1(\sigma)=-\frac{\sigma}{4}+\frac{\sigma}{4}\ln\frac{\sigma}{4}\ , \tag{C6}$$

$$Z_0(0,\sigma)=-1-\ln\frac{1+\sqrt{1+\sigma}}{2}+\frac{1}{\sqrt{1+\sigma}}+\frac{1}{2}\frac{1}{(1+\sigma)^{3/2}}+I_0(\sigma)=-1-\tfrac{1}{2}\ln\frac{\sigma}{4}\ . \tag{C7}$$

These are the equations given in (B11) and (B13).

(II) Small- $|\sigma|$ *limit.* The evaluations of I_n will be similar to part (I), but the expansion in (C2) is modified,

$$(1+it)^2[(1+it)^2+\sigma]^{n-3/2}=\int_0^\infty dx\frac{(1+it)^2}{\Gamma(3/2-n)}x^{1/2-n}e^{-(1+it)^2x}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sigma^k x^k$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\sigma^k}{\Gamma(\frac{3}{2}-n)}\Gamma(\tfrac{3}{2}-n+k)(1+it)^{-1+2n-2k}\ , \tag{C8}$$

$$I_n=i\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\frac{3}{2}-n+k)}{\Gamma(\frac{3}{2}-n)}\sigma^k\int_0^\infty\frac{dt}{e^{2\pi t}-1}[(1+it)^{2n-2k-1}-(1-it)^{2n-2k-1}]\ . \tag{C9}$$

The behaviors of $2n=2k-1>0$ or <0 is quite different, and have to be treated differently. Instead of presenting the most general cases, we consider here only I_0, I_1, I_2 which are the ones used in the text.

$$I_2=i\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\sigma^k\int_0^\infty\frac{dt}{e^{2\pi t}-1}[(1+it)^{3-2k}-(1-it)^{3-2k}]$$

$$=i\left[L(3)+\frac{\sigma}{2}L(1)-\frac{\sigma^2}{8}L(-1)+\sum_{k=3}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\sigma^k L(3-2k)\right]\ , \tag{C10}$$

where

$$L(v)\equiv\int_0^\infty\frac{dt}{e^{2\pi t}-1}[(1+it)^{\nu}-(1-it)^{\nu}]\ .$$

For $v>1$, one can integrate $L(-v)$ as follows,

$$L(-v)=\int_0^\infty\frac{dt}{e^{2\pi t}-1}\frac{1}{\Gamma(\nu)}\int_0^\infty[e^{-(1+it)x}-e^{-(1-it)x}]x^{\nu-1}dx$$

$$=\int_0^\infty dx\frac{x^{\nu-1}}{2\pi}\frac{e^{-x}}{\Gamma(\nu)}\left[\psi\left[1-\frac{ix}{2\pi}\right]-\psi\left[1+\frac{ix}{2\pi}\right]\right]$$

$$=\frac{1}{2\pi\Gamma(\nu)}\int_0^\infty dx\,x^{\nu-1}e^{-x}(-i\pi)\left[\coth\frac{x}{2}-\frac{2}{x}\right]$$

$$=\frac{-i}{\Gamma(v)}\left[\tfrac{1}{2}\int_0^\infty dx\,x^{\nu-1}e^{-x}\coth\frac{x}{2}-\int_0^\infty dx\,x^{\nu-2}e^{-x}\right]$$

$$=-i\left[\zeta(\nu)-\frac{1}{2}-\frac{\Gamma(\nu-1)}{\Gamma(\nu)}\right]\ , \tag{C11}$$

where equations (3.311.11), (8.363.4), and (1.421.4) of Ref. 33 have been used.

The integration of $L(v)$ for other v's can be done easily. The results for relevant $L(v)$'s are listed below:

$$L(3)=i\int_0^\infty\frac{dt}{e^{2\pi t}-1}(6t-2t^3)=i\tfrac{29}{120}\ , \tag{C12}$$

$$L(1)=i\int_0^\infty\frac{dt}{e^{2\pi t}-1}(2t)=\frac{i}{12}\ , \tag{C13}$$

$$L(-1)=\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left[\frac{-2it}{1+t^2}\right]=-i(-\tfrac{1}{2}+\gamma)\ , \tag{C14}$$

$$L(-3)=-i\left[\zeta(3)-\frac{1}{2}-\frac{\Gamma(2)}{\Gamma(3)}\right]=-i[\zeta(3)-1]\ , \tag{C15}$$

$$L(-5)=-i\left[\zeta(5)-\frac{1}{2}-\frac{\Gamma(4)}{\Gamma(5)}\right]=-i[\zeta(5)-\tfrac{3}{4}]\ . \tag{C16}$$

Substituting (C12), (C13), and (C14) into (C10), we get

$$I_2=-\frac{29}{120}-\frac{\sigma}{24}+\frac{\sigma^2}{8}(\tfrac{1}{2}-\gamma)+\sum_{k=3}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k)}\sigma^k\left[\zeta(2k-3)-\tfrac{1}{2}-\frac{1}{2k-4}\right]. \tag{C17}$$

Similarly,

$$I_1=-\tfrac{1}{12}+\frac{\sigma}{2}(\tfrac{1}{2}-\gamma)-\tfrac{3}{8}\sigma^2[1-\zeta(3)]+\sum_{k=3}^{\infty}\frac{(-1)^k}{k!}\sigma^k\Gamma\frac{(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\left[\zeta(2k-1)-\frac{1}{2}-\frac{1}{2k-2}\right], \tag{C18}$$

$$I_0=(\gamma-\tfrac{1}{2})-\tfrac{3}{2}\sigma[\zeta(3)-1]+\tfrac{15}{8}\sigma^2[\zeta(5)-\tfrac{3}{4}]+\sum_{k=3}^{\infty}\frac{(-1)^k}{k!}\sigma^k\frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{3}{2})}\left[\zeta(2k+1)-\frac{1}{2}-\frac{1}{2k}\right]. \tag{C19}$$

With other relevant σ expansions in (C5) to (C7), we get

$$Z_0(2,\sigma)=\tfrac{1}{120}-\frac{\sigma}{24}-\gamma\frac{\sigma^2}{8}+\sum_{k=3}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\zeta(2k-3)\sigma^k\ , \tag{C20}$$

$$Z_0(1,\sigma)=-\tfrac{1}{12}-\frac{\sigma}{2}(1+\gamma)+\tfrac{3}{8}\sigma^2\zeta(3)+O(\sigma^3)\ , \tag{C21}$$

$$Z_0(0,\sigma)=\gamma-\tfrac{3}{2}\zeta(3)\sigma+\tfrac{15}{8}\zeta(5)\sigma^2+O(\sigma^3)\ . \tag{C22}$$

(III) $\sigma=-1+y$, $|y|\ll 1$. We can expand Z_n in orders of y, but the $n=1$ term should be treated separately. Recall (B17),

$$Z'(r,\sigma)=D(r)-rD(r+1)y+\frac{r}{2}(r+1)D(r+2)y^2+\cdots\ . \tag{C23}$$

$D(r)$ can be evaluated similar to $Z(r,\sigma)$,

$$D(r)=\int_0^\infty dt\frac{t^2}{(t^2-1)^r}-\int_0^2\frac{t^2}{(t^2-1)^r}dt+\frac{2}{3^r}+i\int_0^\infty\frac{f(2+it)-f(2-it)}{e^{2\pi t}-1}dt\ ,$$

where (C24)

$$f(x)=\frac{x^2}{(x^2-1)^r}\ .$$

Again, for convenience, we define $r=\frac{3}{2}-n$, so that the pole terms can be expanded out. From (B4)

$$D_{-1}(n)=\frac{\sqrt{\pi}}{4}\frac{1}{\Gamma(\frac{3}{2}-n)}\frac{1}{n!}\quad(\text{for } n=2,1,0)\ ,$$

(C25)

$$D_0(n)=\frac{\sqrt{\pi}}{4}\frac{1}{n!}\frac{1}{\Gamma(\frac{3}{2}-n)}[\psi(n+1)-\psi(\tfrac{3}{2}-n)-\ln(-1)]-\int_0^2\frac{t^2}{(t^2-1)^{3/2-n}}dt+\frac{2}{3^{3/2-n}}+I_n(-1)\ ,$$

and

$$I_n(-1)=i\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left[\frac{(2+it)^2}{[(2+it)^2-1]^{3/2-n}}-\frac{(2-it)^2}{[(2-it)^2-1]^{3/2-n}}\right]. \tag{C26}$$

The problematic term $\ln(-1)$ will be canceled out by the same pathological part in

$$\int_0^2\frac{t^2}{(t^2-1)^{3/2-n}}dt\ .$$

Being finite the $I_n(-1)$ can be obtained by numerical integration [although one can expand about $2+it$ and use small σ approximation for $I_n(\sigma)$ in (C23) to (C25) but it converges very slowly],

$$I_2(-1)=-9.5374\times10^{-1}\,, \tag{C27}$$

$$I_1(-1)=-6.5236\times10^{-2}\,, \tag{C28}$$

$$I_0(-1)=5.8785\times10^{-2}\,. \tag{C29}$$

From (C38), one can obtain $D_0(n)$ as

$$D_0(2)=\tfrac{1}{32}+\tfrac{1}{8}\ln\frac{2+\sqrt{3}}{2}+\frac{\sqrt{3}}{4}+I_2(-1)=-0.4115\,, \tag{C30}$$

$$D_0(1)=\tfrac{1}{4}-\frac{1}{\sqrt{3}}-\tfrac{1}{2}\ln\frac{2+\sqrt{3}}{2}+I_1(-1)=-0.7045\,, \tag{C31}$$

$$D_0(0)=-1+\frac{8}{3\sqrt{3}}-\ln\frac{2+\sqrt{3}}{2}+I_0(-1)=-2.542\times10^{-2}\,. \tag{C32}$$

These values are used in (B18) and (B20). When $n<0$, both Z_n and D_n are convergent summations. One cannot use the splitting as in (B4). The values of $D(-1)$ and $D(-2)$ used in (B20) are obtained by numerical summation.

APPENDIX D: RENORMALIZATION IN COVARIANT GEOMETRIC TERMS

In the text we have used two parameters; the curvature radius a and the deformation parameter α to describe the Taub universe and systems therein. However, for various purposes where covariant expressions are needed, as in the renormalization of massless fields, it is better to use covariant geometric terms like the scalar curvature and Weyl curvature squared.

From Eqs. (A8) and (A10), we have (no approximation on α)

$$R=\frac{6}{a^2}\left[1+\frac{1}{3}\frac{\alpha}{1+\alpha}\right],$$

$$C^2=\frac{64}{3}\frac{1}{a^4}\left[\frac{\alpha}{1+\alpha}\right]^2.$$

These two equations can be solved for a^2 and α to yield

$$a^2=\frac{6}{R}\left\{1+\tfrac{1}{4}\left[\pm\left[\frac{3C^2}{R^2}\right]^{1/2}\right]\right\}^{-1}, \tag{D1}$$

$$\alpha=\tfrac{3}{4}\left[\pm\left[\frac{3C^2}{R^2}\right]^{1/2}\right]\left\{1-\left[\pm\left[\frac{3C^2}{R^2}\right]^{1/2}\right]\right\}^{-1}. \tag{D2}$$

We will omit the $\pm$ sign in front of $(3C^2/R^2)^{1/2}$ with the understanding that the sign of α is carried by that of $(3C^2/R^2)^{1/2}$. Since we are dealing with small $|\alpha|$ and only keeping terms up to $O(\alpha^2)$, the corresponding expansion in powers of $(3C^2/R^2)^{1/2}$ will be terminated at $O(3C^2/R^2)$. Using (D1) and (D2), we can rewrite the bare effective potential Eq. (2.32) as

$$\begin{aligned}V^{(1)}=&\frac{R^2}{144\pi^2}\left[A+\tfrac{3}{4}\left[B-\frac{A}{6}\right]\left[\frac{3C^2}{R^2}\right]^{1/2}+\tfrac{27}{16}(C+\tfrac{7}{6}B+\tfrac{23}{72}A)\frac{C^2}{R^2}\right]\\&-\frac{1}{64\pi^2}\left[\left[M^2-\frac{\xi R}{6}\right]^2+\frac{C^2}{60}\right]\left[2+\ln\frac{3\mu^2}{2R}\right].\end{aligned} \tag{D3}$$

In the limit of large M^2a^2, we can use Eq. (3.4) for A,B,C and drop all terms proportional to M^{-2}. Then (D3) can be written as

$$\begin{aligned}V^{(1)}=\frac{\hbar}{144\pi^2}\Bigg[&-\tfrac{3}{32}\xi^2R^2+\tfrac{9}{8}M^2\xi R-\tfrac{27}{8}M^4+\frac{\sqrt{3C^2}}{64}(\xi^2R-6M^2\xi)\\&+\frac{3C^2}{512}\xi^2+\left[\frac{\xi^2R^2}{16}-\tfrac{3}{4}M^2\xi R+\tfrac{9}{4}M^4+\tfrac{3}{80}C^2\right]\ln\frac{\sigma}{a^2\mu^2}\Bigg].\end{aligned} \tag{D4}$$

The $\ln(\sigma/a^2\mu^2)$ term can be expressed in terms of $M^2-\xi R/6$:

$$\ln\frac{\sigma}{a^2\mu^2}=\ln\frac{-\xi+M^2a^2}{a^2\mu^2}=\ln\left[\frac{M^2-\xi R/6}{\mu^2}\right]+\ln\left[1+\frac{(\xi R/24)(3C^2/R^2)^{1/2}}{M^2-\xi R/6}\right]$$
$$=\ln\frac{M^2-\xi R/6}{\mu^2}+\frac{\xi\sqrt{3C^2}}{24M^2}\left[1+\frac{\xi R}{6M^2}\right]-\frac{\xi^2C^2}{384M^4}\ . \tag{D5}$$

Substituting (D5) into (D4), we get

$$V^{(1)}=\hbar\left[\frac{-3}{4608\pi^2}\xi^2R^2+\frac{M^2\xi R}{128\pi^2}-\frac{3}{128\pi^2}M^4+\left[\frac{\xi^2R^2}{2304\pi^2}-\frac{M^2\xi R}{192\pi^2}+\frac{M^4}{64\pi^2}+\frac{C^2}{3840\pi^2}\right]\ln\frac{M^2-\xi R/6}{\mu^2}\right]\ . \tag{D6}$$

From this point on, we can proceed with the same renormalization scheme given in Ref. 12 and obtain the corresponding counterterms in Ref. 12.

In the massless case, since $M^2=(\lambda/2)\phi^2$ is not a large quantity for the perturbative range of λ, Eq. (D4) is not applicable. We have to go back to the original form of the effective potential (D3) and reevaluate the A,B,C coefficients in this limit. We find the effective potential can be cast into the general form

$$V(\hat\phi)=V(0)+g_1\left[\frac{\hat\phi^2}{R}\right]R^2+g_2\left[\frac{\hat\phi^2}{R}\right]C^2+\tfrac{1}{12}\left[1-\theta\left[\frac{C^2}{R^2}\right]\right]R\hat\phi^2+\frac{\lambda}{4!}\left[1-\eta\left[\frac{C^2}{R^2}\right]\right]\hat\phi^4\ ,$$

where

$$V(0)=-\left[\Lambda+kR+\tfrac{1}{2}\epsilon_1\left[\frac{C^2}{R^2}\right]R^2+\tfrac{1}{2}\epsilon_2\left[\frac{C^2}{R^2}\right]C^2\right]\ . \tag{D7}$$

For the conformal case, using Eq. (3.6), we get

$$\begin{aligned}
&\epsilon_1=\frac{1}{8640\pi^2}\ ,\\
&\epsilon_2=\frac{1}{1920\pi^2}\ln\frac{R}{R_2}\ ,\\
&g_1=g_2=0\ ,\\
&\theta=\frac{\lambda}{96\pi^2}\ ,\\
&\eta=\frac{3\lambda}{32\pi^2}\left[\ln\frac{3\lambda\hat\phi_0^2}{2R}+\tfrac{14}{3}+2\gamma+\tfrac{9}{16}[\tfrac{1}{6}+3\gamma-\tfrac{7}{5}\zeta(3)]\frac{C^2}{R^2}\right]\ .
\end{aligned} \tag{D8}$$

For the minimally coupled case, we get

$$\begin{aligned}
&\epsilon_1=\epsilon_1(R_1)-\frac{1}{4608\pi^2}\ln\frac{R}{R_1}\ ,\\
&\epsilon_2=\epsilon_2(R_2)+\frac{1}{1920\pi^2}\ln\frac{R}{R_2}\ ,\\
&g_1=\frac{1}{144\pi^2}\left[\frac{3\lambda}{R}\hat\phi^2\right]^{1/2}\ ,\\
&g_2=\frac{1}{256\pi^2}\left[\frac{3\lambda}{R}\hat\phi^2\right]^{1/2}\ ,\\
&\theta\left[\frac{C^2}{R^2}\right]=1-\frac{\lambda}{4\pi^2}\left[\frac{1}{8}\ln\frac{R}{R_0}+\tfrac{1}{4}(a_1-\tfrac{1}{2}d_0+\tfrac{1}{4})\left[\frac{3C^2}{R^2}\right]^{1/2}+\tfrac{9}{16}(\tfrac{9}{48}+\tfrac{37}{15}a_1-\tfrac{13}{30}d_0+\tfrac{1}{10}d_1-\tfrac{1}{10}e_0)\frac{C^2}{R^2}\right]\ ,\\
&\eta\left[\frac{C^2}{R^2}\right]=\frac{3\lambda}{32\pi^2}\left[\ln\frac{3\lambda}{4R}\hat\phi^2+\frac{14}{3}+\frac{a_2}{16}+\tfrac{1}{64}(2a_2-\tfrac{1}{2}d_1-\tfrac{1}{4})\left[\frac{3C^2}{R^2}\right]^{1/2}\right.\\
&\qquad\left.-\tfrac{9}{256}(\tfrac{1}{3}+\tfrac{37}{15}a_2+\tfrac{23}{30}d_1-\tfrac{1}{10}d_2+\tfrac{3}{40}e_0-\tfrac{1}{10}e_1)\frac{C^2}{R^2}\right]\ ,
\end{aligned} \tag{D9}$$

where a_n,d_n,e_n are as in Eq. (4.10).

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[27]As pointed out by Coleman and Weinberg (Ref. 28), the $\lambda^4\phi$ theory in flat spacetime has a fake minimum at $\lambda\ln(\phi/M)=-\frac{16}{3}\pi^2$. This is so because it is too far from the origin, causing the $\ln(\phi/M)$ term to be too large to be valid under one-loop perturbation theory. There is no phase transition associated with this minimum. In our case, the one-loop generalized CW potential does have a nonzero minimum at a nonzero value of ϕ which is within the validity range of perturbation theory. This is true for the case of minimum coupling, where classical curvature effect is absent. Phase transition induced by the quantum effects of curvature is thus possible.

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