January 1, 2015

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January 26, 2015

Abstract

Suppose the vertices of a graph G are labeled with real numbers. For each vertex $v \in G$, let $S(v)$ denote the sum of the labels of all vertices adjacent to v. A labeling is called *lucky* if $S(u) \neq S(v)$ for every pair u and v of adjacent vertices in G. The least integer k for which a graph G has a lucky labeling from $\{1, 2, \ldots, k\}$ is called the *lucky number* of the graph, denoted $\eta(G)$. In 2009, Czerwiński, Grytczuk, and Żelazny [6] conjectured that $\eta(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of G. In this paper, we improve the current bounds for particular classes of graphs with a strengthening of the results through a list lucky labeling. We apply the discharging method and the Combinatorial Nullstellensatz to show that for a planar graph G of girth at least 26, $\eta(G) \leq 3$. This proves the conjecture for non-bipartite planar graphs of girth at least 26. We also show that for girth at least 7, 6, and 5, $\eta(G)$ is at most 8, 9, and 19, respectively.

Keywords: lucky labeling, additive coloring, reducible configuration, discharging method, Combinatorial Nullstellensatz.

MSC code: 05C78, 05C15, 05C22, 05C78.

1   Introduction

In this paper we only consider simple, finite, undirected graphs. For such a graph G, let $V(G)$ denote the vertex set and $E(G)$ the edge set of G. When G is planar, let $F(G)$ be the set of faces of G and $l(f)$ be the length of a face f. Unless otherwise specified, we refer the reader to [12] for notation and definitions.

We consider a derived vertex coloring in which each vertex receives a color based on assigned labels of its neighbors. Let $\ell : V(G) \to \mathbb{R}$ be a labeling of the vertices of a graph G. For each $v \in V(G)$, let $S_G(v) = \sum_{u \in N_G(v)} \ell(u)$, where $N_G(v)$ is the open neighborhood of v in G. When the context is clear we use $S(v)$ in place of $S_G(v)$. We say the labeling ℓ is *lucky* if for every pair of adjacent vertices u and v, we have $S(u) \neq S(v)$; that is S induces a proper vertex coloring of G. The least integer k for which a graph G has a lucky labeling using labels from $\{1, \ldots, k\}$ is called the *lucky number* of G, denoted $\eta(G)$.

Determining the lucky number of a graph is a natural variation of a well–studied problem posed by Karoński, Łuczak and Thomason [9], in which edge labels from $\{1, \ldots, k\}$ are summed at incident

vertices to induce a vertex coloring. Karoński, Łuczak and Thomason conjecture that edge labels from $\{1, 2, 3\}$ are enough to yield a proper vertex coloring of graphs with no component isomorphic to K_2. This conjecture is known as the 1,2,3–Conjecture and is still open. In 2010, Kalkowski, Karoński and Pfender [8] showed that labels from $\{1, 2, 3, 4, 5\}$ suffice.

Similar to the lucky number of a graph, Chartrand, Okamoto, and Zhang [5] defined $\sigma(G)$ to be the smallest integer k such that G has a lucky labeling using k distinct labels. They showed that $\sigma(G) \leq \chi(G)$. Note that $\sigma(G) \leq \eta(G)$, since with $\eta(G)$ we seek the smallest k such that labels are from $\{1, \ldots, k\}$, even if some integers in $\{1, \ldots, k\}$ are not used as labels, whereas $\sigma(G)$ considers the fewest distinct labels, regardless of the value of the largest label.

In 2009, Czerwiński, Grytczuk, and Żelazny proposed the following conjecture for the lucky number of G.

Conjecture 1.1 ([6]). *For every graph G, $\eta(G) \leq \chi(G)$.*

This conjecture remains open even for bipartite graphs, for which no constant bound is currently known. Czerwiński, Grytczuk, and Żelazny [6] showed that $\eta(G) \leq k + 1$ for every bipartite graph G having an orientation in which each vertex has out-degree at most k. They also showed that $\eta(G) \leq 2$ when G is a tree, $\eta(G) \leq 3$ when G is bipartite and planar, and $\eta(G) \leq 100, 280, 245, 065$ for every planar graph G. Note that if the conjecture is true, then $\eta(G) \leq 4$ for any planar graph G. The bound for planar graphs was later improved to $\eta(G) \leq 468$ by Bartnicki et al. [3], who also show the following.

Theorem 1.2 ([3]). *If G is a 3-colorable planar graph, then $\eta(G) \leq 36$.*

The *girth* of a graph is the length of its shortest cycle, which is especially useful in giving a measure of sparseness. Knowing the girth of a planar graph gives a bound on the *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, which is the maximum average degree over all subgraphs of G. The following proposition is a simple application of Euler's formula (see [12]) and gives a relationship between these two parameters.

Proposition 1.3. *If G is a planar graph with girth g, then $\mathrm{mad}(G) < \frac{2g}{g-2}$.*

Bartnicki et al. [3] proved the following.

Theorem 1.4 ([3]). *If G is a planar graph of girth at least 13, then $\eta(G) \leq 4$.*

In 2013, Akbari et al. [1] proposed the list version of lucky labeling. A graph is *lucky k-choosable* if whenever each vertex is given a list of at least k available integers, a lucky labeling can be chosen from the lists. The *lucky choice number* of a graph G is the minimum positive integer k such that G is lucky k-choosable, and is denoted by $\eta_\ell(G)$. Akbari et al. [1] showed that $\eta_\ell(G) \leq \Delta^2 - \Delta + 1$ for every graph G with $\Delta \geq 2$. They also proved the following.

Theorem 1.5 ([1]). *If G is a forest, then $\eta_\ell(G) \leq 3$.*

In this paper we improve these results for planar graphs of particular girths. Specifically, we use the Combinatorial Nullstellensatz within reducibility arguments of the discharging method to prove our results. The combination of these two popular techniques is a novel approach that can eliminate a considerable amount of case analysis. Moreover, using the Combinatorial Nullstellensatz in reducibility arguments of coloring problems enables proving choosability results, rather than just colorability.

We show the following improvements on the lucky choice number for planar graphs of given girths.

Theorem 1.6. *Let G be a planar graph with girth g.*

1. *If $g \geq 5$, then $\eta_\ell(G) \leq 19$.*

2. *If $g \geq 6$, then $\eta_\ell(G) \leq 9$.*

3. *If $g \geq 7$, then $\eta_\ell(G) \leq 8$.*

4. *If $g \geq 26$, then $\eta_\ell(G) \leq 3$.*

Various 3–colorings of planar graphs have been obtained under certain girth assumptions. For example, Grötzsch [7] proved that planar graphs with girth at least 4 are 3–colorable and Thomassen [11] proved that planar graphs with girth at least 5 are 3–list–colorable. Combined with Grötzsch's result, our result answers Conjecture 1.1 for non-bipartite planar graphs with girth at least 26.

In Section 2 we introduce the notation and tools that are used throughout the remainder of the paper. We also give an overview of how we use the discharging method and the Combinatorial Nullstellensatz. Section 3 describes certain reducible configurations. Finally, in Section 4 we prove Theorem 1.6.

2 Notation and Tools

Let $N_G(v)$ be the open neighborhood of a vertex v in a graph G. For convenience, a j–*vertex*, j^-–*vertex*, or j^+–*vertex* is a vertex with degree j, at most j, or at least j, respectively. Similarly, a j-*neighbor* (respectively j^-–*neighbor* or j^+–*neighbor*) of v is a j–vertex (respectively j^-–vertex or j^+–vertex) adjacent to v.

For sets A and B of real numbers $A \oplus B$ is defined to be the set $\{a + b \colon a \in A, b \in B\}$. Likewise, $A \ominus B$ is defined to be the set $\{a - b \colon a \in A, b \in B\}$. When $B = \emptyset$, we define $A \oplus B = A \ominus B = A$. We use the following known result from additive combinatorics.

Proposition 2.1. *Let $A_1, \ldots, A_r$ be finite sets of real numbers. We have*

$$|A_1 \oplus \cdots \oplus A_r| \geq 1 + \sum_{i=1}^{r} (|A_i| - 1).$$

Proof. We apply induction on $\sum_{i=1}^{r} |A_i|$. When $\sum_{i=1}^{r} |A_i| = 1$, all but one A_i are empty, so we have $|A_1 \oplus \cdots \oplus A_r| = 1$, as desired.

Now suppose that $\sum_{i=1}^{r} |A_i| = n$. We may suppose that all A_i are nonempty. Let a_i be the minimum element of A_i for $i \in \{1, \ldots, n\}$. Let $A'_1 = A_1 - \{a_1\}$. By the induction hypothesis we have $|A'_1 \oplus A_2 \oplus \cdots \oplus A_r| \geq 1 + \sum_{i=1}^{r} (|A_i| - 1) - 1$. However $|A_1 \oplus \cdots \oplus A_r| \geq |\{a_1 + \cdots + a_r\}| + |A'_1 \oplus A_2 \oplus \cdots \oplus A_r|$. Therefore $|A_1 \oplus \cdots \oplus A_r| \geq 1 + \sum_{i=1}^{r} (|A_i| - 1)$. $\qquad\square$

Note that $A \oplus (-B)$ is the same as $A \ominus B$, where $-B = \{-b : b \in B\}$. This yields the following known corollary.

Corollary 2.2. *Let A and B be nonempty sets of positive real numbers. We have $|A \ominus B| \geq |A| + |B| - 1$.*

Throughout, we consider when endpoints of edges need different sums to yield a lucky labeling. For this reason, if we know $S(u) \neq S(v)$ for an edge uv of G, we say that uv is *satisfied*; uv is *unsatisfied* otherwise.

Our proofs rely on applying the discharging method. This proof technique assigns an initial charge to vertices and possibly faces of a graph and then distributes charge according to a list of discharging rules. A configuration is *k–reducible* if it cannot occur in a vertex minimal graph G with $\eta_\ell(G) > k$. Note that any k–reducible configuration is also $(k+1)$–reducible. When applying the discharging method in Theorem 4.5 we require the following known lemma, which is a simple application of Euler's Formula (see [12]).

Proposition 2.3. *Given a planar graph G,*

$$\sum_{f \in F(G)} (l(f) - 4) + \sum_{v \in V(G)} (d(v) - 4) = -8.$$

We also require a large independent set, which is given from the following theorem.

Theorem 2.4 ([10])**.** *Every planar triangle-free graph on n vertices has an independent set of size at least $\dfrac{n+1}{3}$.*

The main tool we use to determine when configurations are k-reducible is the Combinatorial Nullstellensatz, which is applied to certain graph configurations.

Theorem 2.5 (Combinatorial Nullstellensatz [2])**.** *Let f be a polynomial of degree t in m variables over a field $\mathbb{F}$. If there is a monomial $\prod x_i^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero in $\mathbb{F}$, then f is nonzero at some point of $\prod T_i$, where each T_i is a set of $t_i + 1$ distinct values in $\mathbb{F}$.*

3 Reducible Configurations

In the lemmas in this section, we let $k \in \mathbb{N}$ and introduce k–reducible configurations that will be used to prove our main result. Let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a function on $V(G)$ such that $|\mathcal{L}(v)| = k$ for each $v \in V(G)$. Thus $\mathcal{L}(v)$ denotes a list of k available labels for v. In each proof we take G to be a vertex minimal graph with $\eta_\ell(G) > k$. Then we define a proper subgraph G' of G with $V(G') \subsetneq V(G)$. By the choice of G, G' has a lucky labeling ℓ such that $\ell(v) \in \mathcal{L}(v)$ for all $v \in V(G')$. This labeling of G' is then extended to a lucky labeling of G by defining $\ell(v)$ for $v \in V(G) - V(G')$. We discuss the details of this approach in Lemmas 3.1 and 3.2. The remaining lemmas are similar in approach, so we include fewer details in the proofs.

Lemma 3.1. *The following configurations are k–reducible in the class of graphs with girth at least 5.*

(a) A vertex v with $\displaystyle\sum_{u \in N(v)} d(u) < k$.

(b) A vertex v with r neighbors of degree 1 and a set Q of 2–neighbors $\{v_1, \ldots, v_q\}$ each having a $(k-1)^-$–neighbor other than v, say $v_1', \ldots, v_q'$, respectively, such that $v_1', \ldots, v_q'$ are independent and $1 + r(k-1) + \displaystyle\sum_{v_i \in Q} (k - d(v_i') - 1) > d(v)$.

Proof. Assume G is a vertex minimal graph with $\eta_\ell(G) > k$ containing the configuration described in (a). Let $G' = G - \{v\}$. Since G is vertex minimal, $\eta_\ell(G') \le k$. Let ℓ be a lucky labeling from $\mathcal{L}$ on $V(G')$. Our aim is to choose $\ell(v)$ from $\mathcal{L}(v)$ to extend the lucky labeling of G' to a lucky labeling of G. Note that the only unsatisfied edges of G are those incident to neighbors of v. Let e be an edge incident to a neighbor u of v. If $e = uv$, then e is satisfied when $\ell(v) \ne \sum_{w \in N(v)} \ell(w) - S_{G'}(u)$. If $e = uw$ for some $w \ne v$, then e is satisfied when $\ell(v) \ne S_{G'}(w) - S_{G'}(u)$. Thus picking $\ell(v)$ distinct from at most $\sum_{u \in N(v)} d(u)$ values ensures that all edges of G are satisfied. Since $\sum_{u \in N(v)} d(u) < k$ there exists $\ell(v)$ in $\mathcal{L}(v)$ that can be used to extend the lucky labeling of G' to a lucky labeling of G. Therefore $\eta_\ell(G) \le k$, a contradiction.

Now assume G is a vertex minimal graph with girth at least 5 and $\eta_\ell(G) > k$ containing the configuration described in (b). Let R be the set of r 1–neighbors of v. Let $G' = G - (R \cup Q)$. Since $\text{girth}(G) \ge 5$, Q is independent. Therefore for each $i \in \{1, \ldots, q\}$ there are at least $|\mathcal{L}(v_i)| - d_G(v_i')$ choices for $\ell(v_i)$ that ensure all edges incident to v_i' are satisfied in G. Consider vw in $E(G)$. If $w \in V(G')$, vw is satisfied when $\sum_{x \in R \cup Q} \ell(x) \ne S_{G'}(w) - S_{G'}(v)$. Also, if $w \in R$, then vw is satisfied when $\sum_{x \in R \cup Q} \ell(x) \ne \ell(v) - S_{G'}(v)$. If $w = v_i$ for some $v_i \in Q$, vw is satisfied when $\sum_{x \in R \cup Q} \ell(x) \ne \ell(v) + \ell(w) - S_{G'}(v)$. Therefore, we must avoid at most $d(v)$ values for $\sum_{x \in R \cup Q} \ell(x)$ in order to satisfy all edges incident to v. Recall that each vertex in R and Q have k and $k - d(v_i')$ labels, respectively, that avoid restricted sums. Proposition 2.1 guarantees at least $1 + r(k-1) + \sum_{v_i \in Q'} (k - d(v_i') - 1)$ available values for $\sum_{w \in R} \ell(w) + \sum_{w \in Q} \ell(w)$. Since, by assumption, $1 + r(k-1) + \sum_{v_i \in Q} (k - d(v_i') - 1) > d(v)$, there is at least one choice for $\ell(w)$ for each w in $R \cup Q$ that completes a lucky labeling of G. Thus $\eta_\ell(G) \le k$, a contradiction. $\square$

Lemma 3.2. *The following configurations are 8-reducible in the class of graphs of girth at least 6.*

(a) A 6–vertex v having six 2-neighbors one of which has a 3^-–neighbor.

(b) A 7–vertex v having seven 2-neighbors two of which have 4^-–neighbors.

Proof. Let G be a vertex minimal graph of girth at least 6 with $\eta_\ell(G) > 8$. To the contrary suppose G contains the configuration described in (a). Let u be a 2–neighbor of v having a 3^-–neighbor. Let $G' = G - \{u, v\}$. Let $\ell : V(G') \to \mathbb{R}$ be a lucky labeling of G' such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G)$.

The only unsatisfied edges of G are those incident to neighbors of u or v. To satisfy the unsatisfied edges not incident to u or v, we avoid at most two values from $\mathcal{L}(u)$ and at most five values from $\mathcal{L}(v)$. Note that $|\mathcal{L}(u)| \ge 8$ and $|\mathcal{L}(v)| \ge 8$. Thus there are at least six labels available for u and at least three available for v. To satisfy the edges incident to u or v, $\ell(u) - \ell(v)$ must avoid at most seven values. Corollary 2.2 gives at least eight values for $\ell(u) - \ell(v)$ from available labels. Thus there are labels that complete a lucky labeling of G. Hence $\eta_\ell(G) \le 8$, a contradiction.
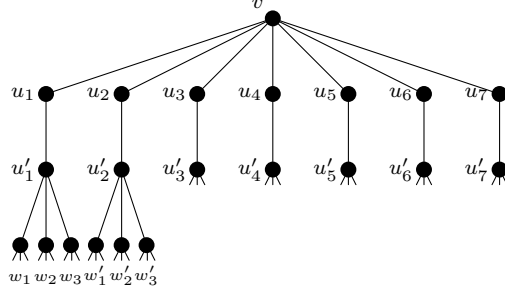
Figure 1: An 8–reducible configuration.

Now, we prove part (b). To the contrary suppose G contains the configuration described in (b). Let $u_1, \ldots, u_7$ be the 2–neighbors of v whose other neighbors are $u'_1, \ldots, u'_7$, respectively, where u'_1 and u'_2 are 4^-–vertices. Since the most restrictions on labels occurs when $d(u'_1) = d(u'_2) = 4$, we assume this is the case. Let $N(u'_1) - \{u_1\} = \{w_1, w_2, w_3\}$ and $N(u'_2) - \{u_2\} = \{w'_1, w'_2, w'_3\}$ (see Figure 1). Consider $G' = G - \{v, u_1, u_2\}$. Let $\ell : V(G') \to \mathbb{R}$ be a lucky labeling of G' such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G)$. The only unsatisfied edges of G are those incident to u'_1, u'_2, and neighbors of v. The following function has factors that correspond to unsatisfied edges, where x, y, and z represent the possible values of $\ell(v)$, $\ell(u_1)$, and $\ell(u_2)$, respectively.

$$f(x, y, z) = \prod_{i=1}^{7} \left(y + z + \sum_{j=3}^{7} \ell(u_j) - x - \ell(u'_i) \right) \cdot \prod_{i=3}^{7} (x + \ell(u'_i) - S_{G'}(u'_i))$$
$$\cdot \prod_{i=1}^{3} (y + S_{G'}(u'_1) - S_{G'}(w_i)) \cdot \prod_{i=1}^{3} (z + S_{G'}(u'_2) - S_{G'}(w'_i))$$
$$\cdot (x + \ell(u'_1) - y - S_{G'}(u'_1)) \cdot (x + \ell(u'_2) - z - S_{G'}(u'_2))$$

The coefficient of $x^7 y^6 z^7$ in $f(x, y, z)$ is equal to its coefficient in $(y + z - x)^7 x^5 y^3 z^3 (x - y)(x - z)$, which is 490. By Theorem 2.5, there is a choice of labels for $\ell(v)$, $\ell(u_1)$, and $\ell(u_2)$ from lists of size at least 8 that make f nonzero. Thus these labels induce a lucky labeling of G. Hence $\eta_\ell(G) \le 8$, a contradiction. $\square$

Lemma 3.3. *A configuration that is an induced cycle with vertices $v_1 v_2 v_3 v_4 v_5$ such that $d(v_1) \le 17$, $d(v_2) = d(v_5) = 2$, $d(v_3) \le 7$, and $d(v_4) \le 7$ is 19–reducible.*
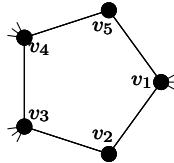

Figure 2: A reducible configuration.

Proof. Let G be a vertex minimal graph with $\eta_\ell(G) > 19$. Suppose to the contrary that G contains the configuration in Figure 2. Since the most restrictions on labels occurs when $d(v_1) = 17$ and

6

$d(v_3) = d(v_4) = 7$, we assume this is the case. Let $G' = G - \{v_2, v_5\}$. Let $\ell : V(G') \to \mathbb{R}$ be a lucky labeling of G' such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G)$. The unsatisfied edges are those incident to $v_1, \ldots, v_5$. The following function has factors corresponding to the unsatisfied edges where x_2 and x_5 represent labels of v_2 and v_5, respectively.

$$
\begin{aligned}
f(x_2, x_5) =& (S_{G'}(v_1) + x_2 + x_5 - \ell(v_1) - \ell(v_3)) \cdot (\ell(v_1) + \ell(v_3) - x_2 - S_{G'}(v_3)) \\
& \cdot (x_2 + S_{G'}(v_3) - x_5 - S_{G'}(v_4)) \cdot (x_5 + S_{G'}(v_4) - \ell(v_1) - \ell(v_4)) \\
& \cdot (\ell(v_1) + \ell(v_4) - x_2 - x_5 - S_{G'}(v_1)) \cdot \prod_{w \in N_{G'}(v_4) - \{v_3\}} (S_{G'}(w) - S_{G'}(v_4) - x_5) \\
& \cdot \prod_{w \in N_{G'}(v_1)} (S_{G'}(w) - S_{G'}(v_1) - x_2 - x_5) \cdot \prod_{w \in N_{G'}(v_3) - \{v_4\}} (S_{G'}(w) - S_{G'}(v_3) - x_2)
\end{aligned}
$$

The coefficient of $x_2^{16} x_5^{14}$ in $f(x_2, x_5)$ is the same as $x_2^{10} x_5^8$ in $-(x_2 + x_5)^{17}(x_2 - x_5)$, which is $\binom{17}{10} - \binom{17}{9}$. Theorem 2.5 gives $\eta_\ell(G) \le 19$, a contradiction. $\qquad\square$

Lemma 3.4. *Let $P(t_2, \ldots, t_{n-1})$ be the path $v_1 \cdots v_n$ such that for each i in $\{2, \ldots, n-1\}$ the vertex v_i has t_i 1–neighbors and $d(v_i) = 2 + t_i$. The configurations $P(1, 0, 1)$, $P(1, 1, 1)$, $P(1, 1, 0, 0)$, $P(0, 1, 0, 0)$, $P(1, 0, 0, 0)$, and $P(0, 0, 0, 0, 0)$ are 3–reducible.*
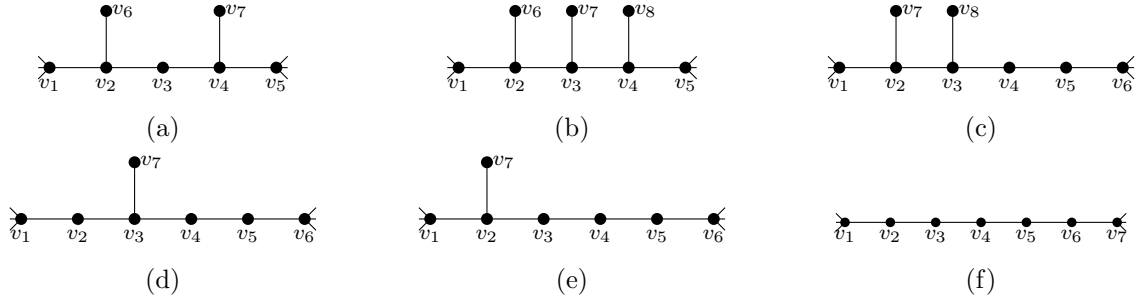

Figure 3: Some 3–reducible configurations.

Proof. Let G be a vertex minimal graph with $\eta_\ell(G) > 3$. We proceed as in earlier proofs presenting the proper subgraph G', the function f derived from the configuration, the monomial, and its coefficient. In each function f, x_i corresponds to the label of v_i.

Suppose G contains $P(1, 0, 1)$, see Figure 3a. Let $G' = G - \{v_3, v_6, v_7\}$.

$$
\begin{aligned}
f(x_3, x_6, x_7) =& (S_{G'}(v_1) - \ell(v_1) - x_3 - x_6) \cdot (\ell(v_1) + x_3 + x_6 - \ell(v_2)) \\
& \cdot (\ell(v_1) + x_3 + x_6 - \ell(v_2) - \ell(v_4)) \cdot (\ell(v_2) + \ell(v_4) - S_{G'}(v_4) - x_3 - x_7) \\
& \cdot (\ell(v_5) + x_3 + x_7 - \ell(v_4)) \cdot (\ell(v_5) + x_3 + x_7 - S_{G'}(v_5))
\end{aligned}
$$

The coefficient of $x_3^2 x_6^2 x_7^2$ is 9.

Suppose G contains $P(1, 1, 1)$, see Figure 3b. Let $G' = G - \{v_3, v_6, v_7, v_8\}$.

$$
\begin{aligned}
f(x_3, x_6, x_7, x_8) =& (S_{G'}(v_1) - \ell(v_1) - x_3 - x_6) \cdot (\ell(v_1) + x_3 + x_6 - \ell(v_2)) \\
& \cdot (\ell(v_1) + x_3 + x_6 - \ell(v_2) - x_7 - \ell(v_4)) \cdot (\ell(v_2) + \ell(v_4) + x_7 - x_3) \\
& \cdot (\ell(v_2) + \ell(v_4) + x_7 - \ell(v_5) - x_3 - x_8) \cdot (\ell(v_5) + x_3 + x_8 - \ell(v_4)) \\
& \cdot (\ell(v_5) + x_3 + x_8 - S_{G'}(v_5))
\end{aligned}
$$

7

The coefficient of $x_3^2 x_6^2 x_7 x_8^2$ is 15.

Suppose G contains $P(1, 1, 0, 0)$, see Figure 3c. Let $G' = G - \{v_3, v_4, v_7, v_8\}$.

$$
\begin{aligned}
f(x_3, x_4, x_7, x_8) = {} & (x_3 + x_7 + \ell(v_1) - S_{G'}(v_1)) \cdot (x_3 + x_7 + \ell(v_1) - \ell(v_2)) \\
& \cdot (x_4 + x_8 + \ell(v_2) - x_3 - x_7 - \ell(v_1)) \cdot (x_4 + x_8 + \ell(v_2) - x_3) \\
& \cdot (x_4 + x_8 + \ell(v_2) - x_3 - \ell(v_5)) \cdot (x_3 + \ell(v_5) - x_4 - \ell(v_6)) \cdot (x_4 + \ell(v_6) - S_{G'}(v_6))
\end{aligned}
$$

The coefficient of $x_3 x_4^2 x_7^2 x_8^2$ is 8.

Suppose G contains $P(0, 1, 0, 0)$, see Figure 3d. Let $G' = G - \{v_3, v_4, v_7\}$.

$$
\begin{aligned}
f(x_3, x_4, x_7) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3) \cdot (\ell(v_1) + x_3 - \ell(v_2) - x_7 - x_4) \\
& \cdot (\ell(v_2) + x_7 + x_4 - x_3) \cdot (\ell(v_2) + x_7 + x_4 - x_3 - \ell(v_5)) \\
& \cdot (x_3 + \ell(v_5) - x_4 - \ell(v_6)) \cdot (x_4 + \ell(v_6) - S_{G'}(v_6))
\end{aligned}
$$

The coefficient of $x_3^2 x_4^2 x_7^2$ is 6.

Suppose G contains $P(1, 0, 0, 0)$, see Figure 3e. Let $G' = G - \{v_3, v_4, v_7\}$.

$$
\begin{aligned}
f(x_3, x_4, x_7) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3 - x_7) \cdot (\ell(v_2) - \ell(v_1) - x_3 - x_7) \\
& \cdot (\ell(v_2) + x_4 - \ell(v_1) - x_3 - x_7) \cdot (\ell(v_2) + x_4 - x_3 - \ell(v_5)) \\
& \cdot (\ell(v_6) + x_4 - x_3 - \ell(v_5)) \cdot (\ell(v_6) + x_4 - S_{G'}(v_6))
\end{aligned}
$$

The coefficient of $x_3^2 x_4^2 x_7^2$ is 7.

Suppose G contains $P(0, 0, 0, 0, 0)$, see Figure 3f. Let $G' = G - \{v_3, v_4, v_5\}$.

$$
\begin{aligned}
f(x_3, x_4, x_5) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3) \cdot (\ell(v_1) + x_3 - \ell(v_2) - x_4) \cdot (\ell(v_2) + x_4 - x_3 - x_5) \\
& \cdot (x_3 + x_5 - x_4 - \ell(v_6)) \cdot (x_4 + \ell(v_6) - x_5 - \ell(v_7)) \cdot (x_5 + \ell(v_7) - S_{G'}(v_7))
\end{aligned}
$$

The coefficient of $x_3^2 x_4^2 x_5^2$ is 7. Theorem 2.5 implies that these configurations are 3–reducible. $\square$

4   Proof of Main Results

Theorem 4.1. *If G is a planar graph with* $\mathrm{girth}(G) \geq 5$, *then* $\eta_\ell(G) \leq 19$.

Proof. Let G be a planar graph with girth at least 5 and suppose that G is vertex minimal with $\eta_\ell(G) > 19$. By Proposition 1.3, $\mathrm{mad}(G) < 10/3$. Assign each vertex v an initial charge $d(v)$, and apply the following discharging rules.

(R1) Each 1–vertex receives $7/3$ charge from its neighbor.

(R2) Each 2–vertex

   (a) with two 8^+–neighbors receives $2/3$ charge from each neighbor.

   (b) with a 4^-–neighbor and a 15^+–neighbor receives $4/3$ charge from its 15^+–neighbor.

   (c) with a 10^+–neighbor and a neighbor of degree 5, 6, or 7 receives 1 charge from its 10^+–neighbor and $1/3$ charge from its other neighbor.

(R3) Each 3–vertex receives $1/3$ charge from a 6^+–neighbor.

A contradiction with $\mathrm{mad}(G) < 10/3$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least $10/3$; we show that this is the case.

By Lemma 3.1 (a), each 1–vertex has a 19^+–neighbor, 2–vertices have neighbors with degree sum at least 19, and 3–vertices have at least one 6^+–neighbor. Thus, by the discharging rules, 3^-–vertices have final charge $10/3$. Since 4–vertices neither give nor receive charge, they have final charge 4.

Vertices of degree d with $d \in \{5, 6, 7\}$ give charge when incident to 3^-–vertices. By the discharging rules, they give away at most $d/3$ charge. This results in a final charge of at least $d - \frac{d}{3} = \frac{2d}{3} \geq \frac{10}{3}$, since $d \geq 5$.

Vertices of degree d with $d \in \{8, 9\}$. By Lemma 3.1 (a) each 9–vertex has at least one 3^+– neighbor. Also, each 8–vertex has at least two 3^+–neighbors or at least one 4^+–neighbor. By the discharging rules, the final charge of any 9–vertex is at least $9 - 8 \cdot \frac{2}{3} - \frac{1}{3} = \frac{10}{3}$ and the final charge of any 8–vertex is at least $\min\{8 - 6 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}, 8 - 7 \cdot \frac{2}{3}\} = \frac{10}{3}$.

Vertices of degree d with $d \in \{10, 11\}$. By Lemma 3.1 (b), these vertices have no 2–neighbors with a 7^-–neighbor. Thus, these vertices have final charge at least $d - \frac{2d}{3} = \frac{d}{3} \geq \frac{10}{3}$, since $d \geq 10$.

Let v have degree d where $d \in \{12, 13, 14\}$. By Lemma 3.1 (b), v has no 2–neighbor with a 4^-–neighbor. By Lemma 3.1 (b) and Lemma 3.3, v has at most two 2–neighbors each having a 7^-–neighbor. By the discharging rules v has final charge at least $d - 2(1) - (d - 2)\left(\frac{2}{3}\right) = \frac{d-2}{3} \geq \frac{10}{3}$, since $d \geq 12$.

Similarly, by Lemma 3.1 (b) and Lemma 3.3 vertices of degree 15, 16, or 17 have at most one 2–neighbor with 7^-–neighbors. Thus these vertices give at most $1(\frac{4}{3}) + (d - 1) \cdot \frac{2}{3}$ charge. Hence they have final charge at least $\frac{d-2}{3} \geq \frac{13}{3}$, since $d \geq 15$.

Finally, consider an 18^+–vertex v of degree d. Let r be the number of 1–neighbors of v. Let $U = \{u_1, u_2, \ldots, u_q\}$ be the set of 2–neighbors of v. For each u_i let $N(u_i) - \{v\} = \{u_i'\}$. Let $T = \{u_i' \in U : d(u_i') \leq 7\}$ and let $|T| = t$. Since $G[T]$ is planar with girth at least 5, Theorem 2.4 guarantees at least $\frac{t+1}{3}$ vertices in T that form an independent set. By Lemma 3.1 (b), $d \geq 18r + 11(\frac{t+1}{3}) + 1$. Thus

$$d \geq 18r + \frac{11}{3}t + \frac{14}{3}. \tag{1}$$

The final charge of v is at least $d - \frac{7}{3}r - \frac{4}{3}t - \frac{2}{3}(d - r - t)$. Hence v has final charge at least $\frac{d}{3} - \frac{5}{3}r - \frac{2}{3}t$. From (1), $\frac{d}{3} - \frac{5}{3}r - \frac{2}{3}t \geq \frac{13}{3}r + \frac{5}{9}t + \frac{14}{9}$. When $r \geq 1$ or $t \geq 4$, the final charge is at least $\frac{10}{3}$. When $r = 0$ and $t \leq 3$, the vertex v has final charge at least $d - \frac{4}{3}t - \frac{2}{3}(d - t) \geq \frac{d-6}{3} \geq \frac{12}{3}$, since $d \geq 18$. $\square$

Theorem 4.2. *If G is a planar graph with* $\mathrm{girth}(G) \geq 6$*, then $\eta_\ell(G) \leq 9$.*

Proof. Let G be a planar graph with girth at least 6 and suppose G is vertex minimal with $\eta_\ell(G) > 9$. By Proposition 1.3, $\mathrm{mad}(G) < 3$. Assign each vertex v an initial charge of $d(v)$ and apply the following discharging rules.

(R1) Each 1–vertex receives 2 charges from its neighbor.

(R2) Each 2–vertex

(a) with one 8^+–neighbor and one 5^-–neighbor receives 1 charge from its 8^+–neighbor.

(b) with one 7^+–neighbor and one 4^-–neighbor receives 1 charge from its 7^+–neighbor.

(c) with one 6^+–neighbor and one 3^-–neighbor receives 1 charge from its 6^+–neighbor.

(d) receives 1/2 charge from each neighbor, otherwise.

A contradiction with $\mathrm{mad}(G) < 3$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 3; we show this is the case.

By Lemma 3.1 (a) each 1–vertex has a 9^+–neighbor and each 2–vertex has neighbors with degree sum at least 9. Under the discharging rules, 1–vertices and 2–vertices gain charge 2 and 1, respectively, and 3–vertices neither gain nor lose charge. Thus, 3^-–vertices have final charge 3.

By Lemma 3.1 (b) each 4–vertex v has no 1–neighbor and has at most one 2–neighbor whose other neighbor is a 6^-–vertex. Therefore each 4–vertex has final charge at least $4 - \frac{1}{2}$. Similarly, each 5–vertex has no 1–neighbor and has at most four 2–neighbors having another 7^-–neighbor. Therefore each 5–vertex has final charge at least $5 - 4(\frac{1}{2})$, as desired.

If v is a 6–vertex, then by Lemma 3.1, v has no 1–neighbor. Moreover, by Lemma 3.2, if v has six 2–neighbors, at most one of them has a 3^-–neighbor. Hence v has charge at least $6 - \max\{1 + 4(\frac{1}{2}), 6(\frac{1}{2})\}$, which is 3 as desired.

Similarly by Lemma 3.1, a 7–vertex v has no 1–neighbor. Moreover, by Lemma 3.2, if v has seven 2–neighbors, at most one of them has a 4^-–neighbor. Thus v has charge at least $7 - \max\{2 + 4(\frac{1}{2}), 1 + 6(\frac{1}{2}), 7(\frac{1}{2})\}$, which is at least 3 as desired.

Finally, if v is a d–vertex with $d \geq 8$, then by Lemma 3.1 (b) we have

$$d \geq 8r + 3q + 1, \tag{2}$$

where r is the number of 1–neighbors and q is the number of 2–neighbors having a 5^--neighbor. The final charge on v is at least $d - 2r - q - \frac{1}{2}(d - r - q) = \frac{d}{2} - \frac{3}{2}r - \frac{1}{2}q$. Thus by (2) v has final charge at least $\frac{1}{2}(8r + 3q + 1) - \frac{3}{2}r - \frac{1}{2}q = \frac{5}{2}r + q + \frac{1}{2}$. When $r \geq 1$ or $q \geq 3$, this final charge is at least 3. If $r = 0$ and $q \leq 2$ then v has final charge at least $d - 2 - \frac{1}{2}(d - 2) = \frac{d-2}{2} \geq 3$, since $d \geq 8$. $\qquad\square$

Theorem 4.3. *If G is a planar graph with* $\mathrm{girth}(G) \geq 7$, *then* $\eta_\ell(G) \leq 8$.

Proof. Let G be a planar graph with girth at least 7 and suppose G is a vertex minimal planar graph with $\eta_\ell(G) > 9$. By Proposition 1.3, $\mathrm{mad}(G) < 14/5$. Assign each vertex v an initial charge of $d(v)$ and apply the following discharging rules.

(R1) Each 1–vertex receives 9/5 charge from its neighbor.

(R2) Each 2–vertex

(a) with one 3^-–neighbor and one 6^+–neighbor receives 4/5 charge from its 6^+–neighbor.

(b) with one 3–neighbor and one 5–neighbor receives 1/5 and 3/5 charge, respectively.

(c) with two 4–neighbors receives 2/5 charge from each neighbor.

(d) with one 4–neighbor and one 5^+–neighbor receives 1/5 and 3/5 charge, respectively.

(e) with two 5^+–neighbors receives 2/5 charge from each neighbor.

A contradiction with $\mathrm{mad}(G) < 14/5$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 14/5; we show this is the case.

By Lemma 3.1 (a) each 1–vertex has an 8^+–neighbor and each 2–vertex has neighbors with degree sum at least 8. Under the discharging rules, 1–vertices and 2–vertices gain 9/5 and 4/5 charge, respectively. If v is a 3–vertex, then by Lemma 3.1 (b), v has at most one 2–neighbor with a 5–neighbor other than v. Thus v gives at most 1/5 charge. Hence, 3^-–vertices have final charge at least 14/5.

If v is a 4–vertex, then by Lemma 3.1 (b) v has at most one 2–neighbor with a 4^-–neighbor other than v. Thus v has final charge at least $4 - 1\left(\frac{2}{5}\right) - 3\left(\frac{1}{5}\right) = 3$.

If v is a 5–vertex, then by Lemma 3.1 (b) v has at most one 2–neighbor with a 4^-–neighbor. Thus v has final charge at least $5 - 1\left(\frac{3}{5}\right) - 4\left(\frac{2}{5}\right) \geq \frac{14}{5}$.

If v is a 6–vertex, then by Lemma 3.1 (b) v has at most one 2–neighbor with a 3^-–neighbor, and at most one 2–neighbor having a 4^-–neighbor. Thus v has final charge at least $6 - 1\left(\frac{4}{5}\right) - 1\left(\frac{3}{5}\right) - 4\left(\frac{2}{5}\right) = 3$.

If v is a 7–vertex, then by Lemma 3.1 (b) v has at most one 2–neighbor with a 3^-–neighbor, and has at most two 2–neighbors with a 4^-–neighbor. Thus v has final charge at least $7 - 1\left(\frac{4}{5}\right) - 2\left(\frac{3}{5}\right) - 4\left(\frac{2}{5}\right) = \frac{17}{5}$.

If v is an 8–vertex, then by Lemma 3.1 (b) v has at most one 1–neighbors, at most two 2–neighbors with a 3^-–neighbor, and at most two 2–neighbors with a 4^-–neighbor. Moreover, if v has a 1–neighbor, then v does not have a 2–neighbor with a 3^-–neighbor. Since the discharging rules allocate charge to neighbors with these constraints, v has final charge at least $8 - \max\left\{1\left(\frac{9}{5}\right) + 7\left(\frac{2}{5}\right), 2\left(\frac{4}{5}\right) + 6\left(\frac{2}{5}\right)\right\} = \frac{17}{5}$.

If v is a d–vertex with $d \geq 9$, then by Lemma 3.1 (b) v has at most $\frac{d}{8}$ 1–neighbors, at most $\frac{d}{4}$ neighbors that are either a 1–vertex or a 2–vertex with a 3^-–neighbor, and at most $\frac{d}{3}$ neighbors that are either a 1–vertex or a 2–vertex with a 4^-–neighbor. Since v gives more charge to neighbors of low degree, we assume v has as many low degree neighbors as possible. Hence v has final charge at least $d - \frac{d}{8}\left(\frac{9}{5}\right) - \left(\frac{d}{4} - \frac{d}{8}\right)\left(\frac{4}{5}\right) - \left(\frac{d}{3} - \frac{d}{4}\right)\left(\frac{3}{5}\right) - \left(d - \frac{d}{3}\right)\left(\frac{2}{5}\right) = \frac{43}{120}d$, which is at least 3 since $d \geq 9$. Therefore, all vertices have final charge at least 14/5 and we obtain a contradiction. $\square$

We call a d-vertex *lonely* if it is in exactly one face of G. We say that a non–lonely 3^+–vertex v is *unique* to a face f of G if it is incident to a cut–edge uv such that $d(u) > 1$ and uv is also in f.

Lemma 4.4. *Let f be a face in a planar graph G with e_c cut–edges such that f has s lonely vertices, and t 3^+–vertices unique to f. We have $s + \frac{t}{2} \leq e_c$.*

Proof. We apply induction on e_c. If $e_c = 0$, then $s = t = 0$ and the inequality holds. In the following two cases, given some face f containing a cut–edge uv, let G' be the graph obtained by contracting the edge uv to a vertex w. Let f' be the face in G' corresponding to f. Let s' and t' be the number of lonely vertices in f' and the number of 3^+–vertices unique to f', respectively.

Case 1: u or v is lonely.
Without loss of generality assume u is lonely. If v is also lonely, then w is lonely and therefore $s' = s - 1$. If v is not lonely, then w is not lonely and still $s' = s - 1$. Vertices unique to f are not affected by the contraction, thus $t' = t$. Since f' has $e_c - 1$ cut–edges, by the induction hypothesis $s' + \frac{t'}{2} \leq e_c - 1$. Therefore, $s + \frac{t}{2} \leq e_c$.

Case 2: u and v are unique to f.
Since u and v are not lonely, w is not lonely and $s' = s$. After contracting uv, either w is unique to f and $t' = t - 1$ or w is not unique to f and $t' = t - 2$, which yields $t' + 1 \leq t \leq t' + 2$. By the induction hypothesis, $s' + \frac{t'}{2} \leq e_c - 1$. Since $t \leq t' + 2$, we have $s + \frac{t}{2} \leq e_c$, as desired. $\square$

Theorem 4.5. *If G is a planar graph with* $\mathrm{girth}(G) \geq 26$*, then* $\eta_\ell(G) \leq 3$*.*

Proof. Let G be planar with girth at least 26 and suppose G is vertex minimal with $\eta_\ell(G) > 3$. Assign each vertex v an initial charge $d(v)$, each face f an initial charge $l(f)$, and apply the following discharging rules.

(R1) Each 1-vertex receives 2 charges from its incident face and 1 charge from its neighbor.

(R2) Each 2-vertex receives 2 charges from its incident face if it is lonely; it receives 1 from each incident face otherwise.

(R3) Each 3-vertex with a 1-neighbor and

    (a) incident to two faces receives 1 charge from each incident face.

    (b) incident to one face receives 2 charges from its face.

(R4) Each 3-vertex without a 1-neighbor and

    (a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.

    (b) incident to two faces receives $\frac{1}{2}$ charge from each incident face.

    (c) incident to one face receives 1 charge from its face.

(R5) Each 4-vertex that has a 1-neighbor and is

    (a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.

    (b) lonely or unique to some face f receives 1 charge from f.

(R6) Each 5-vertex that has two 1-neighbors and is

    (a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.

    (b) lonely or unique to some face f receives 1 charge from f.

A contradiction with Proposition 2.3 occurs if the discharging rules reallocate charge so that every vertex and face has charge at least 4; we show this is the case.

By Lemma 3.1 (a) a 1–vertex has a 3^+–neighbor. By Lemma 3.1 (b) a 4^-–vertex has at most one 1–neighbor, a 5–vertex has at most two 1–neighbors, and in general a d–vertex has at most $\frac{d-1}{2}$ neighbors of degree 1. Since vertices only give charge to 1–neighbors, 6^+–vertices have final charge at least 4. Note that if v is a d–vertex with $d \in \{3,4,5\}$, at most $d-3$ neighbors of degree 1, and incident to at most two faces, then v is unique to a face. Thus all vertices have final charge at least 4 under the discharging rules.

We turn our attention to the final charge of faces. By Theorem 1.5 and the choice of G, G is connected and each face contains at least one cycle. Therefore, each face has length at least 26. Let R_f be the set of vertices incident to a face f that are either a 2–vertex or a 3–vertex that is not lonely and has one 1–neighbor. Let f be a face with s lonely vertices, t unique vertices, and r vertices in R_f. By Lemma 4.4 f has at least $s + \frac{t}{2}$ cut edges. Thus,

$$l(f) \geq 26 + 2s + t. \tag{3}$$

A combination of the reducible configurations in Lemma 3.3 implies that there are at most four consecutive vertices from R_f in any cycle of f. Thus

$$r \leq \left\lfloor \frac{4}{5}(l(f) - 2s - t) \right\rfloor. \tag{4}$$

By the discharging rules, f has final charge at least

$$l(f) - 2s - t - r - \frac{1}{3}(l(f) - 2s - t - r) = \frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}r.$$

By (4),

$$\frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}r \geq \frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}\left\lfloor \frac{4}{5}(l(f) - 2s - t) \right\rfloor. \tag{5}$$

Therefore the final charge of f is at least

$$\frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}\left(\frac{4}{5}(l(f) - 2s - t) \right) = \frac{2}{15}(l(f) - 2s - t),$$

which is at least 4 when $l(f) - 2s - 5 \geq 30$. When $l(f) \in \{26, \ldots, 29\}$, (5) gives final charge at least 4. $\qquad\square$

Since [5] shows $\sigma(C_n) = \chi(C_n)$ for $n \geq 3$ and $\sigma(G) \leq \eta(G)$, we have $\eta(C_{2n+1}) \geq 3$ for $n \geq 13$. By Theorem 4.5, we have the following immediate corollary.

Corollary 4.6. *If $n \geq 13$, then $\eta_\ell(C_{2n+1}) = 3$.*

5 Acknowledgements

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