Quality Enhancing Network Effect and Endogenous Market Structure*

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Abstract

In many digital products markets, network externality plays an important role to affect the overall quality of the product. However, the parameter which measures the strength or the degree of demand network externality is assumed to be fixed in most of the literature. We propose a model of vertical product differentiation with two competing firms where the strength of the network externality is endogenized as a strategic choice of the high quality firm. This leads to interesting market structures and market coverage in the equilibrium. We find that whether the market will be partially or fully covered and whether the resulting market structure will be monopoly or duopoly come out endogenously in the model. The equilibrium outcomes depend on how costly the investment on the network externality is and the relative difference in respective qualities of the products. In the comparative statics analysis the relation between the optimal degree of network externalities and the relative quality differences of the products is studied for various levels of costliness of investment on the network externality by the high quality firm.

Keywords: Vertical product differentiation, Network externality, Market structure, Market coverage, Investment cost

JEL Classifications: D23, D43, L13, L86

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1. Introduction

It is well known that in a vertically differentiated product market, firms not only compete in prices or quantities, but also compete in other attributes of the product which can be summarized in one measure as overall or effective quality of the product. For any consumer, apart from the price, the overall quality of the product is also an important factor in the decision making process. For the producers, generally, there are two ways to enhance the quality of a product. One is to improve (say through R&D) the intrinsic quality of the existing product so that users get more value from new improved version, and the other is to augment more services directly or indirectly through supporting products and environments to the existing version (without actually enhancing intrinsic quality) to improve the overall or effective usage of the product. The former approach is more conventional type for quality enhancement already studied in details in standard models of vertical product differentiation where the quality is endogenously chosen by the competing firms. The latter approach of quality enhancement is relatively new and getting more relevant in recent times, particularly for products with the feature of network externalities. Most digital products fall in this category where bringing new accessories, supporting devices and suitable applications (apps) to the core product that connect different users enhance the strength of the network effect and hence improve the effective quality of the core product. For example, leading producers of digital products (we call their core products as high quality because typically their products embody the latest and best technology) of smart phones, smart TVs, tablets etc. add various features to their core products over time to improve the impact of network externality among its users. Relatively small or generic producers (we call their core products as low quality in this context, because to make their products relatively cheap they use older technology or embed limited functionality in their product) in the same core product categories also get the benefits of this enhanced strength of network externality for their products; however, due to lower quality of the core products, the absorption capacity of these network effects can be limited. But at the same time, low quality producers completely free ride on the benefit of the enhanced strength of network externality which comes as a result of costly investments from the high quality producers.[1] On the other hand, we will see that the high quality

[1] It is understood that developing new features, apps or supporting devices for the core product needs costly investment over time in the form of R&D. We assume that it is only done by the leading producers (high quality producers) in the industry who have deep pockets for investments.

producers can also use their choice of desired network strength as a strategic device to compete more successfully with the low quality producers and control the market outcomes.

To capture this situation for the analysis, we consider a two-stage game where in the first stage, the high quality firm makes costly investment to improve or enhance the impact of the network externality of its core product. In the second stage it competes with a low quality producer in the same product category. The increase in the impact of network externality is enjoyed by both products (high and low), although the absorption capacity of the network externality of the high quality product is more than that of the low quality product due to the reasons discussed before. Hence the high quality product generates more value (apart from its inherent high quality) to its users compared to the low quality product. Thus, it also gives adequate incentive for the high quality firm to invest in increasing the strength of network externality in stage one.

We solve the two-stage game in this economic environment and find out the precise impact of various levels of costliness of investment on the network externality by the high quality firm for any given level of relative quality difference of the high and low quality product. To simplify the analysis, we normalize the quality of the high quality firm's product to one and low quality firm's product to a parameter less than one. To characterize the equilibrium outcomes of the game, first we find out what would be the optimal levels of the degree of the network externality for the high quality firm for various levels of costliness of investment on the network externality, namely when costs are low, low-medium, medium or high in a relative scale. Then for an optimal degree of network externality, we also find out what would be the ensuing equilibrium market structure and whether the market will be partially or fully covered in that equilibrium.

Our main qualitative finding is when the investment on the network is cheap and the quality difference between the two products is high, the market is monopolized and fully covered by the high quality firm, otherwise the market mostly remains duopoly which may be partially or fully covered. For the comparative statics analysis, we find that if the investment on the network is very cheap or very costly, the optimal degree of network externality is always increasing in quality difference of the products, but for any other levels of costliness of the investment the relationship is generally non-monotonic. The results have interesting strategic and policy implications as we will discuss later.

To relate this study with the conventional literature of vertical product differentiation and place our study in an appropriate context, we also observe the following. In the standard models of vertical product differentiation, one issue that has always received some attention is the extent of market coverage, namely whether it is partially or fully covered. Typically, in most of the economic analysis, the extent of the market coverage whether full or partial is assumed to be exogenous. To the best of our knowledge the only exception is Wauthy (1996) where he explains that whether markets are covered or not should be an endogenous outcome of the quality choice game, if quality is chosen endogenously before the price competition in a two-stage model of vertical product differentiation. He says "covering the market or not is at the heart of the strategic problem for firms". We also extend and reinforce this theme in our proposed model of vertical product differentiation with the additional feature of demand network externality. In our analysis, we will see the market coverage (partial or full) is endogenously determined in the model along with the market structures (i.e. whether it will be a monopoly or duopoly market). However, unlike Wauthy (1996), in our model, we do not allow the firms to choose the intrinsic quality of their respective products, they are assumed to be fixed, but allow the high quality firm to control or choose the degree or strength of the network externality to affect the overall quality of the product. This also enables us to depart from the typical assumption of a given degree of network externality (as parameter) as is assumed in most models with network externality. We endogenize the degree of network externality as a strategic choice of a firm.

The impact of network externality or network effects on various economic situations is studied in the literature in several contexts. There is a sizeable research discussing the effects of network externality in economic contexts which originated from the studies by Rohlfs (1974), Katz and Shapiro (1985), followed by Chou and Shy (1990), Church and Gandal (1993) among many others.[2] In another area, where the impact of network externality is getting a renewed interest is copyright violations or piracy in the digital products market (see the work done by Conner and Rumelt 1991, Takeyama 1994, Shy and Thisse 1999, Banerjee 2003, 2013 among others). However, the research dealing with the impact of network externality on consumer and

[2] Grilo et. al (2001) modelled social pressures such as conformity or vanity in terms of consumption externalities in a model of spatial duopoly and characterized various possible equilibrium outcomes and market structures. Lambertini and Orsini (2005) considered a model of vertical product differentiation with the feature of positive network externality and focused on the existence of quality-price equilibrium. However, in their model, network externality is only product specific and does not have any cross effect on the other product.

producer behaviour in all the above studies assumes the degree or strength of the network effect as an exogenous parameter in the analysis. We believe this is not always satisfactory, and needs to be endogenized in the model to capture relevant economic aspects. In the modern environment of digital products market with the feature of network externality, a leading firm in this industry is not only just interested to increase the size of its users-base i.e. the network size of its products, it is also interested to enhance the *strength* of the network effect by making appropriate investments, which consequently increases the overall value of the products. In the context of our analysis, adding new apps, features and supporting devices to the core product over time exactly does that and enhances the strength of network externality. It is an alternative way to enhance the effective quality of an existing product without investing to improve the intrinsic quality of the product which may not also be feasible in many circumstances. In this paper, we capture this aspect in detail through a model of vertical product differentiation.

The rest of the paper is organized as follows. In section 2, we present the basic model. Sections 3 and 4 analyze the price competition between the two firms when the market is partially and fully covered respectively, assuming the degree of demand network externality as a parameter. In section 5, we endogenize the degree of network externality. The high quality firm strategically chooses the optimal degree of network externality; and the market structure and market coverage are endogenously determined in the equilibrium. We summarize our main results. In section 6 we conclude with a discussion.

2. The Model

A two-stage game is considered in a vertically differentiated product market with two firms. Each firm is producing a core product. The high quality product is denoted by H and the low quality product is denoted by L. We assume both products are in the same product category.[3] There is a continuum of consumers distributed uniformly over a unit interval with heterogeneous preferences towards product quality. The products also exhibit the feature of positive network externality.[4] However, as explained before, the impact of the network externality is *asymmetric* between the users of the high quality product and low quality product. The users of the high

[3] For example, say the product category is smart phone.

[4] We rule out the case of any negative network externality in this model.

quality product enjoy a higher level of network externality compared to the users of the low quality product as the absorption capacity of the network effect of the high quality product is more than that of the low quality product.[5]

In terms of utility, the consumer who buys the high quality product, first of all, gets all the intrinsic benefit from the product due to its high quality; secondly she also enjoys the full extent of the network externality generated by those users who also buy the high quality product, plus the (limited) network externality generated by the low quality product users. The buyers of the low quality product can enjoy all the value of the product (intrinsic as well as network) subject to limitation that the lower quality can permit. We normalize the quality of the high quality firm's product to one. The low quality product is indexed by q, $q \in (0,1)$ where q captures the quality depreciation.

Formally, the utility of a typical consumer X ($X \in [0,1]$) is given as follows: [6]

$$U = \begin{cases} X + \gamma D_H + q\gamma D_L - p_H & \text{if buys high quality product,} \\ q\left(X + \gamma D_H + \gamma D_L\right) - p_L & \text{if buys low quality product,} \\ 0 & \text{if buys none,} \end{cases}$$

where D_H, p_H and D_L, p_L are the demand and prices for the high quality and low quality products respectively. $\gamma > 0$ is the coefficient which measures the *degree or strength* of network externalities. For example, higher γ implies stronger effect of network externality, whereas when γ is close to zero, it implies almost no effect of network externality. In our model only the high quality firm can influence γ. The interpretation is choosing a higher level of γ is same as choosing more useful features, apps or supporting devices for the core product to enhance the strength of the network externality.

[5] High quality product has all latest features functional to absorb all the network effect, while the low quality product has limited functionality and absorption capacity of the network effect.

[6] The utility representation is borrowed from the standard model of vertical product differentiation in the literature (see Shaked and Sutton, 1982; Tirole, 1988, see also Banerjee 2003, 2013). The parameter q can be interpreted as a quality index for the low quality product. If q is interpreted as the probability that the low quality product is functional, then we make the additional assumption that if the low quality product is operational for one consumer then it is also operational for every other consumer. Note that $q = 0$ will eliminate the low quality product, while $q = 1$ will make the two products identical. We eliminate these two uninteresting cases from the outset. Also technically, $q \in (0,1)$ is needed so that demands, prices and profits are not indeterminate.

We consider a two-stage game where in stage 1, the high quality firm chooses how much to invest to improve the strength of network externality, i.e. chooses a level of γ. In stage 2, the high quality firm and the low quality firm compete in prices in the product market for a given level of γ. The high quality firm can choose the degree of network externality γ by incurring a cost of $c(\gamma)=\frac{1}{2}k\gamma^2$, where $k>0$ measures how costly it is to invest in enhancing the degree of network externality. For simplicity, we assume the costs of production for the firms are zero. The size of the market is normalized to 1. We look for the subgame perfect Nash equilibrium of this game and work backward.

Consider the price competition stage. We will first deal with cases when the market is partially covered and when it is fully covered separately. The market coverage aspect gets endogenized later when γ is chosen.

3. Partial Market Coverage

The marginal consumer X_m, who is indifferent between buying the high quality and low quality product, is given by

$$X_m+\gamma D_H+q\gamma D_L-p_H=q\left(X_m+\gamma D_H+\gamma D_L\right)-p_L.$$

This gives us $X_m=\frac{p_H-p_L}{1-q}-\gamma D_H$. The marginal consumer Y_m, who is indifferent between buying the low quality product and buying none, is given by

$$q\left(Y_m+\gamma D_H+\gamma D_L\right)-p_L=0.$$

We thus have $Y_m=\frac{p_L}{q}-\gamma\left(D_H+D_L\right)$.

The demand for the high quality product is given by $D_H=1-X_m$; we then obtain

$$D_H=\frac{1}{1-\gamma}\left[1-\frac{p_H-p_L}{1-q}\right]. \tag{1}$$

The demand for the low quality product is given by:

$$D_L = X_m - Y_m = \frac{p_H q - p_L}{q(1-q)(1-\gamma)}.^{7} \tag{2}$$

The high and low quality firms compete by choosing prices strategically. The Nash equilibrium prices and demands are

$$p_H^* = \frac{2(1-q)}{4-q}, \; p_L^* = \frac{q(1-q)}{4-q}, \; D_H^* = \frac{2}{(1-\gamma)(4-q)}, \; D_L^* = \frac{1}{(1-\gamma)(4-q)}.$$

The profits of the high and low quality firms are respectively

$$\pi_H^* = \frac{4(1-q)}{(1-\gamma)(4-q)^2}, \; \pi_L^* = \frac{q(1-q)}{(1-\gamma)(4-q)^2}.$$

Under this case, the market structure is always duopoly.

Next we deal with the case of full market coverage.

4. Full Market Coverage

Note that we have normalized the size of the market to be 1. There will be an upper bound of the degree of network effect γ for which $D_H^* + D_L^* = 1$. Denote that upper bound by $\hat{\gamma}$. From the previous analysis, we find $\hat{\gamma} = \frac{1-q}{4-q}$. So when $\gamma \geq \hat{\gamma}$, the market is always fully covered.

Now, when the market is fully covered, we have the following analysis. First, in the case of full market coverage, we can distinguish two cases: (i) both the firms are active (i.e. duopoly); and (ii) only the high quality firm serves the whole market (i.e. monopoly) Case (ii) arises when γ is above a threshold value denoted by $\tilde{\gamma}$ (which will be defined later), and case (i) realizes when $\hat{\gamma} \leq \gamma < \tilde{\gamma}$. When case (ii) arises the market actually gets back to monopoly from duopoly as we will see that the low quality firm cannot sell its product even if it sets its price at marginal cost. Thus, market monopolization happens endogenously here as the strength of the network effect γ crosses a certain threshold.

Now we will analyze case (i) in Section 4.1 and case (ii) in Section 4.2 under the full market coverage in detail.

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[7] To be precise, this demand system is correct only when $p_H q \geq p_L$. However, it can be shown that in equilibrium we do have $p_H q > p_L$. For the sake of simplicity, here we omit the demand system when $p_H q < p_L$.

4.1 $\hat{\gamma} \le \gamma < \tilde{\gamma}$ (Moderate degree of network externality)

Here, the demand for the high quality firm remains same as in the previous case, i.e. equation (1), while the demand for the low quality firm is

$$D_L = 1 - D_H = \frac{1}{1-\gamma}\left(\frac{p_H - p_L}{1-q} - \gamma\right). \tag{3}$$

Note that in this case, full coverage of the market implies the consumer with *X=0* also obtains nonnegative surplus, which can be expressed as $q\gamma - p_L \ge 0$. It turns out we need to distinguish two subcases (a) and (b). In subcase (a) the consumer with *X=0* has no surplus while the consumer gets positive surplus in subcase (b).

The Nash equilibrium prices, demands and each firm's profit are given below. The detailed analysis is presented in Appendix A.

Subcase (a) When $\frac{1-q}{4-q} = \hat{\gamma} \le \gamma \le \gamma = \frac{1-q}{2+q}$:

$$p_H^* = \frac{1-q+q\gamma}{2}, \quad p_L^* = q\gamma, \; D_H^* = \frac{(1-q)+q\gamma}{2(1-q)(1-\gamma)}, D_L^* = \frac{(1-q)-(2-q)\gamma}{2(1-q)(1-\gamma)}, \; \pi_H^* = \frac{\left((1-q)+q\gamma\right)^2}{4(1-q)(1-\gamma)},$$

$$\pi_L^* = q\gamma\frac{(1-q)-(2-q)\gamma}{2(1-q)(1-\gamma)};$$

Subcase (b) When $\frac{1-q}{2+q} = \gamma < \gamma < \gamma = \frac{1}{2}$:

$$p_H^* = \frac{1-q}{3}(2-\gamma), \quad p_L^* = \frac{1-q}{3}(1-2\gamma), \; D_H^* = \frac{2-\gamma}{3(1-\gamma)}, \; D_L^* = \frac{1-2\gamma}{3(1-\gamma)}, \; \pi_H^* = \frac{(1-q)(2-\gamma)^2}{9(1-\gamma)},$$

$$\pi_L^* = \frac{(1-q)(1-2\gamma)^2}{9(1-\gamma)}.$$

4.2 $\gamma \ge \tilde{\gamma}$ (High degree of network externality)

When $\gamma = \tilde{\gamma} = \frac{1}{2}$, from the expression of the price and demand of the low quality firm (in the previous section 4.1, subcase (b)), we find that they go to zero (hence the profit as well). This means when γ reaches that threshold or beyond, the low quality firm is unable to compete

profitably, market becomes monopoly and all consumers buy the high quality product even if the low quality product is free (strictly speaking, sold at marginal cost). To ensure the consumer with the lowest $X(=0)$ buys the high quality product, the following condition has to be satisfied: $\gamma D_H - p_H \geq q(\gamma D_H + \gamma D_L) - p_L \Leftrightarrow \gamma - p_H \geq q\gamma$ (since $D_H = 1, D_L = 0, p_L = 0$). It is not hard to see that the high quality firm charges a price $(1-q)\gamma$ to maximize its profit and its total profit is also $(1-q)\gamma$ (recall that the size of the market is normalized to 1). This is true irrespective of the quality of the low quality product.

5. Choice of the Optimal Degree of Network Externality

The high quality firm's profit in the second stage can be summarized as

$$\pi_H^* = \begin{cases} \dfrac{4(1-q)}{(1-\gamma)(4-q)^2} & \text{if } \gamma \leq \dfrac{1-q}{4-q} \\ \dfrac{((1-q)+q\gamma)^2}{4(1-q)(1-\gamma)} & \text{if } \dfrac{1-q}{4-q} \leq \gamma \leq \dfrac{1-q}{2+q} \\ \dfrac{(1-q)(2-\gamma)^2}{9(1-\gamma)} & \text{if } \dfrac{1-q}{2+q} \leq \gamma \leq \dfrac{1}{2} \\ (1-q)\gamma & \text{if } \gamma \geq \dfrac{1}{2} \end{cases}$$

In Stage 1, the high quality firm chooses γ to maximize its net profit $\pi^N(\gamma) = \pi_H^*(\gamma) - \frac{1}{2}k\gamma^2$.

Denote $f(\gamma) = \dfrac{4(1-q)}{(1-\gamma)(4-q)^2} - \dfrac{1}{2}k\gamma^2$ for $\gamma \leq \dfrac{1-q}{4-q}$, $g(\gamma) = \dfrac{((1-q)+q\gamma)^2}{4(1-q)(1-\gamma)} - \dfrac{1}{2}k\gamma^2$ for $\dfrac{1-q}{4-q} \leq \gamma \leq \dfrac{1-q}{2+q}$, $h(\gamma) = \dfrac{(1-q)(2-\gamma)^2}{9(1-\gamma)} - \dfrac{1}{2}k\gamma^2$ for $\dfrac{1-q}{2+q} \leq \gamma \leq \dfrac{1}{2}$, $j(\gamma) = (1-q)\gamma - \dfrac{1}{2}k\gamma^2$ for $\gamma \geq \dfrac{1}{2}$.

5.1 Preliminary analysis and preliminary results

Consider $f(\gamma)$ first. We can get $f'(\gamma)=\gamma\left(\frac{4(1-q)}{\gamma(1-\gamma)^2(4-q)^2}-k\right)$. Since $\frac{1}{\gamma(1-\gamma)^2}$ is decreasing in γ for $\gamma\le\frac{1-q}{4-q}$, in this range of γ, $\frac{4(1-q)}{\gamma(1-\gamma)^2(4-q)^2}$ is minimized when evaluated at $\gamma=\frac{1-q}{4-q}$ and the minimum is $\frac{4(4-q)}{9}$. Therefore, $f(\gamma)$ is maximized at $\gamma_1^*=\frac{1-q}{4-q}$ if $k\le\frac{4(4-q)}{9}$; otherwise, $f(\gamma)$ is maximized at an interior $\gamma_1^*<\frac{1-q}{4-q}$.[8]

Next we examine $g(\gamma)$. We can get $g'(\gamma)=\gamma\left(\frac{1-q^2(1-\gamma)^2}{4\gamma(1-\gamma)^2(1-q)}-k\right)$. It can be shown that $\frac{1-q^2(1-\gamma)^2}{4\gamma(1-\gamma)^2(1-q)}$ is decreasing in γ for $\frac{1-q}{4-q}\le\gamma\le\frac{1-q}{2+q}$ when $q\ge 0.26185$. Hence, when $q\ge 0.26185$, in this range of γ, $\frac{1-q^2(1-\gamma)^2}{4\gamma(1-\gamma)^2(1-q)}$ is maximized when evaluated at $\gamma=\frac{1-q}{4-q}$ and the maximum is $\frac{2(4-q)(2+q)}{9(1-q)}$ while it is minimized when evaluated at $\gamma=\frac{1-q}{2+q}$ and the minimum is $\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}$. Therefore, when $q\ge 0.26185$, $g(\gamma)$ is maximized at $\gamma_2^*=\frac{1-q}{2+q}$ if $k\le\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}$, is maximized at $\gamma_2^*=\frac{1-q}{4-q}$ if $k\ge\frac{2(4-q)(2+q)}{9(1-q)}$, while is maximized at an interior $\gamma_2^*\in\left(\frac{1-q}{4-q},\frac{1-q}{2+q}\right)$ if $k\in\left(\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2},\frac{2(4-q)(2+q)}{9(1-q)}\right)$. On the contrary, when $q\le 0.26185$, it can be shown that in the range of $\frac{1-q}{4-q}\le\gamma\le\frac{1-q}{2+q}$, $\frac{1-q^2(1-\gamma)^2}{4\gamma(1-\gamma)^2(1-q)}$ is decreasing in γ first and then

[8] The second-order condition can be verified. This also applies in the analysis below when relevant.

increasing. Let k denote the minimum of $\frac{1-q^2(1-\gamma)^2}{4\gamma(1-\gamma)^2(1-q)}$ in this range of γ. Therefore, when $q \le 0.26185$, $g(\gamma)$ is maximized at $\gamma_2^* = \frac{1-q}{2+q}$ if $k \le k$, is maximized at $\gamma_2^* = \frac{1-q}{4-q}$ if $k \ge \max\left\{\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}, \frac{2(4-q)(2+q)}{9(1-q)}\right\}$, [9] while is maximized at an interior $\gamma_2^* \in \left(\frac{1-q}{4-q}, \frac{1-q}{2+q}\right)$ or at one of the limits of this range if $k \in \left(k, \max\left\{\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}, \frac{2(4-q)(2+q)}{9(1-q)}\right\}\right)$.

We now examine $h(\gamma)$. We can get $h'(\gamma) = \gamma\left(\frac{(1-q)(2-\gamma)}{9(1-\gamma)^2} - k\right)$. Since $\frac{2-\gamma}{(1-\gamma)^2}$ is increasing in γ for $\frac{1-q}{2+q} \le \gamma \le \frac{1}{2}$, in this range of γ, $\frac{(1-q)(2-\gamma)}{9(1-\gamma)^2}$ is minimized when evaluated at $\gamma = \frac{1-q}{2+q}$ and the minimum is $\frac{(2+q)(1+q)(1-q)}{3(1+2q)^2}$ while it is maximized when evaluated at $\gamma = \frac{1}{2}$ and the maximum is $\frac{2(1-q)}{3}$. Therefore, $h(\gamma)$ is maximized at $\gamma = \frac{1}{2}$ if $k \le \frac{(2+q)(1+q)(1-q)}{3(1+2q)^2}$, while it is maximized at $\gamma = \frac{1-q}{2+q}$ if $k \ge \frac{2(1-q)}{3}$. If $k \in \left(\frac{(2+q)(1+q)(1-q)}{3(1+2q)^2}, \frac{2(1-q)}{3}\right)$, $h(\gamma)$ is decreasing first and then increasing in γ. By comparing $h\left(\frac{1-q}{2+q}\right)$ and $h\left(\frac{1}{2}\right)$ and combining the results stated earlier in this paragraph, we

[9] Note that $\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2} < \frac{2(4-q)(2+q)}{9(1-q)}$ for $q > 0.08082$ while the reverse is true for $q < 0.08082$.

conclude that $h(\gamma)$ is maximized at $\gamma=\frac{1}{2}$ if $k\le\frac{4(2+q)(1-q)}{3(1+2q)(4-q)}$, while it is maximized at $\gamma=\frac{1-q}{2+q}$ if $k\ge\frac{4(2+q)(1-q)}{3(1+2q)(4-q)}$.

Finally we examine $j(\gamma)$. Since $j'(\gamma)=(1-q)-k\gamma$, $j(\gamma)$ is maximized at $\gamma=\frac{1-q}{k}$ if $k\le 2(1-q)$, and is maximized at $\gamma=\frac{1}{2}$ if $k\ge 2(1-q)$.

As we can see from the above analysis, it is impossible to derive the optimal level of network externality for general values of *k* and *q*. Hence, we turn to numerical examples and simulation.

Before we turn to numerical examples and simulation, let's point out some facts which are useful for further analysis.

(i) The net profit function $\pi^N(\gamma)=\pi_H^*(\gamma)-\frac{1}{2}k\gamma^2$ is continuous in γ.

(ii) For any $q\in(0,1)$, $\frac{4}{3}<\frac{4(4-q)}{9}<\frac{16}{9}$, $\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}>\frac{4(4-q)}{9}$, $\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}$ is decreasing in *q* first and then increasing, is minimized at $q=0.0636$ and the minimum is 1.970, $\frac{2(4-q)(2+q)}{9(1-q)}$ is increasing in *q* and the minimum is $\frac{16}{9}=1.778$.

(iii) $k>\frac{27}{16}$; this is because

$$k=\min_{\frac{1-q}{4-q}\le\gamma\le\frac{1-q}{2+q}}\frac{1-q^2(1-\gamma)^2}{4\gamma(1-\gamma)^2(1-q)}>\min_{\gamma}\frac{1}{4\gamma(1-\gamma)^2(1-q)}-\max_{\frac{1-q}{4-q}\le\gamma\le\frac{1-q}{2+q}}\frac{q^2}{4\gamma(1-q)}=\frac{27}{16(1-q)}-\frac{q^2(4-q)}{4(1-q)^2}$$
$$=\frac{27-27q-16q^2+4q^3}{16(1-q)^2}>\frac{27}{16}.$$

We also present four preliminary results in the form of lemmas before we move to numerical examples and simulation.

Lemma 1: $\gamma = \frac{1}{2}$ can never be optimal.

Proof. See Appendix B.

This result tells us that we can ignore subcase (b) mentioned in Section 4.1 since any level of network externality in the range $\left(\frac{1-q}{2+q}, \frac{1}{2}\right]$ can never be optimal for the high quality firm.[10] So, to determine the optimal level of network externality, we only need to compare the high quality firm's profits when γ takes the values of γ_1^*, γ_2^* and $\frac{1-q}{k}$ respectively; namely, we need to compare $f\left(\gamma_1^*\right)$, $g\left(\gamma_2^*\right)$ and $j\left(\frac{1-q}{k}\right)$, where γ_1^* and γ_2^* denote the optimal levels of network externality in the range $\left(0, \frac{1-q}{4-q}\right]$ and $\left[\frac{1-q}{4-q}, \frac{1-q}{2+q}\right]$ respectively.

Lemma 2: $\gamma = \frac{1-q}{4-q}$ can never be optimal.

Proof. See Appendix B.

This result tells us that we can ignore the case of $\gamma \le \frac{1-q}{4-q}$ for $k \le \frac{4(4-q)}{9}$. When the investment on network externality is relatively cheap, the high quality firm will make sufficiently large investment such that the market is fully covered.

Lemma 3: When $k \le \frac{4}{3}$, we only need to compare $g\left(\frac{1-q}{2+q}\right)$ and $j\left(\frac{1-q}{k}\right)$ to determine the optimal level of network externality.[11]

Proof. See Appendix B.

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[10] Also because the net profit function is continuous, $\gamma = \frac{1-q}{2+q}$ can be regarded as a special case of subcase (a).

[11] Of course, this applies only when $j(\gamma)$ is maximized at $\gamma = \frac{1-q}{k}$, i.e., when $k \le 2(1-q)$.

Lemma 4: A necessary condition for $\gamma = \frac{1-q}{k}$ to be optimal is $k < 1.8057$.

Proof. See Appendix B.

This result tells us that when the investment on enhancing network externality is sufficiently costly, the high quality firm will not choose a sufficiently high level of network externality such that it becomes a monopolist.

Based on these facts and preliminary results, we will now provide the numerical examples for *k*=1, 1.5, 1.8, 2, 6. These particular values of *k* are chosen to capture all the relevant situations necessary for the complete analysis. When the cost of investment on the network is low then *k*=1, 1.5; when the cost is low-medium then *k*=1.8; when it is medium then *k*=2; and when it is high then *k*=6.

5.2 Numerical Examples

5.2.1 *k*=1

According to Lemma 3, we only need to compare $g\left(\frac{1-q}{2+q}\right)$ and $j\left(\frac{1-q}{k}\right)$. This applies only when $q \le 1-\frac{k}{2}$. When $q \ge 1-\frac{k}{2}$ instead, the high quality firm's profit is maximized at $\gamma = \frac{1-q}{2+q}$. Straightforward computation yields

$$j\left(\frac{1-q}{k}\right) - g\left(\frac{1-q}{2+q}\right)$$
$$= \frac{1-q}{2k(1+2q)(2+q)^2}\left[(2-k)^2 + \left(k^2-10k+8\right)q - \left(2k^2+8k+3\right)q^2 - (2k+7)q^3 - 2q^4\right].$$

Define $z(q) = (2-k)^2 + \left(k^2-10k+8\right)q - \left(2k^2+8k+3\right)q^2 - (2k+7)q^3 - 2q^4$. It is straightforward to get $z(0) = (2-k)^2 > 0$, $z(1) = -24k < 0$ and $z''(q) = -2\left[\left(2k^2+8k+3\right) + 3(2k+7)q + 12q^2\right] < 0$. So when $q < q^c(k)$, $\gamma = \frac{1-q}{k}$ is optimal;

when $q > q^c(k)$, $\gamma = \frac{1-q}{2+q}$ is optimal, where $q^c(k)$ denotes the quality q such that $z(q^c(k)) = 0$ and comparative static analysis tells us that $q^c(k)$ is decreasing in *k*.

Plugging *k*=1 into the above expression of $z(q)$, we get $z(q) = 1 - q - 13q^2 - 9q^3 - 2q^4$. Then we can get the result summarized in the following proposition.

Proposition 1: When *k*=1, the optimal level of network externality chosen by the high quality firm is

$$\gamma^* = \begin{cases} 1-q & \text{if } q \le q^c(k) = 0.226 \\ \frac{1-q}{2+q} & \text{if } q \ge q^c(k) = 0.226 \end{cases}.$$

Equilibrium Characterization and Comparative Statics

Under this, the market is always fully covered and the market structure can be a monopoly or duopoly. The high quality firm is a monopolist when *q* is relatively small (as defined above) while both the high quality firm and the low quality firm share the market (duopoly) when *q* is relatively big (as defined above). When the market is shared, the optimal level of network externality chosen by the high quality firm is the upper limit of the level of network externality such that the consumer with X=0 earns zero surplus.

The optimal level of network externality is always monotonically decreasing in *q*. In each range, the optimal level of network externality (γ) is decreasing in *q*. Also note that when $q = q^c(k)$=0.226, the high quality firm is indifferent between $\gamma = 1 - q$ and $\gamma = \frac{1-q}{2+q}$, we have $1 - q > \frac{1-q}{2+q}$. Therefore, overall, the optimal level of network externality is decreasing in *q*.

5.2.2 *k*=1.5

When *k*=1.5, we have the following facts: (1) $f(\gamma)$ is maximized at $\gamma = \frac{1-q}{4-q}$ if $q \le 0.625$; otherwise, $f(\gamma)$ is maximized at an interior $\gamma_1^* < \frac{1-q}{4-q}$; (2) $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{2+q}$; (3)

$j(\gamma)$ is maximized at $\gamma = \frac{2(1-q)}{3}$ if $q \le 0.25$, and is maximized at $\gamma = \frac{1}{2}$ if $q \ge 0.25$. Therefore, when $0.25 \le q \le 0.625$, the high quality firm's profit is maximized at $\gamma = \frac{1-q}{2+q}$; when $q \le 0.25$, we need to compare $g\left(\frac{1-q}{2+q}\right)$ and $j\left(\frac{2(1-q)}{3}\right)$; when $q \ge 0.625$, we need to compare $g\left(\frac{1-q}{2+q}\right)$ and $f\left(\gamma_1^*\right)$.

The comparison between $g\left(\frac{1-q}{2+q}\right)$ and $j\left(\frac{2(1-q)}{3}\right)$ is easy. After plugging *k*=1.5 into the expression of $z(q)$, we find that when $q < 0.0444$, $\gamma = \frac{2(1-q)}{3}$ is optimal; when $0.0444 < q < 0.25$, $\gamma = \frac{1-q}{2+q}$ is optimal.

The comparison between $g\left(\frac{1-q}{2+q}\right)$ and $f\left(\gamma_1^*\right)$ is more complicated since we cannot write down a closed-form of γ_1^*. However, numerical calculations show that when *k*=1.5, for any $q \ge 0.625$, $g\left(\frac{1-q}{2+q}\right) > f\left(\gamma_1^*\right)$ always holds true.

We thus have,

Proposition 2: When *k*=1.5, the optimal level of network externality chosen by the high quality firm is

$$\gamma^* = \begin{cases} \frac{2(1-q)}{3} & \text{if } q \le q^c(k) = 0.0444 \\ \frac{1-q}{2+q} & \text{if } q \ge q^c(k) = 0.0444 \end{cases}.$$

This result implies that when *k* is bigger than but sufficiently close to 4/3, even though $f(\gamma)$ is maximized at $\gamma_1^* < \frac{1-q}{4-q}$ for big values of *q*, $g\left(\frac{1-q}{2+q}\right)$ is still greater than $f\left(\gamma_1^*\right)$ for all

values of q. We may expect this inequality does not hold true after k crosses a threshold. Indeed this is true.

5.2.3 *k*=1.8

When k=1.8, we have the following facts: (1) $f(\gamma)$ is maximized at an interior $\gamma_1^* < \frac{1-q}{4-q}$; (2) $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{2+q}$ if $q \geq 0.26185$, but if $q < 0.26185$, we need to calculate at which value of γ, $g(\gamma)$ is maximized for specific values of q; (3) $j(\gamma)$ is maximized at $\gamma = \frac{5(1-q)}{9}$ if $q \leq 0.1$, and is maximized at $\gamma = \frac{1}{2}$ if $q \geq 0.1$. Therefore, when $q \leq 0.1$, we need to compare $f(\gamma_1^*)$, $g(\gamma_2^*)$ and $j\left(\frac{5(1-q)}{9}\right)$; when $0.1 \leq q \leq 0.26185$, we need to compare $f(\gamma_1^*)$ and $g(\gamma_2^*)$; when $q \geq 0.26185$, we need to compare $g\left(\frac{1-q}{2+q}\right)$ and $f(\gamma_1^*)$.

As we have seen in Lemma 4, a necessary condition for $\gamma = \frac{1-q}{k}$ to be optimal is $k < 1.8057$. Since 1.8 is so close to 1.8057 (this is exactly the reason why we choose the case of k=1.8 as one numerical example), it can be expected that $\gamma = \frac{5(1-q)}{9}$ is optimal only for very small values of q or it is never optimal for any value of q. Indeed, after plugging k=1.8 into the expression of $z(q)$, we find that only when $q < 0.005798$, $\gamma = \frac{5(1-q)}{9}$ is preferred to $\gamma = \frac{1-q}{2+q}$. We can also derive an even stricter necessary condition by comparing $j\left(\frac{5(1-q)}{9}\right)$ and $f\left(\frac{1-q}{4-q}\right)$: $q < 0.001784$. Numerical analysis shows that $\gamma = \frac{5(1-q)}{9}$ is optimal for $q \leq q_1^c \approx 0.0017$.

We can also derive a necessary condition for $\gamma = \frac{1-q}{2+q}$ to be optimal by comparing $g\left(\frac{1-q}{2+q}\right)$ and $f\left(\frac{1-q}{4-q}\right)$: $q > 0.02255$. Numerical analysis shows that $\gamma = \frac{1-q}{2+q}$ is optimal for $q > q_2^c \approx 0.023$.

We thus have,

Proposition 3: When k=1.8, the optimal level of network externality chosen by the high quality firm is

$$\gamma^* = \begin{cases} \frac{5(1-q)}{9} & \text{if } q \le q_1^c \approx 0.0017 \\ \gamma_1^*(q) < \frac{1-q}{4-q} & \text{if } q_1^c \le q \le q_2^c \\ \frac{1-q}{2+q} & \text{if } q \ge q_2^c \approx 0.023 \end{cases}.$$

Equilibrium Characterization and Comparative Statics

Here the market may be fully or partially covered depending on the quality of the low quality product and market structure can be a monopoly or duopoly. When q is sufficiently small (as defined above), the high quality firm chooses a high level of network externality and becomes a monopolist and the market is fully covered; when q is small but not sufficiently small, the high quality firm chooses a low level of network externality, both the high quality firm and the low quality firm share the market (duopoly), and the market is partially covered. After q crosses a bigger threshold, the high quality firm chooses a moderate level of network externality, both the high quality firm and the low quality firm share the market (duopoly), and the market is fully covered again, and in this case, the optimal level of network externality chosen by the high quality firm is the upper limit of the level of network externality such that the consumer with X=0 earns zero surplus.

The relationship between the optimal level of network externality and q is not monotonic. In each range, the optimal level of network externality (γ) is decreasing in q. Also note that when

$q = q_1^c$, the high quality firm is indifferent between $\gamma = \frac{5(1-q)}{9}$ and $\gamma = \gamma_1^*(q)$, we have $\frac{5(1-q)}{9} > \gamma_1^*(q)$, and that when $q = q_2^c$, the high quality firm is indifferent between $\gamma = \gamma_1^*(q)$ and $\gamma = \frac{1-q}{2+q}$, we have $\gamma_1^*(q) < \frac{1-q}{2+q}$. Therefore, overall, the relationship between the optimal level of network externality and q is not monotonic.

5.2.4 *k*=2

When *k*=2, we have the following facts: (1) $f(\gamma)$ is maximized at an interior $\gamma_1^* < \frac{1-q}{4-q}$; (2) $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{2+q}$ if $q \geq 0.26185$, $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{4-q}$ if $q \leq 0.09167$,but if $0.09167 < q < 0.26185$, we need to calculate at which value of γ, $g(\gamma)$ is maximized for specific values of *q*; (3) $j(\gamma)$ is maximized at $\gamma = \frac{1}{2}$. Therefore, when $q \leq 0.09167$, the high quality firm's profit is maximized at $\gamma_1^* < \frac{1-q}{4-q}$; when $0.09167 < q < 0.26185$, we need to compare $f(\gamma_1^*)$ and $g(\gamma_2^*)$; when $q \geq 0.26185$, we need to compare $f(\gamma_1^*)$ and $g\left(\frac{1-q}{2+q}\right)$.

We can derive a necessary condition for $\gamma = \frac{1-q}{2+q}$ to be optimal by comparing $g\left(\frac{1-q}{2+q}\right)$ and $f\left(\frac{1-q}{4-q}\right)$: $q > 0.14772$. Numerical analysis shows that $\gamma = \frac{1-q}{2+q}$ is optimal for $q \geq 0.1644$.

We thus have,

Proposition 4: When *k*=2, the optimal level of network externality chosen by the high quality firm is

$$\gamma^* = \begin{cases} \gamma_1^*(q) < \dfrac{1-q}{4-q} & \text{if } q \le q^c(k) = 0.1644 \\ \dfrac{1-q}{2+q} & \text{if } q \ge q^c(k) = 0.1644 \end{cases}.$$

Equilibrium Characterization and Comparative Statics

The market may be fully or partially covered depending on the quality of the low quality product, however it is always shared between the two firms (i.e. market structure is always duopoly). When q is small, the high quality firm chooses a low level of network externality and the market is partially covered; when q becomes bigger, the high quality firm chooses a moderate level of network externality and the market is fully covered, and in this case, the optimal level of network externality chosen by the high quality firm is the upper limit of the level of network externality such that the consumer with X=0 earns zero surplus.

The relationship between the optimal level of network externality and q is not monotonic. In each range, the optimal level of network externality (γ) is decreasing in q. Also note that when $q = q^c(k)$=0.1644, the high quality firm is indifferent between $\gamma = \gamma_1^*(q)$ and $\gamma = \dfrac{1-q}{2+q}$, $\gamma_1^*(q) < \dfrac{1-q}{2+q}$. Therefore, overall, the relationship between the optimal level of network externality and q is not monotonic.

5.2.4 *k*=6

From the numerical examples above, we observe that as q increases from close to 0 to close to 1, the high quality firm chooses the optimal level of network externality such that the market is always fully covered when k is less than a critical value k_1^c which lies between 1.78 and 1.79; [12] the high quality firm chooses the optimal level of network externality such that the market changes from fully covered to partially covered and then to fully covered again when k is greater than k_1^c but less than another critical value k_2^c which lies between 1.8 and 1.8057; when k is ever greater than k_2^c, the high quality firm chooses the optimal level of network externality such that

[12] More numerical analysis yields this.

the market changes from partially covered to fully covered. One may expect that when the investment on network externality is sufficiently costly (k is bigger than k_3^c), the high quality firm chooses the optimal level of network externality such that the market is always partially covered. This is indeed true.

Since when k is relatively big (but not sufficiently big), the full market coverage arises when the high quality firm chooses $\gamma^* = \frac{1-q}{2+q}$, we are going to find a sufficiently big value of k such that $\gamma = \frac{1-q}{2+q}$ is not optimal. This can be done by comparing the high quality firm's profit between $g\left(\frac{1-q}{2+q}\right)$ and $f(0)$. Straightforward computation yields

$$f(0) - g\left(\frac{1-q}{2+q}\right)$$
$$= \frac{(1-q)^2}{2(1+2q)(2+q)^2(4-q)^2}\left[16k - 32 + (24k-64)q - (15k+44)q^2 - (6-2k)q^3 + 2q^4\right].$$

Define $y(q) = 16k - 32 + (24k-64)q - (15k+44)q^2 - (6-2k)q^3 + 2q^4$. It is straightforward to get $y(0) = 16k - 32$, $y(1) = 27k - 144$ and $y''(q) = 2\left[(6q-15)k + 12q^2 - 18q - 44\right] < 0$. When $k > \frac{16}{3}$, since $y(0) < 0$, $y(1) < 0$ and $y(q)$ is concave, $y(q) < 0$ for all $q \in (0,1)$ and thus $\gamma = \frac{1-q}{2+q}$ is never optimal.

We now consider the case of k=6. When k=6, we have the following facts: (1) $f(\gamma)$ is maximized at an interior $\gamma_1^* < \frac{1-q}{4-q}$; (2) $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{2+q}$ if $q \geq 0.70821$, $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{4-q}$ if $q \leq 0.67068$, but if $0.67068 < q < 0.70821$, we need to calculate at which value of γ, $g(\gamma)$ is maximized for specific values of q; (3) $j(\gamma)$ is maximized at $\gamma = \frac{1}{2}$. Therefore, when $q \leq 0.67068$, the high quality firm's profit is maximized at $\gamma_1^* < \frac{1-q}{4-q}$; when

$0.67068 < q < 0.70821$, we need to compare $f\left(\gamma_1^*\right)$ and $g\left(\gamma_2^*\right)$; when $q \geq 0.70821$, since $f\left(\gamma_1^*\right) > f(0) > g\left(\frac{1-q}{2+q}\right)$, the high quality firm's profit is maximized at $\gamma_1^* < \frac{1-q}{4-q}$.

Numerical analysis shows that when $0.67068 < q < 0.70821$, $f\left(\gamma_1^*\right) > g\left(\gamma_2^*\right)$. Hence, we have,

Proposition 5: When *k*=6, the optimal level of network externality chosen by the high quality firm is $\gamma^* = \gamma_1^*(q) < \frac{1-q}{4-q}$.

Equilibrium Characterization and Comparative Statics

Here the market structure is always a duopoly and it is always partially covered. This is true for all values of *q*. The optimal level of network externality is monotonically decreasing in *q*.

5.3 Simulation

In Section 5.2, we have considered five values of *k* representing different cost coefficients for investment on network externality by the high quality firm. These values are *k*=1, 1.5, 1.8, 2 and 6. Lower values of *k* imply lower cost of investment on network externality, i.e., the investment is less costly.

Recall that

$$\pi_H^* = \begin{cases} \dfrac{4(1-q)}{(1-\gamma)(4-q)^2} & \text{if } \gamma \leq \dfrac{1-q}{4-q} \\ \dfrac{\left((1-q)+q\gamma\right)^2}{4(1-q)(1-\gamma)} & \text{if } \dfrac{1-q}{4-q} \leq \gamma \leq \dfrac{1-q}{2+q} \\ \dfrac{(1-q)(2-\gamma)^2}{9(1-\gamma)} & \text{if } \dfrac{1-q}{2+q} \leq \gamma \leq \dfrac{1}{2} \\ (1-q)\gamma & \text{if } \gamma \geq \dfrac{1}{2} \end{cases}.$$

We will refer the four ranges of γ as the first, second, third, and fourth range respectively.

We have observed four patterns of the range distribution of the optimal level of network externality, γ^*. When k is small (e.g. k=1, 1.5), γ^* lies in the fourth range when q is small, and when q is big, it is equal to the critical value between the second range and the third range, i.e., $\gamma^* = \frac{1-q}{2+q}$. When k is small-medium (e.g. k=1.8), as q increases from close to 0 to close to 1, γ^* moves from the fourth range to the first range and then to the critical value between the second range and the third range. When k is medium (e.g. k=2), as q increases from close to 0 to close to 1, γ^* moves from the first range to the critical value between the second range and the third range. When k is big (e.g. k=6), as q increases from close to 0 to close to 1, γ^* always lies in the first range.

The numerical analysis suggests that there are three critical values of k such that k can be divided into four ranges, each corresponding to one pattern of the range distribution of the optimal level of network externality, γ^*. Simulation verifies the results obtained from the numerical analysis. Figure 1 presents the main result from the simulation, in which three colour-shaded (blue, green, brown) areas represent the first range, the critical value between the second range and the third range, and the fourth range respectively. More simulation analysis shows that the three critical values are around 1.784, 1.805 and 5.051.

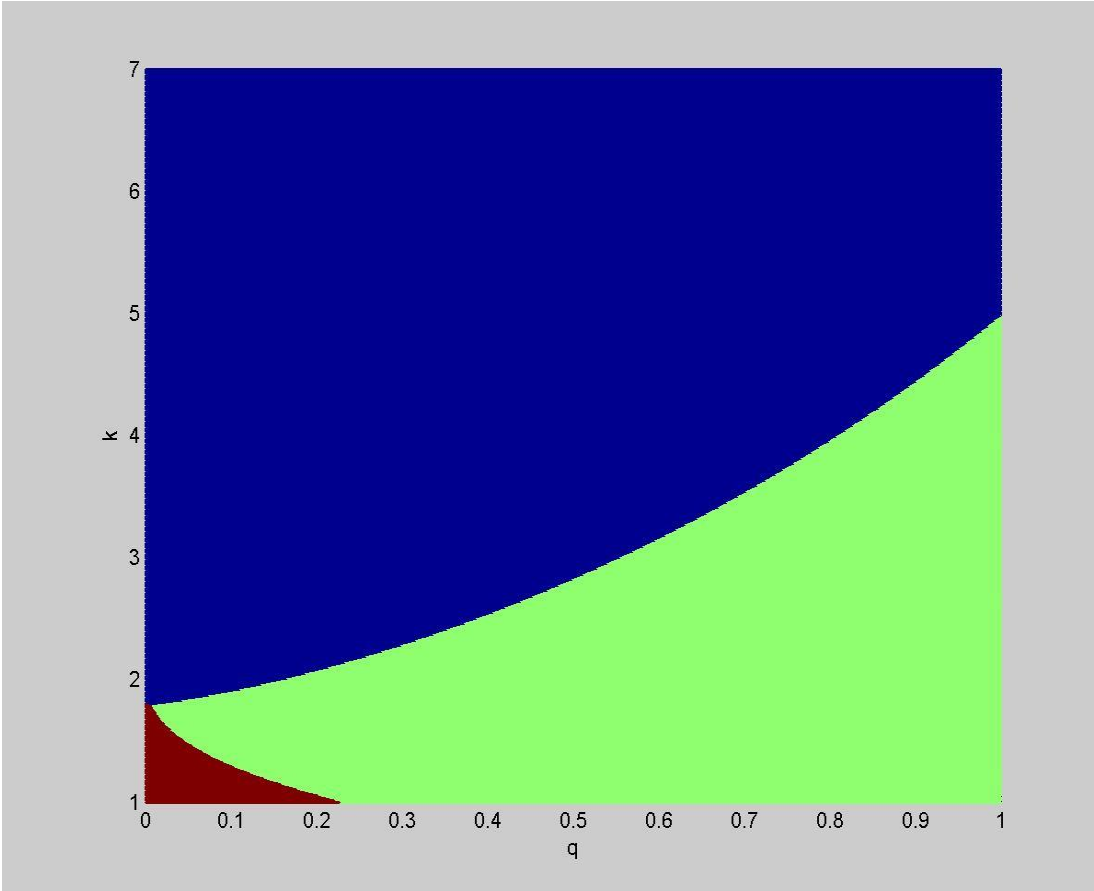

Figure 1 Range distribution of the optimal level of network externality ($1 \le k \le 7$ and $0 < q < 1$**)**

The interval between *k*=1.784 and *k*=1.805 is very narrow, which makes it difficult to distinguish the second pattern and the third pattern. However, if we look at Figure 1 more closely, we can distinguish these two patterns: The brown-shaded area intersects the *k*-axis at a point above the point at which the brown-, green-, and blue-shaded areas intersect. Alternatively, we can draw another figure to show this difference more clearly. Figure 2 presents the range distribution of γ^* for $1.6 \le k \le 2$ and $0 < q \le 0.1$.

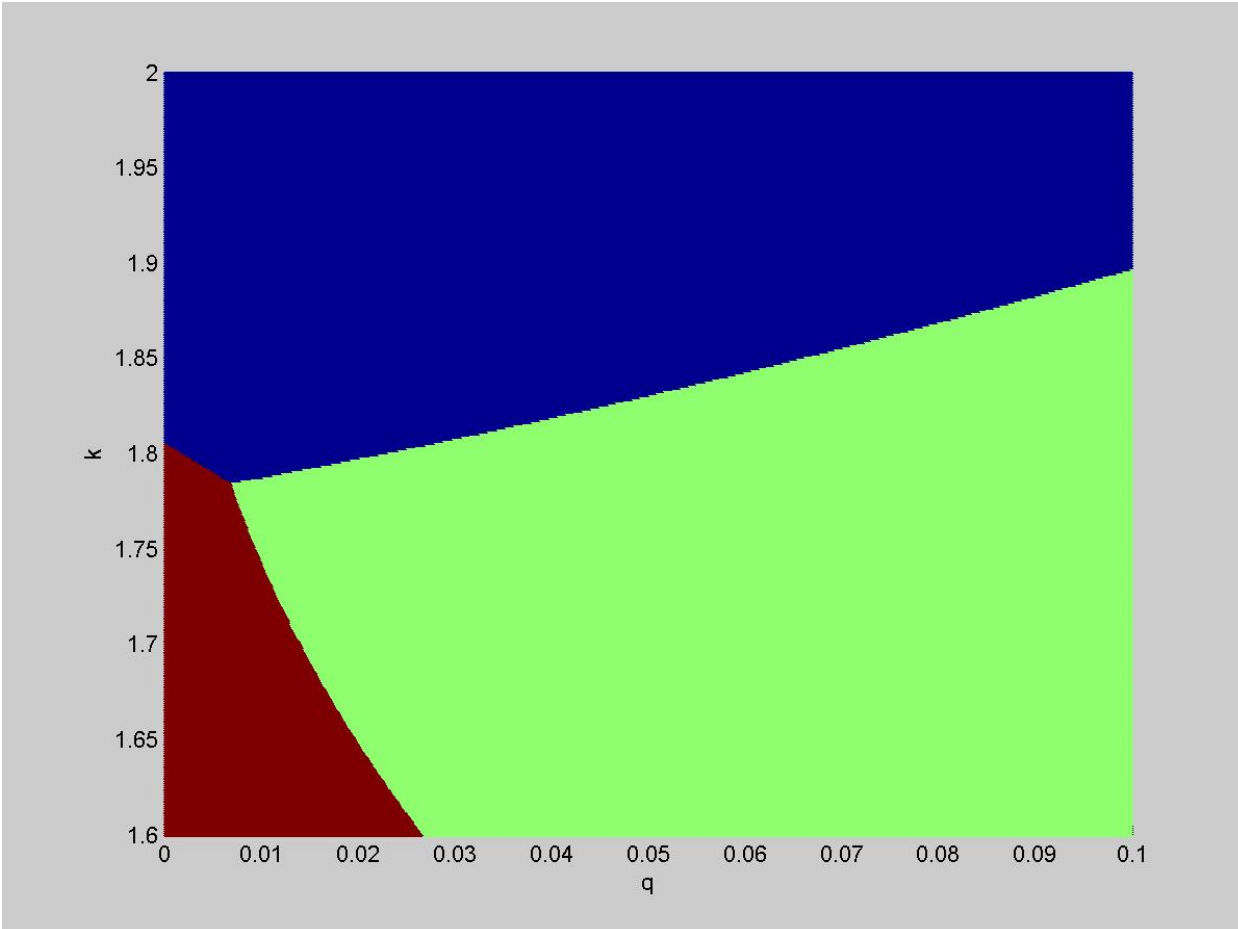

Figure 2 Range distribution of the optimal level of network externality ($1.6 \le k \le 2$ and $0 < q \le 0.1$**)**

5.4 Summary Findings

From the above analysis, we find that three possible optimal values of γ will actually be chosen by the high quality firm in equilibrium: $\gamma_L^* = \gamma_1^*(q) < \frac{1-q}{4-q}$ (low level of γ), $\gamma_M^* = \frac{1-q}{2+q}$ (medium level of γ), and $\gamma_H^* = \frac{1-q}{k}$ (high level of γ), where $k < k_2^c \approx 1.805$.

Now corresponding to each optimal value of γ an equilibrium market structure (monopoly or duopoly) is emerged and the market coverage (full or partial) is also determined endogenously.

When the optimal value of γ is γ_L^* which corresponds to a low degree of network externality, the market structure is duopoly i.e. shared by the high quality and the low quality firm and the market is partially covered.

When the optimal value of γ is γ_M^* which corresponds to a medium degree of network externality, the market structure is duopoly i.e. shared by the high quality and the low quality firm and the market is fully covered.

When the optimal value of γ is γ_H^* which corresponds to a high degree of network externality, the market is monopolized by the high quality firm and the market is fully covered.

In the table below, we list all possible equilibrium outcomes, where $q^c(k)$, $q_1^c(k)$ and $q_2^c(k)$ are critical values of q as defined above in Propositions 1-5.

Table 1 Equilibrium outcomes for different investment costs and relative quality differences[13]

Investment cost (*k*)	**Low quality Index (*q*)**	**Equilibrium Outcome**		
		Optimal degree of network externality	**Market Structure**	**Market coverage**
Low (k<1.784)	$q \le q^c(k)$	High (γ_H^*)	Monopoly	Full
	$q \ge q^c(k)$	Medium (γ_M^*)	Duopoly	Full
Low-medium (1.784<k<1.805)	$q \le q_1^c(k)$	High (γ_H^*)	Monopoly	Full
	$q_1^c(k) \le q \le q_2^c(k)$	Low (γ_L^*)	Duopoly	Partial
	$q \ge q_2^c(k)$	Medium (γ_M^*)	Duopoly	Full
Medium (1.805<k<5.051)	$q \le q^c(k)$	Low (γ_L^*)	Duopoly	Partial
	$q \ge q^c(k)$	Medium (γ_M^*)	Duopoly	Full
High (k>5.051)	$0 < q < 1$	Low (γ_L^*)	Duopoly	Partial

[13] The relative quality difference is defined to be (1- *q*).

Below we summarize our main qualitative result from the above analysis.

Theorem

In a model of vertical product differentiation with demand network externalities, when the degree of the network externality is a strategic choice of the high quality firm,

(i) the market can be monopolized and fully covered by the high quality firm when the investment cost on the network is low or low-medium and the quality difference between the two products is high, otherwise the market mostly remains duopoly which may be partially or fully covered.

(ii) if the investment cost on the network is very low or very high, the optimal degree of network externality is always increasing in quality difference of the products, but for any other levels of investment cost the relationship is non-monotonic.

6. Concluding Discussion

In the modern environment of digital products market with strong features of network externalities, a leading firm in this industry is not only just interested to increase the size of its users-base i.e. the network size of its products, it is also interested to enhance the *strength* of the network effect by making appropriate investments, which consequently increases the overall value of the products. Adding new apps, features and supporting devices to its core product over time enhances the usage and overall quality of the product. This is an alternative way to enhance the effective quality of an existing product without investing to improve the intrinsic quality of the product which may not also be feasible in many circumstances. In this paper, we study this particular aspect in detail through a model of vertical product differentiation with two competing firms. [14]

The other aspect we address here is; typically in the models of product differentiation with network externality, the degree or the strength of the network effect is mostly assumed as a

[14] In the situations, where the products do not have the explicit feature of network externality, the producers as part of their marketing strategy can invest to create a network for the existing and potential users of the product through an interactive digital platform. For example, a firm can invest to create a digital forum of its core products where the existing users can discuss and share their experiences about the products and thereby intensify the effect of network externality of the relevant products. A product developer can also pay to use a popular social networking site, (say like Facebook) to promote its product as this would engage the potential buyers to communicate more effectively and seamlessly about the product through that platform.

parameter. We depart from this and endogenize the degree of network externality. We explore the situation where the strength of the network externality is a strategic choice of a firm. This leads to interesting endogenous market structures and market coverage in the equilibrium. We find whether the market will be partially or fully covered and whether the resulting market structure is monopoly or duopoly come out as strategic outcome of the game between two firms; and depends on how costly is the investment on the network and relative difference in respective qualities of the products. Our main qualitative findings are when investment cost on the network is low and the quality difference between the two products is high, the market is monopolized and fully covered by the high quality firm, otherwise the market mostly remains duopoly which may be partially or fully covered. In our comparative statics analysis, we find that if the investment cost on the network is very low or very high, the optimal degree of network externality is always increasing in quality difference of the products, but for any other levels of investment cost the relationship is generally non-monotonic.

We conclude our discussion with a particular scenario where the findings of this analytical model can be useful. These days piracy of digital goods became quite prevalent due to the easy availability of copying technology. It is understood that rampant piracy of digital goods is not desirable from the innovators' as well as society's point of view. In this context, consider our high quality firm as the original firm or the copyright holder of the concerned product and the low quality firm as the commercial pirate. Then our model will suggest that increasing the degree of network effect strategically can eliminate the pirate from the market in some situations (see Table 1). Thus, in the markets where commercial piracy is a serious threat, we propose piracy can be fought by influencing the demand network externality as an effective instrument apart from other regular instruments (like monitoring and/or imposing lump-sum fine to the pirate which are already discussed in detail in the literature of stopping piracy). We think this is an additional instrument which can be used by the copyright holder or monitoring authority to effectively limit or stop rampant piracy.

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Appendix A

The high-quality producer maximizes $p_H D_H = \frac{1}{1-\gamma} p_H \left(1 - \frac{p_H - p_L}{1-q}\right)$. The first-order condition is

$$1 - q - 2p_H + p_L = 0. \tag{A1}$$

The low-quality producer maximizes

$p_L D_L = \frac{1}{1-\gamma} p_L \left(\frac{p_H - p_L}{1-q} - \gamma\right) = \frac{1}{(1-\gamma)(1-q)} p_L \left(p_H - p_L - \gamma(1-q)\right)$ subject to $q\gamma - p_L \geq 0$. Define the Lagrangian $L = p_L \left(p_H - p_L - \gamma(1-q)\right) + \lambda\left(q\gamma - p_L\right)$. The first-order and slackness conditions are

$$\begin{gathered} \frac{\partial L}{\partial p_L} = p_H - 2p_L - \gamma(1-q) - \lambda = 0, \\ \lambda \geq 0,\ q\gamma - p_L \geq 0,\ \lambda\left(q\gamma - p_L\right) = 0. \end{gathered} \tag{A2}$$

If $p_L = q\gamma$, then we must have $\lambda = p_H - 2p_L - \gamma(1-q) \geq 0$. Given $p_L = q\gamma$, it follows that

$$p_H = (1 - q + p_L)/2 = (1 - q + q\gamma)/2, \tag{A3}$$

and

$$\lambda = p_H - 2p_L - \gamma(1-q) = (1 - q + q\gamma)/2 - 2q\gamma - \gamma(1-q) = \tfrac{1}{2}\left(1 - q - (2+q)\gamma\right).$$

Thus, $\lambda \geq 0 \Leftrightarrow \gamma \leq (1-q)/(2+q) = \hat{\hat{\gamma}}$.

If $q\gamma - p_L \geq 0$, then we must have $\lambda = 0$ and

$$p_H - 2p_L - \gamma(1-q) = 0. \tag{A4}$$

The solution to the system of equations (A.1) and (A.4) is

$$p_H = (1-q)(2-\gamma)/3,\ \ p_L = (1-q)(1-2\gamma)/3. \tag{A5}$$

It follows that

$$q\gamma - p_L = q\gamma - (1-q)(1-2\gamma)/3 = \left(\gamma(2+q) - (1-q)\right)/3.$$

Thus,

$$q\gamma - p_L \geq 0 \Leftrightarrow \gamma \geq (1-q)/(2+q) = \hat{\hat{\gamma}}.$$

Note that for (A5) to be the solution, γ must not exceed $1/2 : \gamma \leq \gamma = 1/2$, so that the price of the low-quality product is nonnegative: $p_L \geq 0$.

The demands and profits for both firms are then obtained straightforwardly. The expressions are given in Section 4.1 and will not be repeated here.

Appendix B

Proof of Lemma 1: This is obvious given that $\frac{4(2+q)(1-q)}{3(1+2q)(4-q)} < 2(1-q)$, since this means that once $h(\gamma)$ is maximized at $\gamma = \frac{1}{2}$, $j\left(\frac{1-q}{k}\right) > j\left(\frac{1}{2}\right) = h\left(\frac{1}{2}\right)$ and that once $j(\gamma)$ is maximized at $\gamma = \frac{1}{2}$, $h\left(\frac{1-q}{2+q}\right) > h\left(\frac{1}{2}\right) = j\left(\frac{1}{2}\right)$.

■

Proof of Lemma 2: We have shown that when $k > \frac{4(4-q)}{9}$, $f(\gamma)$ is maximized at an interior $\gamma_1^* < \frac{1-q}{4-q}$ and when $k \le \frac{4(4-q)}{9}$, $f(\gamma)$ is maximized at $\gamma_1^* = \frac{1-q}{4-q}$. We have also shown that when $q \le 0.26185$, $g(\gamma)$ is maximized at $\gamma_2^* = \frac{1-q}{2+q}$ if $k \le k$, and when $q \ge 0.26185$, $g(\gamma)$ is maximized at $\gamma_2^* = \frac{1-q}{2+q}$ if $k \le \frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2}$. Since $\frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2} > \frac{4(4-q)}{9}$ for $q \ge 0.26185$ and $k > \frac{4(4-q)}{9}$ for $q \le 0.26185$, once $f(\gamma)$ is maximized at $\gamma_1^* = \frac{1-q}{4-q}$, $g(\gamma)$ is maximized at $\gamma_2^* = \frac{1-q}{2+q}$, we have $g\left(\frac{1-q}{2+q}\right) > g\left(\frac{1-q}{4-q}\right) = f\left(\frac{1-q}{4-q}\right)$.

■

Proof of Lemma 3: When $k \le \frac{4}{3}(< \frac{4(4-q)}{9})$, $f(\gamma)$ is maximized at $\gamma = \frac{1-q}{4-q}$; When $k \le \frac{4}{3}(< \frac{27}{16} < k)$, $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{2+q}$ when $q \le 0.26185$; When

$k \le \frac{4}{3}(< \frac{(2+q)(1+q)(1+q+q^2)}{(1-q)(1+2q)^2})$, $g(\gamma)$ is maximized at $\gamma = \frac{1-q}{2+q}$ when $q \ge 0.26185$. Since the net profit function is continuous, we have $f\left(\frac{1-q}{4-q}\right) = g\left(\frac{1-q}{4-q}\right) < g\left(\frac{1-q}{2+q}\right)$. Given Lemma 1, Lemma 3 is correct when $k \le \frac{4}{3}$.

■

Proof of Lemma 4: Clearly, if $j\left(\frac{1-q}{k}\right) < f\left(\frac{1-q}{4-q}\right)$, then $\gamma = \frac{1-q}{k}$ is not optimal. Straightforward computation yields

$$j\left(\frac{1-q}{k}\right) - f\left(\frac{1-q}{4-q}\right)$$
$$= \frac{1-q}{6k(4-q)^2}\left[\left(3k^2 - 32k + 48\right) - \left(3k^2 - 8k + 72\right)q + 27q^2 - 3q^3\right].$$

Define $x(q) = \left(3k^2 - 32k + 48\right) - \left(3k^2 - 8k + 72\right)q + 27q^2 - 3q^3$. It is straightforward to get $x(0) = 3k^2 - 32k + 48$, $x(1) = -24k < 0$, $x'(q) = -\left(3k^2 - 8k + 72\right) + 9q(6-q)$ and $x''(q) = 18(3-q) > 0$. So $x'(q)$ is increasing in q and the maximum is $x'(1) = -\left(3k^2 - 8k + 27\right) < 0$, which in turn implies that $x(q)$ is decreasing in q.

Note that if $1.8057 < k < 8.861$, $x(0) = 3k^2 - 32k + 48 < 0$. Also note that $j(\gamma)$ is maximized at $\gamma = \frac{1-q}{k}$ only when $k \le 2(1-q)$. So a necessary condition for $\gamma = \frac{1-q}{k}$ to be optimal is $k < 1.8057$.

■