Liquidity Supply by a Risk-Averse Market Maker*

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Market makers bear enormous uncertainty of the values of their portfolios and their attitude towards risk should not be completely ignored. In this article, we analyse a quote-driven market of a risky financial asset, where a risk-averse market maker supplies liquidity to traders. We characterise the equilibrium of the market for trading based on liquidity demand/diverse opinions on the value of the risky asset or based on information asymmetry. We find that risk aversion of the market maker is likely to increase the non-participation range of traders and the bid-ask spread.

I Introduction
Risk aversion is one of the most essential and common attributes of agents in financial markets. Their attitudes towards risk significantly affect the values of risky securities and the market equilibrium. Despite this, in the literature on financial markets, in particular, information-based models, market makers are often assumed to be risk neutral (see, for instance, Kyle, 1985 for an order-driven market, and Glosten, 1989, 1994 and Biais et al., 2000 for a quote-driven market). This assumption can facilitate analysis but is not totally satisfactory because market makers bear enormous uncertainty of the values of their portfolios due to random price and quantity variation of the assets they are trading. In models of quote-driven markets, such as Glosten (1989, 1994) and Biais et al. (2000), market makers are broadly defined as liquidity suppliers, submitting limit orders, while investors/traders are those who submit market orders and are assumed to be risk-averse. In the real world, the same agent submits limit orders under certain circumstances but submits market orders under other circumstances. It seems impossible that an agent changes his/her risk attitude frequently and is risk neutral when submitting limit orders but risk-averse when submitting market orders. A more reasonable and plausible assumption is that both market makers and other investors are risk-averse. In this article, we analyse a quote-driven market of a risky financial asset, where a risk-averse market maker supplies liquidity to traders. Apart from risk aversion on the side of the market maker, our model resembles the basic setup of Biais et al. (2000). We believe that traders, informed and speculative traders in particular, are likely to be less risk-averse than the market maker, since by its very nature this occupation attracts individuals relatively tolerant to risk. The model also departs from the standard information asymmetry assumption in the literature on quote-driven markets by analysing the trading mechanism based on the liquidity demand of investors.

Our model can be formally transformed into the model of optimal screening by a risk-averse market maker. The authors would like to thank the participants of seminar at the University of Melbourne for constructive comments.

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principal (i.e. the market maker in this article), developed by Basov and Yin (2010). In contrast to their prediction that the trade distortion decreases in the degree of risk aversion of the market maker, we find that it is likely to increase the range where the market maker does not supply liquidity. In the meantime, the bid-ask spread of the asset price increases as well. Intuitively, the undersupply of liquidity can be understood as a trade-off between two different objectives: to minimise the risk to income and to increase the total surplus of the transaction. The optimal non-linear pricing is the outcome of trading off these two effects. The weight put by the market maker on the decrease of total surplus, and corresponding possible breakup of the relationship, decreases as he becomes more risk-averse. This induces him to increase the ask-bid spread, even though it drives some investors out of the market.

Our model is also related to inventory models in the literature on financial microstructure (e.g. Ho and Stoll, 1983; Biais, 1993; Yin, 2005). Similar to these models, the holding cost caused by the risk aversion of the market maker is one of the reasons responsible for bid-ask spread in our model. However, we allow for non-linear pricing when the market maker quotes his offer, while the standard inventory-based models restrict price to being linear. Non-linear pricing is likely to be more realistic as the quotes provided by specialists/dealers or limit orders are usually quantity dependent. Non-linear pricing implies a quantity-dependent bid-ask spread, although in the formal analysis we focus on the spread when the quantity tends to zero. Moreover, in the full model analysed in Section IV, the spread is the outcome of the interaction between risk version of the market maker and information asymmetry.

The rest of the article is organised in the following way. Section II presents the basic model. In Section III, a more specific model of trading based on liquidity motives or diverse opinions on the value of the risky asset is given. Section IV accommodates information asymmetry and the market maker’s learning of the investors’ superior information. Section V concludes.

II The Model

We analyse a quote-driven market for a risky asset, where a risk-averse market maker supplies liquidity to investors by posting a price schedule (or a limit order), $T_i$, for the risky asset. The value of the risky asset, $v$, is random, and the investor believes that it can be characterised by the following relationship

$$v = s + e,$$

where $e$ is independent of $s$ and is normally distributed disturbance with mean zero and variance $\sigma^2$. We present here two different interpretations for $s$. First, following Biais et al. (2000), $s$ is a private signal of the value of the risky asset, which is observed only by the investor but not by the market maker. Thus, the investor has superior information over the market maker, who is keen to discover this information through the investor’s order. In the second interpretation, $s$ is just the investor’s opinion of the mean of $v$, formed based on public information. Therefore, the investor does not have any informational advantage over the market maker, and the market maker does not necessarily update his belief of asset value upon the receipt of the investor’s order. These two interpretations correspond to the two trading mechanisms this article focuses on, trade based on an informational advantage and trade based on liquidity/inventory requirement or diverse opinions. We will discuss some justifications for assuming different opinions at equilibrium in Section III.

The trader (or investor) is also endowed with initial asset holdings, $I$, which is unobservable to the market maker and is treated as a random variable. If the investor sends a market order of buying $q$ units of the risky asset to the market (or the market maker), she has to pay $T(q)$. Following Biais et al. (2000), the utility of the investor who buys $q$ units of the asset is assumed to be

$$2 \text{ For the justification of adopting } v = s + e \text{ with } e \text{ independent of } s \text{ rather than the standard assumption } s = v + e \text{ with } e \text{ independent of } v, \text{ see footnote 7 of Biais et al. (2000).}$$

$$3 \text{ When } q \text{ is negative, it means that the investor sells to the market maker and } T(q) \text{ is the amount of money she receives from the market maker.}$$

$$4 \text{ Alternatively, one can start with the assumption that the investor is an expected utility maximiser with the coefficient of absolute risk aversion equal to } \gamma. \text{ Then, Equation (2) is the investor’s certainty equivalent wealth, provided } e \text{ is normal. For details, see the Appendix.}$$
Thus, if the surplus is known, the implementing equation (5) can be re-written as:

\[ T(\theta) = \max_\theta \left( \theta q - \frac{1}{2} \gamma \sigma^2 q^2 - S(\theta) \right) \]  

(5)

where \( \theta = s - \gamma \sigma^2 I \) reflects the marginal valuation of the risk asset of the investor. This marginal valuation is higher if the investor observes a good signal or has a strong opinion of the asset (i.e., a greater \( s \)). It decreases in the initial position \( I \) because of risk aversion of the investor. The first term in Equation (2) is the investor’s utility if she does not trade, which affects only her participation decision. As Equation (2) shows, the optimal order quantity depends on \( \theta \) rather than \( s \), therefore the market maker is unable to extract the entire private information the trader might possess. With given \( T(q) \), investor \( \theta \) buys \( q(\theta) \) units of the risky asset to maximise Equation (2), that is,

\[ q(\theta) = \arg \max \left\{ \theta q - \frac{1}{2} \gamma \sigma^2 q^2 - T(q) \right\}, \]

subject to participation condition that \( U(q; \theta) \geq U(0; \theta) \), which can be simplified as \( \theta q(\theta) - \frac{1}{2} \gamma \sigma^2 q^2(\theta) - T(q(\theta)) \geq 0 \).

Define the gross utility generated from owning \( q \) units of the risky asset as

\[ u(q, \theta) = \theta q - \frac{1}{2} \gamma \sigma^2 q^2, \]

(3)

and the net surplus from trading as

\[ S(\theta) = u(q(\theta), \theta) - T(q(\theta)). \]  

(4)

Thus, if the surplus is known, the implementing tariff can be found from (see Basov, 2005):

\[ T(q) = \max_\theta \left( \theta q - \frac{1}{2} \gamma \sigma^2 q^2 - S(\theta) \right). \]

Since term \( \frac{1}{2} \gamma \sigma^2 q^2 \) does not depend on \( \theta \), Equation (5) can be re-written as:

\[ T(q) + \frac{1}{2} \gamma \sigma^2 q^2 = \max_\theta (\theta q - S(\theta)), \]

which implies that the total trading cost for the investor, defined as \( T(q) + \frac{1}{2} \gamma \sigma^2 q^2 \), is convex. Therefore, the investor always faces a concave optimisation problem.

By the envelope theorem, we get from Equation (4) that

\[ S'(\theta) = u_0(q(\theta), \theta) = q(\theta). \]  

(6)

Let \( Q \) be the market maker’s initial holding of the risky asset. His problem is then to determine price schedule \( T(q) \) to maximise the expected utility,

\[ \int E(V(T(q) + v(Q - q)) | \Omega) f(\theta) d\theta, \]

where \( V(\cdot) \) is the Bernoulli utility function of the market maker, \( E(\cdot | \Omega) \) is the conditional expectation based on his information set \( \Omega \) and \( f(\theta) \) is the probability density function of \( \theta \). We assume that the market maker’s Bernoulli utility is given by

\[ V(y) = \frac{1 - \exp(-Ry)}{R}. \]

Then, the expected utility of the market maker who sells \( q \) units of the asset to investor \( \theta \) at a total price of \( T(q) \) is

\[ E(V(T(q) + v(Q - q)) | \Omega) \]

\[ = \frac{1 - \exp(-R(T(q) + h(q; \Omega)))}{R} \]

\[ \equiv W(T(q) + h(q; \Omega)), \]

(7)

where

\[ h(q; \Omega) = \frac{1}{R} \ln E(\exp(-Rv(Q - q)) | \Omega). \]  

(8)

If the market maker believes that the investor has no superior information about the asset value, he does not update his belief upon his receipt of the market order from the investor. Assuming that the market maker believes that \( v \) is a normal variable with mean \( v_0 \) and variance \( \tau^2 \), and following the steps similar to ones used in derivation of Equation (2) given in the Appendix, \( h(q; \Omega) \) can be simplified as

\[ h(q) = (Q - q)v_0 - \frac{R\tau^2}{2} (Q - q)^2. \]  

(9)

On the contrary, if the market maker believes that the investor has superior information, he will infer the value of random variable \( s \) through the quantity of the investor’s order, or equivalently \( \theta \), via Equation (1). Because of

To be general, we allow the market maker to have a different belief from the investor. \( v_0 = s \) and \( \tau^2 = \sigma^2 \) can be considered as a special case in our model.
The Hamiltonian for the maximisation problem is the beliefs of the market maker concerning the evolution of the co-state variable $\lambda(t)$, whereas Equation (13) governs the control variable delivering the maximum to the market maker. The Hamiltonian, whereas Equation (13) governs the evolution of the co-state variable $\lambda(t)$.\(^7\)

Applying Equations (7) and (3), the first-order conditions (12) and (13) can be transformed into:

$$\exp(-R(u-S+h)) (\theta - \gamma \sigma^2 q + h_q(q;\Omega)) f(\theta) + \lambda(\theta) = 0,$$

$$\exp(-R(u-S+h)) f(\theta) = \lambda'(\theta). (15)$$

Define $\xi(q,\theta) \equiv \theta - \gamma \sigma^2 q + h_q(q;\Omega)$, then Equation (14) can be rewritten as:

$$\lambda(\theta) = -\exp(-R[u(q(\theta), \theta) - S(\theta) + h(q(\theta);\Omega)]) \xi(q(\theta), \theta) f(\theta). (16)$$

Differentiating the above expression, one obtains:

$$\lambda'(\theta) = -\exp(-R(u-S+h))$$

$$\times \left( \frac{d(\xi f)}{d\theta} - R \xi f(u_q q'_q + u_0 - S' + h_0 + h_q q'_q) \right). (17)$$

From Equation (6), one immediately obtains $u_q = S'$. Moreover, it can be easily shown that $u_q + h_q = \xi$. Using these observations to simplify Equation (17) and substituting it into Equation (15) yield:

$$R \xi f(q' q + h_0) - \frac{d(\xi f)}{d\theta} = f. (18)$$

Dividing by $f(\theta)$ and slightly rearranging the terms, one finally obtains:

$$R \xi^2(q, \theta) q'_q(\theta) + R h_0(q;\Omega) \xi(q, \theta)$$

$$- \frac{d}{f(\theta) d\theta} (\xi(q, \theta) f(\theta)) = 1. (18)$$

In Equation (18), $h(q;\Omega) = h(q, \theta)$, if $\Omega = \theta$ (i.e. the trader has private information), whereas $h(q;\Omega) = 0$, if $\Omega$ is the set of public information (i.e. the trader has no private information).

Let the support of $\theta$ be $[\theta, \bar{\theta}]$, which can be either finite or infinite. Since there are no constraints on trading when the investor’s type is either $\theta = \theta$ or $\theta = \theta$, the market maker faces a free-end optimisation problem; that is, the trading mechanism should satisfy the transversality conditions $\lambda(\bar{\theta}) = \lambda(\theta) = 0$. From Equation (14), allocation $q(\theta)$ satisfies the following boundary conditions:

$$\xi(q(\bar{\theta}), \bar{\theta}) f(\bar{\theta}) = [\bar{\theta} - \gamma \sigma^2 q(\bar{\theta}) + h_q(q(\bar{\theta});\Omega)] f(\bar{\theta}) = 0,$$

$$\xi(q(\theta), \theta) f(\theta) = [\theta - \gamma \sigma^2 q(\theta) + h_q(q(\theta);\Omega)] f(\theta) = 0. (19)$$

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6 Although the value of $s$ is unobservable by the market maker, the relationship (1) and variance $\sigma^2$ are assumed to be publicly known.

7 In general, the evolution of the co-state variable is governed by differential equation $\lambda'(\theta) = -\theta H \lambda \theta$. 

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\[ \zeta(q(\bar{v}), \bar{q}) f(\bar{q}) = \left[ \bar{q} - \gamma \sigma^2 q(\bar{v}) + h_q(q(\bar{v}); \Omega) \right] f(\bar{q}) = 0. \]  

**III Traders do not have Superior Information**

In an important scenario that often arises when trading financial assets, traders have different beliefs from the market maker but no one has superior or private information about the value of the risky asset. In this case, \( s \) can be considered as the trader’s belief of the mean of \( v \) while the market maker has a belief of \( v_0 \). Therefore, both parties may have speculative motive to trade. Alternatively, the trader may have the same belief as the market maker on the value of the risky asset (i.e. \( s = v_0 \) and \( \sigma^2 = \tau^2 \)), in which case the trade is motivated by risk sharing or liquidity requirement. We assume that under both scenarios neither the market maker nor the investor updates their beliefs conditional on the other party’s willingness to trade on specific terms. Such difference in beliefs seems to be a realistic feature of the market and can be traced to divergent priors. The early papers that assume difference in beliefs are Arrow (1964), Lintner (1969), and Harrison and Kreps (1978). Morris (1994) investigates under what conditions difference in prior beliefs leads to speculative trade.

To obtain a close-form solution for the optimal price schedule by the market maker, assume that \( \theta \) is distributed uniformly on \( [\bar{\theta}, \tilde{\theta}] \), that is, \( f(\bar{q}) = 1/(\tilde{\theta} - \bar{\theta}) \) and \( F(\bar{q}) = (\tilde{\theta} - \bar{q})/(\tilde{\theta} - \bar{\theta}) \). Moreover, to ensure that the investor can be both buyer and seller of the risky asset, we assume investors are sufficiently diversified that \( \bar{\theta} < v_0 - R\tau^2 Q < \tilde{\theta} \). Since \( h_\theta = 0 \) and \( \zeta(q, \bar{\theta}) = \bar{\theta} - (\gamma \sigma^2 + R\tau^2) q - v_0 + R\tau^2 Q \), Equation (18) can be rewritten as:

\[ 2 - (\gamma \sigma^2 + R\tau^2) q'(\theta) = R[\theta - v_0 - \gamma \sigma^2 q + R\tau^2 (Q - q)]^2 q'(\theta). \]  

Boundary conditions (19) and (20) can be simplified as:

\[ q(\bar{\theta}) = \frac{\bar{\theta} - v_0 + R\tau^2 Q}{\gamma \sigma^2 + R\tau^2} \quad \text{and} \quad q(\tilde{\theta}) = \frac{\tilde{\theta} - v_0 + R\tau^2 Q}{\gamma \sigma^2 + R\tau^2}. \]  

**Proposition 1.** If there is no learning between investors and the market maker, and the type of investors is uniformly distributed, there is a unique separating equilibrium, where equilibrium liquidity is given by

\[ q_L(\theta) = \begin{cases} 
q_L(\theta), & \text{if } \theta \in [\bar{\theta}, \tilde{\theta}] \\
0, & \text{if } \theta \in [\bar{\theta}, \tilde{\theta}] \\
q_L(\theta), & \text{if } \theta \in (\bar{\theta}, \tilde{\theta}],
\end{cases} \]  

where \( q_L \) and \( q_R \) are given by Equations (45) and (46) in the Appendix, respectively, and

\[ \begin{aligned}
\bar{\theta}_L &= v_0 - R\tau^2 Q + \frac{\tilde{z} - 1}{\sqrt{aR(\tilde{z} + 1)}}, \\
\bar{\theta}_R &= v_0 - R\tau^2 Q - \frac{1 - \tilde{z}}{\sqrt{aR(\tilde{z} + 1)}},
\end{aligned} \]

where \( \tilde{z} \equiv \exp(\sqrt{aR(\theta - v_0 \pm R\tau^2 Q)}) \). The region where the market maker does not supply liquidity, \( \bar{\theta}_L - \bar{\theta}_R \), approaches to \( \frac{1}{2}(\bar{\theta} - \tilde{\theta}) \) if market maker is risk neutral.

While the exact form of the solution given in the proposition is driven by the particular distribution of investor type, several of its features are highly intuitive and hold more generally. The supply of liquidity (i.e. \( q_L(\theta) \) in Eqn 23) is determined by the trader’s marginal valuation of the risk asset. More specifically, \( q_L(\theta) \) increases in \( \theta \) although the increase is not strict. If valuation is high (i.e. \( \theta > \bar{\theta}_L \)), the market maker sells more to a trader with a greater valuation, since he can set a higher marginal price for each unit sold. In contrast, if the trader’s marginal valuation is low (\( \theta < \bar{\theta}_L \)), the market maker buys the asset from the Trader, and the lower is the marginal value the more he buys. In the intermediate level (\( \bar{\theta}_L \leq \theta \leq \bar{\theta}_R \)), the market maker does not supply liquidity. Note that \( v_0 - R\tau^2 Q \) can be interpreted as the market maker’s marginal valuation of the risky asset. Thus, the condition \( \bar{\theta}_L \leq \theta \leq \bar{\theta}_R \) implies that the marginal valuations of the market maker and investor are very close, and there is no substantial gain from trade between them because both speculation and risk-sharing motivations for trading disappear. In addition to adverse selection induced non-provision of liquidity in asymmetric information models (see Glosten, 1989, 1994; Biais et al., 2000), there are two additional reasons for the existence of non-trading region \([\bar{\theta}_L, \bar{\theta}_R]\) in our model. First, because of risk aversion, the market maker bears a holding cost when he provides liquidity services as his holding deviates from the optimal position. If a trader cannot pay sufficient compensation for this cost, the market maker has no incentive to serve. However, the non-trading region exists even when the market
maker is risk neutral, as can be seen that \(\theta_L - \theta_L = \frac{1}{2}(\bar{\theta} - \underline{\theta})\) when \(R \to 0\). This reflects the second reason; that is, the market maker utilises his monopoly power to price discriminate traders. He forgoes the trading opportunities with traders who have marginal valuation close to his to gain larger payoffs from traders with significantly different marginal valuation of the risky asset. It is not hard to infer from the findings of other inventory-based models\(^8\) that such a non-trading region will disappear if market makers are competitive and risk neutral.

The risk aversion of the market maker also yields a bigger bid-ask spread of the asset price than is the case when the market maker is risk neutral. To illustrate the claim formally, let us first find the marginal tariff that implements Equation (23). We first consider the region \(\bar{\theta} \in (\bar{\theta}_L, \bar{\theta})\), which implies \(q > 0\). Note that the first-order condition for the investor’s optimisation is \(T'(q) = \bar{\theta} - \gamma\sigma^2 q\). Therefore, Equation (45) in the Appendix implies

\[
\sqrt{aR} q = \bar{z} + \ln \left( \frac{1 - \sqrt{aR}[T'(q) - v_0 + R\gamma^2(Q - q)]}{1 + \sqrt{aR}[T'(q) - v_0 + R\gamma^2(Q - q)]} \right).
\]

Routine algebra gives the marginal price of trading \(q > 0\) units of the risky asset as

\[
T'(q) = \frac{1}{aR} \frac{\bar{z} - \exp(\sqrt{(R/a)q}) + v_0 - R\gamma^2(Q - q)}{\exp(\sqrt{(R/a)q}) + v_0 - R\gamma^2(Q - q)}.
\]  

(24)

Similarly, for \(q < 0\) one obtains

\[
T'(q) = \frac{1}{aR} \frac{\bar{z} - \exp(\sqrt{(R/a)q}) + v_0 - R\gamma^2(Q - q)}{\exp(\sqrt{(R/a)q}) + v_0 - R\gamma^2(Q - q)}.
\]

(25)

The bid-ask spread can be defined as \(B \equiv \lim_{q \to 0} T'(q) - \lim_{q \to -0} T'(q)\). Applying Equations (24) and (25) and noting \(\bar{z} \geq 1 \geq \underline{z}\), we find

\[
B = \frac{1}{aR} \frac{2(\bar{z} - \underline{z})}{(\bar{z} + 1)(\underline{z} + 1)} > 0.
\]  

(26)

This bid-ask spread corresponds to the non-trading region of \(\bar{\theta} \in [\underline{\theta}_L, \bar{\theta}_L]\) specified in Proposition 1. Although in our model there is no holding cost for the market maker when he is risk neutral, his pricing schedule still implies a bid-ask spread. It can be easily seen from Equation (26) that \(B = \frac{1}{2}(\bar{\theta} - \underline{\theta})\) when \(R \to 0\); that is, the bid-ask spread is equal to half of the difference between the highest and lowest types of investors. This spread is attributable to the market maker’s monopoly power, who receives economic rents, while sharing risk with investors. When the market maker is risk-averse, he is more conservative in taking a long or a short position of the risky asset. To compensate him for risk taking, he would impose a larger bid-ask spread. The spread increases in the market maker’s risk aversion, as demonstrated in Proposition 2.

**Proposition 2.** Under the assumptions of Proposition 1, the market maker imposes a bid-ask spread of \(\frac{1}{2}(\bar{\theta} - \underline{\theta})\) even if he is risk neutral. The bid-ask spread monotonically increases in the market maker’s risk aversion as long as \(R\) is sufficiently small.

As we will see next, this property of the price schedule holds even when the information between the market maker and investors is asymmetric. The positive spread under a risk-neutral market maker agrees with the findings from inventory-based models with linear pricing (e.g. O’Hara and Oldfield, 1986). But the spreads derived from these models are constant, independent of trading quantity. In contrast, the spread given in Equation (26) is the marginal price difference of selling and buying when the quantity tends to zero. The spread of trading a non-zero quantity in our model can be defined as \(\ln(T(q) - T(-q))/\ln|q|\). Clearly, it varies as quantity \(q\) changes.

Integrating Equation (24) or Equation (25), one can easily obtain the equilibrium price schedule specified in the following proposition:

**Proposition 3.** Under the assumptions of Proposition 1, the market maker imposes an equilibrium price schedule

\[
T(q) = \left( v_0 + \frac{1}{aR} \frac{R\gamma^2}{2}(Q - 2Q) \right) q
- \frac{2}{R} \ln \left( \frac{z + \exp(\sqrt{(R/a)q})}{z + 1} \right),
\]

(27)

where \(z = \bar{z}\) for \(q > 0\) and \(z = \underline{z}\) for \(q < 0\).

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The non-linear pricing of Equation (27) distinguishes our model from existing inventory-based models. In these models, market makers are usually restricted to using a constant price for buying or selling. Non-linear pricing is arguably more realistic as it captures the feature of quantity-dependent quotes provided by specialists/dealers or limit orders.

IV Optimal Pricing Strategy when the Market Maker Extracts the Private Information of Investors

When investors have superior information over the market maker, the latter will try to extract the private information from the former through the order for the risky asset. Because relationship (1) is common knowledge, the key for the market maker is to estimate \( s \) by observing \( \theta \).

(i) A Risk-Neutral Market Maker

Before dealing with the pricing problem a risk-averse market maker faces, let us consider the case of a risk-neutral market maker, which is the scenario the existing literature has focused on. It provides us with a benchmark for our results of a risk-averse market maker. The main results under risk neutrality can be found in Biais et al. (2000) but our treatment is simpler. The assumption of risk neutrality on the market maker side can substantially reduce the complexity of the problem, since in this case \( h(q, \theta) = (Q - q)v(\theta) \) as shown in Equation (11). Because \( h_q(q, \theta) = -v(\theta) \) and \( R = 0 \), Equation (18) collapses to

\[
\frac{d((\theta - \gamma \sigma^2 q(\theta) - v(\theta))f(\theta))}{d\theta} = -\frac{dF(\theta)}{d\theta},
\]

which yields

\[
[\theta - \gamma \sigma^2 q(\theta) - v(\theta)]f(\theta) = -F(\theta) + \text{const}. \tag{28}
\]

Assume the support of \( \theta \), \( [\underline{\theta}, \bar{\theta}] \), satisfies that \( \underline{\theta} < v(\theta) < \bar{\theta} \), then Equations (19) and (20) yields

\[
[\theta - \gamma \sigma^2 q(\theta) - v(\theta)]f(\theta) = 0, \tag{29}
\]

\[
[\bar{\theta} - \gamma \sigma^2 q(\theta) - v(\theta)]f(\bar{\theta}) = 0.
\]

Applying the first condition in Equation (29) to Equation (28), we find \( \text{const} = 1 \) and the optimal amount of the risky asset sold to investor \( \theta \) is

\[
q_N(\theta) = q^*(\theta) - \frac{1 - F(\theta)}{\gamma \sigma^2 f(\theta)}, \quad \theta \in (\bar{\theta}_N, \bar{\theta}), \tag{30}
\]

where \( \bar{\theta}_N = \arg\{q^*(\theta) = (1 - F(\theta))/\gamma \sigma^2 f(\theta)\} \) and \( q^*(\theta) = (\theta - v(\theta))/\gamma \sigma^2 \) is the efficient quantity. Similarly, applying the second condition of Equation (29) to Equation (28), we find \( \text{const} = 0 \) and

\[
q_N(\theta) = q^*(\theta) + \frac{F(\theta)}{\gamma \sigma^2 f(\theta)}, \quad \theta \in [\underline{\theta}, \underline{\theta}_N), \tag{31}
\]

where \( \underline{\theta}_N = \arg\{q^*(\theta) = F(\theta)/\gamma \sigma^2 f(\theta)\} \). For an investor whose type falls within interval \([\underline{\theta}_N, \bar{\theta}_N] \), the market maker does not supply liquidity; that is

\[
q_N(\theta) = 0, \quad \theta \in [\underline{\theta}_N, \bar{\theta}_N]. \tag{32}
\]

Formulae (30)–(32) describe the solution to the problem of optimal liquidity supply, provided \( q_N(\theta) \geq 0 \) in the points where it is differentiable. Sufficient conditions for this are:

\[
\frac{d((1 - F(\theta))/f(\theta))}{d\theta} < 0, \quad \frac{d(F(\theta)/f(\theta))}{d\theta} > 0, \quad 0 \leq v'(\theta) < 1.
\]

These conditions hold in the specification of our analysis below.

(ii) A Risk-Averse Market Maker

When the market maker is risk-averse, a closed-form solution to the first-order condition (18) with boundary conditions (19) and (20) cannot be found. Therefore, we focus on the two polar cases where risk aversion of the market makers is either extremely high or very low.

Let us first consider the case when the market maker is extremely risk-averse, that is, \( R \rightarrow \infty \). To find out the equilibrium supply of liquidity in this case, we first define a constant \( c \) by:

\[
\frac{1}{R} \ln E(\exp(-Rs(Q - q))|\theta) = O(R^c),
\]

when \( R \rightarrow \infty \). Second, define \( b = 1 + 2 \max\{c, 1\} \). Then, divide both sides of Equation (18) by \( R^b \) and pass to the limit \( R \rightarrow \infty \). Since the trade volume should remain bounded, we find that \( q'(\theta) = 0 \). Boundary conditions (19) and (20) require \( q(\theta) = q(\bar{\theta}) = Q \) when \( R \rightarrow \infty \). Thus, for an extremely risk-averse market maker, the equilibrium supply of liquidity to all investors is \( q(\theta) = Q \).
This result is quite intuitive. If the market maker is very risk-averse, he would like to hold a zero net position after trading to avoid any uncertainty caused by the risky asset. This means that he wants to sell his holding of the risky asset, \( Q \), to the investor (or buy \(-Q\) from the investor if \( Q \) is negative). Because of this extreme risk aversion, the value of the risky asset is not important in his decision making. All he wants is to get rid of his long position or recover his short position of the risky asset.

The problem a risk-averse market maker faces is similar to the principal-agent model studied in Basov and Yin (2010) if \( h(q, \theta) \) is considered as the effective cost. However, the key difference is that here the effective cost \( h(q, \theta) \) explicitly depends on \( R \). In Basov and Yin (2010), as \( R \) increases the distortion decreases, and when \( R \to \infty \) the effort supplied by the agent reaches the efficient level. These results are driven by the fact that the principal tries to avoid the worst scenario – failing to hire an agent. In the current model, the market maker can potentially make losses. As a result, the trading volume converges to \( Q \) when \( R \to \infty \).

Let us now turn to a market maker with a finite degree of risk aversion. To explicitly address the learning process of the market maker, we assume that he has a prior belief that \( s \) and \( l \) are independently normally distributed with means \( s_0 \) and \( l_0 \), and variances \( \sigma_0^2 \) and \( \sigma_f^2 \), respectively. Therefore, \( \theta \) is also normal with mean \( s_0 - \gamma \sigma^2 l_0 \) and variance \( \sigma_\theta^2 = \sigma_0^2 + \gamma^2 \sigma^4 \sigma_f^2 \). Upon the observation of \( \theta \), the market maker’s posterior belief is that \( s \) has mean and variance

\[
\nu(\theta) = \left( \frac{s_0}{\sigma_0} + \frac{\theta}{\sigma_\theta^2} \right) \sigma_s^2, \quad \sigma_s^2 = \frac{1}{(1/\sigma_0^2) + (1/\sigma_\theta^2)}.
\]

The conditional normality of \( s \) implies that Equation (10) can be written as:

\[
h(q, \theta) = (Q - q)\nu(\theta) - \frac{R(Q - q)^2}{2} (\sigma_s^2 + \sigma_\theta^2).
\]

Note, \( E(s|\theta) = E(\nu|\theta) = \nu(\theta) \) and \( \sigma_s^2 + \sigma_\theta^2 = \text{var}(\nu|\theta) \) is the conditional variance of \( \nu \). Then,

\[
h_q(q, \theta) = R(Q - q)(\sigma_s^2 + \sigma_\theta^2) - \nu(\theta), \quad h_0 = \frac{(Q - q)\sigma_s^2}{\sigma_\theta^2},
\]

\[
\xi(q, \theta) = 0 - \gamma \sigma^2 q + R(Q - q)(\sigma_s^2 + \sigma_\theta^2) - \nu(\theta).
\]

Substituting these values into the optimal condition (18) yields:

\[
\frac{d}{d\theta} [\xi(q, \theta) f(\theta) + F(\theta)] = Rf(\theta) \xi(q, \theta) \left[ \xi(q, \theta) q'(\theta) + \frac{(Q - q(\theta))\sigma_s^2}{\sigma_\theta^2} \right].
\]

Substituting the normal distribution function and density for \( F(\cdot) \) and \( f(\cdot) \), respectively, one obtains:

\[
(R\sigma_s^2 + (\gamma + R)\sigma^2 + R\sigma_f^2)q' = 2 - \frac{\sigma_s^2}{\sigma_\theta^2} - R\xi(\frac{Q - q(\theta)\sigma_s^2}{\sigma_\theta^2})
\]

\[
- \frac{\xi - \theta}{\sigma_\theta^2}.
\]

As we pointed out earlier, we cannot provide a closed-form solution for the general case. Therefore, we will limit ourselves to computing the first-order correction to the risk-neutral solution for small \( R \). The solution to Equation (37) can be formally written as:

\[
q(\theta) = \sum_{n=0}^{\infty} q_n(\theta) R^n.
\]

Restricting ourselves to the first-order approximation with respect to \( R \), one can write:

\[
q(\theta) = q_0(\theta) + R q_1(\theta) + o(R).
\]

To get the condition on \( R \) that justifies keeping only the first-order term of \( R \) in Equation (39), notice that the ratio of the consecutive terms in the perturbation series expansion Equation (38) is of the order \( R(\theta q - (\gamma \sigma_s^2/2)q^2 - T(q)) \). Since

\[
\left| \theta q - \frac{\gamma \sigma_s^2}{2} q^2 - T(q) \right| \leq \left| \theta q - \frac{\gamma \sigma_s^2}{2} q^2 \right| \leq \frac{\theta^2}{2 \sigma^2},
\]

the first-order correction provides a good approximations as long as

\[
9 \text{This approach is known as perturbation theory. A remarkable observation about the perturbation theory analysis is that, while the differential equation for } q_0(\cdot) \text{ is in general non-linear, the differential equations for } q_n(\cdot) \text{ for } n \geq 1 \text{ are always linear.}
\]
where << means much smaller. Condition (40) should hold for all $\theta$. A sufficient condition for that is
\[ \frac{R\theta^2}{2\gamma \sigma^2} << 1, \]  
where $\theta$ is the signal she receives. In contrast, if the magnitude of the signal is much smaller, Condition (41) is
\[ \frac{R[\max(|\theta|, 0)]^2}{2\gamma \sigma^2} << 1. \]

Suppose that the market maker is as risk-averse as investors, that is, $R = \gamma$, then the sufficient condition (41) collapses to $R[\max(|\theta|, 0)]^2 / 2\gamma \sigma^2 << 1$, which requires that the investor’s marginal valuation of the risky asset must be small enough relative to the standard deviation of the signal so that the non-trading range set by a risk-neutral market maker is much less risk-averse than the investors. The precise requirement for smallness of $R[\max(|\theta|, 0)]^2 / 2\gamma \sigma^2$ depends on the precision we want to achieve.

**Proposition 4.** When investors have superior information and the market maker is risk-averse but $R$ is sufficiently small, $q_0(\theta)$ in the optimal price schedule (39) is equal to $q_N(\theta)$ given by Equations (30)–(32) and the non-trading range determined by $q_0(\theta) + Rq(\theta)$ is not smaller than the non-trading range set by a risk-neutral market maker if $Q = 0$.

Turning to bid-ask spread, write the marginal price of the risky asset as $\tau(\theta) \equiv T'(q(\theta))$. As we noticed earlier, the trader’s utility maximisation implies that the marginal price of the risky asset is $\tau(\theta) = \theta - \gamma \sigma^2 q(\theta)$. Because the non-trading range under risk aversion, $[\bar{\theta}_R, \theta_N]$, is larger than the non-trading range under a risk neutrality, $[\bar{\theta}_N, \theta_N]$, one obtains $\bar{\theta}_R > \theta_N$ and in turn,
\[ \tau_R(\bar{\theta}_R) = \bar{\theta}_R - \gamma \sigma^2 q_R(\bar{\theta}_R) > \bar{\theta}_N - \gamma \sigma^2 q_N(\bar{\theta}_N) = \tau_N(\bar{\theta}_N). \]

Similarly, we can show that $\tau_R(\bar{\theta}_R) < \tau_N(\bar{\theta}_N)$. Therefore, the following corollary is implied by Proposition 4.

**Corollary 1.** With asymmetric information, risk aversion on the market maker’s side enlarges bid-ask spread if the degree of the market maker’s risk aversion and his initial holding of the risky asset are sufficiently small.

Clearly, $\tau_N(\bar{\theta}_N) - \tau_N(\bar{\theta}_N)$ can be interpreted as the bid-ask spread when the market maker is risk neutral, whereas $\tau_R(\bar{\theta}_R) - \tau_R(\bar{\theta}_R)$ is the bid-ask spread when the market maker is risk-averse. Therefore, our analysis shows that risk aversion on the market maker’s side increases the bid-ask spread. Spread $(\tau_N(\bar{\theta}_N) - \tau_N(\bar{\theta}_N))$ is due to information asymmetry, which has been discussed in the literature (see Glosten, 1989, 1994; Biais et al., 2000). Spread $(\tau_R(\bar{\theta}_R) - \tau_R(\bar{\theta}_R))$ is due to the risk aversion of the market maker. Risk aversion induced spread has been analysed by inventory-based models (see O’Hara, 1995). However, our model combines information asymmetry and risk aversion, and spread $(\tau_R(\bar{\theta}_R) - \tau_R(\bar{\theta}_R))$ reflects both effects. It is greater than the spread reflecting only one effect.

**V Conclusions**

A monopolistic market maker in a financial market can use non-linear pricing to screen traders and capture more economic rents by price discrimination than he would have captured by offering a uniform price. The strategy pays even when the market maker is risk neutral and he does not have informational disadvantage. Both risk aversion and informational disadvantage make non-trading range more non-linear, at least for small volumes of trade. Indeed, because of risk aversion, he sets a larger bid-ask spread to compensate for the income risk he bears. If on top of that he is also informationally disadvantaged, he will enlarge the bid-ask spread further to cover his payment of information rents to the informed traders.

As we have found, the total trading cost to an investor, $T(q) + \frac{1}{2} \gamma \sigma^2 q^2$, is convex with respect to $q$. However, it is generally impossible to exclude the possibility that $T(q)$ is concave.

A convex $T(q)$ implies an increasing unit price, which provides traders an incentive to split their orders. Even when unit price is not increasing, traders with superior information still have the incentive to split their orders to hide their information (see, for instance, Kyle, 1985). Our simple one-shot static model cannot deal with such an issue. An alternative setup is to consider a different set of assumptions. For instance, the optimal non-linear schedule implementing Equations (30)–(32) is concave on $\theta \in (\bar{\theta}_N, \theta]$ when
\[ \frac{\partial}{\partial \theta} T'(\theta) < -\frac{1}{\partial \theta} \frac{F'(\theta)}{F(\theta)} \]  
and on $\theta \in (\bar{\theta}_N, \theta]$ when
\[ \frac{\partial}{\partial \theta} T'(\theta) > \frac{1}{\partial \theta} \frac{F'(\theta)}{F(\theta)} \]. It is convex on these intervals when $v'(\theta)$ is strictly larger. The non-existence of an unambiguous result of convexity or concavity is not specific to our model but a general property of monopolistic screening models (see Maskin and Riley, 1989; Spulber, 1989).
a number of rounds of trading taking place sequentially. Moreover, in this dynamic setup some investors may have private information while others do not (see Glosten and Milgrom, 1985). In a separate study, we have used a static model to analyse the optimal non-linear pricing by competitive market makers facing heterogeneous traders with and without private information by extending the models of Glosten and Milgrom (1985) and Madhavan (1992). It is found that many of the main results of Glosten and Milgrom (1985) and Madhavan (1992) remain unchanged. Therefore, we conjecture that in the proposed dynamic model, the effects of the market maker’s risk aversion discovered in this paper are likely to prevail. However, formal analysis is required to prove such a claim. This is a topic for our future research.

REFERENCES


Appendix I

In this appendix, we present the proofs of all the Propositions and define and compute the certainty equivalent for an individual with a CARA Bernoulli utility function who faces a normal distribution of payoffs.

(i) Proof of Proposition 1

Let us first consider the solution in the region \( \theta \geq v_0 - R^2 Q \) and introduce

\[
\gamma(\theta) = \xi(q(\theta), \theta) = \theta - v_0 - \gamma\sigma^2 q(\theta) + R^2 (Q - q(\theta)).
\]

Differentiating the above expression implies:

\[
y'(\theta) = 1 - (\gamma \sigma^2 + R^2) q'(\theta).
\]  

(42)

Then Equation (22) implies \( y(\theta) = y(\theta) = 0 \) and substituting Equation (42) into Equation (21) one obtains

\[
1 + y'(\theta) = \frac{R^2 y(\theta)(1 - y'(\theta))}{\gamma \sigma^2 + R^2}.
\]

Rearranging the terms one obtains:

\[
y'(\theta) \left(1 + \frac{R^2 y}{\gamma \sigma^2 + R^2}\right) = \frac{R^2 y}{\gamma \sigma^2 + R^2} - 1.
\]

Finally, defining \( a = 1/(\gamma \sigma^2 + R^2) \) and substituting it into the above equation yield:

\[
y' = \frac{a R^2 y - 1}{a R^2 y + 1}.
\]  

(43)
Equation (43) implies that \( y'(\theta) \leq 1 \), which in turn implies that \( q'(\theta) = a(1 - y'(\theta)) \geq 0 \). Therefore, the solution does not have bunches in the region \( \theta \geq v_0 - R \tau^2 Q \). It can be verified that the solution to Equation (43) with boundary condition \( y(\theta) = 0 \) is given by

\[
\theta - \tilde{\theta} = y + \frac{1}{\sqrt{aR}} \ln \left( \frac{1 - \sqrt{aR} \theta}{1 + \sqrt{aR} \theta} \right). \tag{44}
\]

From Equation (43), there is \( y'(\theta) < 0 \) for \( y < 1/\sqrt{aR} \). Since \( y(\theta) = 0 \), the above equation implies that \( \theta \to -\infty \) when \( y \to 1/\sqrt{aR} \). In other words, there is \( 0 \leq y(\theta) < 1/\sqrt{aR} \) for \( \theta \in (-\infty, \theta_0) \). Consequently, the solution on \( \theta \in [v_0 - R \tau^2 Q, \theta_0] \) should be

\[
\theta - \tilde{\theta} = y + \frac{1}{\sqrt{aR}} \ln \left( \frac{1 - \sqrt{aR} \theta}{1 + \sqrt{aR} \theta} \right). \tag{44}
\]

Returning to the original variable \( q(\cdot) \), Equation (44) implies an implicit function \( q_L(\cdot) \):

\[
q_L = a(\theta - v_0 + R \tau^2 Q) + \sqrt{\frac{a}{R}} \ln \left( \frac{1 - \sqrt{aR} (\theta - v_0 + R \tau^2 Q)}{1 + \sqrt{aR} (\theta - v_0 + R \tau^2 Q)} \right). \tag{45}
\]

The above discussion implies that the function is increasing. The solution for \( \theta \geq v_0 - R \tau^2 Q \) is given by \( q_L(\theta) = \max \{ q_L(0), 0 \} \). Putting \( q_L = 0 \) into Equation (45), one can find the exclusion region to be \( (v_0 - R \tau^2 Q, \theta_L) \). Applying \( \tilde{z} \) to Equation (45) yields

\[
\frac{1}{\tilde{z}} = \frac{1 - \sqrt{aR} \theta_L - v_0 + R \tau^2 Q}{1 + \sqrt{aR} \theta_L - v_0 + R \tau^2 Q}.
\]

Thus, we get \( \theta_L \) in the proposition. For \( \theta < v_0 - R \tau^2 Q \) function \( q(\cdot) \) solves the same ordinary differential equation, but now subject to the boundary condition of the second equation in (22). Going through similar calculations, one obtains \( q_L(\theta) = \min \{ q_L(0), 0 \} \), where \( q_L(\theta) \) is the function implicitly defined by

\[
q_L = a(\theta - v_0 + R \tau^2 Q) + \sqrt{\frac{a}{R}} \ln \left( \frac{1 - \sqrt{aR} (\theta - v_0 + R \tau^2 Q) + (R/a) q_L}{1 + \sqrt{aR} (\theta - v_0 + R \tau^2 Q) - (R/a) q_L} \right). \tag{46}
\]

The exclusion region is \( (\theta_L, v_0 - R \tau^2 Q) \) and can be determined in the same way as \( (v_0 - R \tau^2 Q, \theta_L) \).

Finally, since \( \tilde{z} \to 1 \) and \( z \to 1 \) as \( R \to 0 \), we have

\[
\lim_{R \to 0} \frac{\tilde{\theta}_L - \theta_L}{\sqrt{\sigma R}} = \lim_{R \to 0} \frac{2(\tilde{z} - z)}{\sqrt{2(\tilde{z} + 1)(\tilde{z} + z + 1)}} = \frac{1}{2} (\tilde{\theta} - \theta).
\]

(ii) Proof of Proposition 2

Since \( \tilde{z} \to 1 \) and \( z \to 1 \) as \( R \to 0 \), we have

\[
\lim_{R \to 0} B = \frac{1}{2} \lim_{R \to 0} \frac{\tilde{z} - z}{\sqrt{2(\tilde{z} + 1)}} = \frac{\tilde{\theta} - \theta}{2}.
\]

Moreover,

\[
\frac{dB}{dR} = \frac{(v^2 + R^2)}{R} \frac{2(\tilde{z} - z)}{(\tilde{z} + 1)(\tilde{z} + z + 1)} + \frac{2}{\sqrt{aR}} \frac{(\tilde{z'} + z')^2 + (\tilde{z} + z + 1)^2}{(\tilde{z} + 1)^2 (\tilde{z} + z + 1)^2}.
\]

Applying \( \tilde{z} \to 1 \) and \( z \to 1 \) when \( R \to 0 \) again, there is

\[
\frac{dB}{dR} \sim -\frac{2\gamma^2 \tilde{z'}(\tilde{z} - \tilde{z'})}{3R^2} + \frac{2}{4} \frac{\tilde{z} - \tilde{z'}}{\sqrt{aR}} \sim \left(-\frac{1}{6R^2} + \frac{1}{2R^2}\right) (\tilde{z'} - \tilde{z'}) > 0.
\]

(iii) Proof of Proposition 3

Integrating Equation (24) or Equation (25), we obtain the optimal price schedule:

\[
T(q) = \left(v_0 + \frac{1}{\sqrt{aR}}\right) q + \frac{R \tau^2}{2} (Q - q)^2 - \frac{2}{R} \ln \left( z + \exp \left( \frac{R}{a} q \right) \right) + \text{const},
\]

where \( z = \tilde{z} \) for \( q > 0 \) and \( z = \tilde{z} \) for \( q < 0 \). Applying \( T(0) = 0 \) to it, we get

\[
\text{const} = -\frac{R \tau^2}{2} Q^2 + \frac{2}{R} \ln(1 + z)
\]

and in turn Equation (27). The bid-ask spread has shown in the text.

(iv) Proof of Proposition 4

Substituting Equation (39) into Equation (37) and collecting terms proportional to the same power of \( R \), one obtains

\[
\frac{d}{d\theta} \left[ (\theta - \gamma \sigma^2 q_0(\theta) - v(\theta)) f(\theta) + F(\theta) \right] = 0, \tag{47}
\]

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\[
\frac{d}{d\theta} \left[ (\gamma \sigma^2 q_1 + (\sigma_1^2 + \sigma^2)q_0) f \right] + (\theta - \gamma \sigma^2 q_0 - v(\theta))
\times f \left( \theta - \gamma \sigma^2 q_0 - v(\theta) \right) q_0 + \frac{\sigma_1^2}{\sigma_0^2} (Q - q_0) = 0.
\]

(48)

For the initial conditions of these ordinary equations, Equation (19) implies
\[
[\bar{\theta} - \gamma \sigma^2 q(\bar{\theta}) + R(Q - q(\bar{\theta}))(\sigma_1^2 + \sigma^2) - v(\bar{\theta})] f(\bar{\theta}) = 0.
\]

Using Equation (39) and collecting terms proportional to the same power of \( R \), one obtains
\[
[\bar{\theta} - \gamma \sigma^2 q_1(\bar{\theta}) - v(\bar{\theta})] f(\bar{\theta}) = 0,
\]

(49)

\[
[-\gamma \sigma^2 q_1(\bar{\theta}) + R(Q - q_0(\bar{\theta}))(\sigma_1^2 + \sigma^2)] f(\bar{\theta}) = 0.
\]

(50)

Thus, the solution to Equation (47) with boundary condition (49) and its counterpart \( \bar{\theta} \)
\( q_0(\bar{\theta}) = q_N(\bar{\theta}). \) This result is not surprising at all. \( q_0(\bar{\theta}) \) is the component of \( q(\bar{\theta}) \) which is independent of \( R \), it should be equal to the the allocation determined by a risk-neutral market maker. As we discussed earlier, \( q_0(\bar{\theta}) \) is continuous but it is kink at \( \theta = \theta_N \) and \( \theta = \bar{\theta}_N \) such that \( q_0(\bar{\theta}) = q_0(0) = 0 \) on \( (\bar{\theta}_N, \theta_N) \) but \( q_0(\bar{\theta}) > 0 \) and \( q_0(\bar{\theta}) > 0 \) in \( (\bar{\theta}_N, \theta_N) \). Substituting \( q_0(\bar{\theta}) = q_0(0) = 0 \) into Equation (48) yields
\[
\frac{d}{d\theta} \left[ (\gamma \sigma^2 q_1(\theta) f(\theta)) + (\theta - v(\theta)) f(\theta) \frac{\sigma_1^2}{\sigma_0^2} Q \right] = 0.
\]

(51)

Let us first assume that the market maker’s holding of the risky asset is zero so that Equation (51) implies that \( q_1(\theta) f(\theta) = \text{constant} \) on \( [\bar{\theta}_N, \theta_N] \). Recalling that participation constraint implies that \( q(\theta) = 0 \) and \( f(\theta) > 0 \), there must be \( q_1(\theta) = 0 \) on \( [\bar{\theta}_N, \theta_N] \). However, if \( Q = 0 \), Equation (48) shows \( \frac{d}{d\theta} \left[ (\gamma \sigma^2 q_1 + (\sigma_1^2 + \sigma^2)q_0) f \right] < 0 \) on \( (\bar{\theta}_N, \theta_N + \delta) \) for a sufficiently small \( \delta > 0 \) although \( q_0(\theta) > 0 \) in this area. Therefore, we cannot rule out the possibility that \( q_1(\theta) < 0 \) on \( (\bar{\theta}_N, \theta_N + \delta) \). If it does occur, we need to ‘iron’ \( q_R(\theta) = q_0(\theta) + R q_1(\theta) \) (because of non-participation constraint) to make it zero in the area. In other words, there is \( \theta_R \geq \theta_N \) that \( q_R(\theta) = 0 \) on \( [\theta_N, \theta_R] \). Similar argument is applicable to the area around \( \theta_N \). This means that risk aversion on the market maker side is likely to enlarge non-participation range from \( [\bar{\theta}_N, \theta_N] \) to \( [\theta_R, \theta_R] \), where \( \theta_R \geq \theta_N \) and \( \theta_R \leq \bar{\theta}_N \).

(v) Certainty Equivalent Wealth
Suppose an investor has a random wealth \( w \) with cumulative distribution \( F(w) \) and her Bernoulli utility function is \( v(w). \)

\[
v(w^c) = \int v(w) dF(w)
\]

(52)

(see, e.g. Mas-Colell et al., 1995). The right-hand side of Equation (52) is the von Neumann-Morgenstern expected utility of the investor facing lottery \( F(\cdot) \). Therefore, if the Bernoulli utility function is strictly increasing, the certainty equivalent wealth is a strictly increasing transformation of the von Neumann-Morgenstern utility and the maximisation of the latter is equivalent to maximising the former. In particular, assuming Bernoulli utility is a CARA utility function, that is,
\[
v(w) = \frac{1 - \exp(-\gamma w)}{\gamma},
\]

where \( \gamma > 0 \) is the Arrow–Pratt coefficient of absolute risk aversion (note that \( u(w; \gamma) \rightarrow w \) as \( \gamma \rightarrow 0 \)). Then, if \( w \) is normal with mean \( \mu \) and variance \( \sigma^2_w \) one obtains:

\[
\exp(-\gamma w^c) = \frac{1}{\sqrt{2\pi\sigma^2_w}} \int_{-\infty}^{+\infty} \exp(-\gamma w) \exp\left(-\frac{(w-\mu)^2}{2\sigma^2_w}\right) dw
\]

\[
= \exp\left(-\gamma \mu + \frac{\gamma^2 \sigma^2_w}{2}\right),
\]

which implies
\[
w^c = \mu - \frac{\gamma \sigma^2_w}{2} = E(w) - \frac{\gamma}{2} \text{var}(w).
\]

(54)

For an investor of Section II, if she buys \( q \) units of the asset and pays \( T(q) \) her wealth is:
\[
w = (q + I)(s + \varepsilon) - T(q).
\]

Therefore,
\[
E(w) = (q + I)s - T(q),
\]

(55)

\[
\text{var}(w) = (q + I)^2 \sigma^2.
\]

(56)

Substituting Equations (55) and (56) into Equation (54) yields Equation (2).

\footnote{It is standard in microeconomic theory to refer to the utility function defined on wealth realisation as Bernoulli utility and to its expectation as von Neumann-Morgenstern utility (see, e.g. Mas-Colell et al., 1995).}