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ADDITIVE LIST COLORING OF PLANAR GRAPHS WITH GIVEN GIRTH

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Abstract

An *additive coloring* of a graph G is a labeling of the vertices of G from $\{1, 2, \ldots, k\}$ such that two adjacent vertices have distinct sums of labels on their neighbors. The least integer k for which a graph G has an additive coloring is called the *additive coloring number* of G, denoted $\chi_\Sigma(G)$. Additive coloring is also studied under the names lucky labeling and open distinguishing. In this paper, we improve the current bounds on the additive coloring number for particular classes of graphs by proving results for a list version of additive coloring. We apply the discharging method and the Combinatorial Nullstellensatz to show that every planar graph G with girth at least 5 has $\chi_\Sigma(G) \leq 19$, and for girth at least 6, 7, and 26, $\chi_\Sigma(G)$ is at most 9, 8, and 3, respectively. In 2009, Czerwiński, Grytczuk, and Żelazny conjectured that $\chi_\Sigma(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of G. Our result for

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the class of non-bipartite planar graphs of girth at least 26 is best possible and affirms the conjecture for this class of graphs.

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1. Introduction

In this paper we only consider simple, finite, undirected graphs. For such a graph G, let $V(G)$ denote the vertex set and $E(G)$ the edge set of G. When G is a plane graph, let $F(G)$ be the set of faces of G and $l(f)$ be the length of a face f. For brevity when discussing a planar graph G, we will abuse notation by assuming $F(G)$ refers to the faces of a fixed plane embedding of G. Unless otherwise specified, we refer the reader to [21] for notation and definitions.

An *additive coloring* of a graph G is a labeling of the vertices of G with positive integers such that two adjacent vertices have distinct sums of labels on their neighbors. The least integer k for which a graph G has an additive coloring using labels in $\{1, \ldots, k\}$ is called the *additive coloring number* of G, denoted $\chi_\Sigma(G)$. The complexity of this coloring has been investigated in [1, 2, 12, 13].

We briefly mention that additive coloring was introduced in the literature as *lucky labeling* by Czerwiński, Grytczuk, and Żelazny [11]. Another name for this coloring, *open distinguishing*, has been suggested by Axenovich *et al.* [5]. The authors have chosen the name and notation from the survey by Seamone [17].

Determining the additive coloring number of a graph is a natural variation of a well-studied problem posed by Karoński, Łuczak, and Thomason [16], in which edge labels from $\{1, \ldots, k\}$ are summed at incident vertices to induce a vertex coloring. Karoński, Łuczak and Thomason conjectured that edge labels from $\{1, 2, 3\}$ are enough to yield a proper vertex coloring of graphs with no component isomorphic to K_2. This conjecture is known as the 1,2,3-Conjecture and is still open. In 2010, Kalkowski, Karoński and Pfender [15] showed that labels from $\{1, 2, 3, 4, 5\}$ suffice.

In 2009, Czerwiński, Grytczuk, and Żelazny proposed the following conjecture for the additive coloring number of G.

Conjecture 1.1 [11]**.** *For every graph G, $\chi_\Sigma(G) \le \chi(G)$.*

If true, complete graphs imply that this conjecture is best possible [18]. This conjecture remains open even for bipartite graphs, for which no constant bound is currently known. However, a result of Czerwiński, Grytczuk, and Żelazny [11] implies that $\chi_\Sigma(G) \le 2$ when G is a tree or a unicyclic graph. They also show that $\chi_\Sigma(G) \le 100,280,245,065$ for every planar graph G. Note that if Conjecture

1.1 is true, then $\chi_\Sigma(G) \leq 4$ for any planar graph G. The bound for planar graphs was later improved by Bartnicki *et al.* [6].

Theorem 1.2 [6]**.** *If G is a planar graph, then $\chi_\Sigma(G) \leq 468$.*

The *girth* of a graph is the length of its shortest cycle, which is especially useful in giving a measure of sparseness. In the same paper, Bartnicki *et al.* [6] prove the following.

Theorem 1.3 [6]**.** *If G is a planar graph of girth at least 13, then $\chi_\Sigma(G) \leq 4$.*

Their proof provides a labeling for an *I,F-partition* of a graph, a partition of the vertex set in which I is a set of vertices that have pairwise distance greater than 2 and the vertices in F induce a forest. After providing an additive labeling for any I, F-partition, they cite a result of Bu *et al.* [8] that guarantees the existence of an I, F-partition for planar graphs with girth at least 13. Referencing a more recent result on the existence of I, F-partitions gives a stronger result; Brandt *et al.* [7] guarantee the existence of an I, F-partition for all graphs G with $\text{mad}(G) < \frac{5}{2}$, where $\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$ denotes the *maximum average degree* of G. This result implies that $\chi_\Sigma(G) \leq 4$ for all G with $\text{mad}(G) < \frac{5}{2}$. A known relationship between girth and maximum average degree gives the following.

Theorem 1.4. *If G is a planar graph with girth at least 10, then $\chi_\Sigma(G) \leq 4$.*

Since the bound on maximum average degree is tight in the sense that there are graphs with maximum average degree $\frac{5}{2}$ that do not have an I, F-partition, our main result focuses on determining bounds on χ_Σ given by various girth assumptions. We prove the result using the Combinatorial Nullstellensatz within reducibility arguments of the discharging method, which eliminates a considerable amount of case analysis. This approach also provides bounds not only on the additive coloring number, but also a list version introduced by Akbari *et al.* [3] in 2013.

A graph is *additively k-choosable* if whenever each vertex is given a list of at least k available positive integers, then an additive coloring can be chosen from the lists. The *additive choice number* of a graph G is the minimum positive integer k such that G is additively k-choosable, and is denoted by $\text{ch}_\Sigma(G)$. Ahadi and Dehghan [2] show that χ_Σ and ch_Σ can be arbitrarily far apart. Axenovich *et al.* [5] show that $\text{ch}_\Sigma(G) \leq 5\Delta(G) + 1$ for all planar G, which improves Theorem 1.2 when $\Delta(G) \leq 93$. The following are also known.

Theorem 1.5 [11]**.** *If G is a bipartite graph with an orientation in which each vertex has out-degree at most k, then $\text{ch}_\Sigma(G) \leq k + 1$.*

Theorem 1.6 [3]**.** *For every graph G with $\Delta(G) \geq 2$, $\text{ch}_\Sigma(G) \leq \Delta(G)^2 - \Delta(G) + 1$.*

Our main result is the following.

Theorem 1.7. *Let G be a planar graph with girth g.*

1. *If $g \geq 5$, then* $\text{ch}_\Sigma(G) \leq 19$.
2. *If $g \geq 6$, then* $\text{ch}_\Sigma(G) \leq 9$.
3. *If $g \geq 7$, then* $\text{ch}_\Sigma(G) \leq 8$.
4. *If $g \geq 26$, then* $\text{ch}_\Sigma(G) \leq 3$.

Various 3-colorings of planar graphs have been obtained under certain girth assumptions. For example, Grötzsch [14] proved that planar graphs with girth at least 4 are 3-colorable and Thomassen [20] proved that planar graphs with girth at least 5 are 3-list-colorable. Combined with Grötzsch's result, our result answers Conjecture 1.1 in the affirmative for non-bipartite planar graphs with girth at least 26.

Similar to the additive coloring number of a graph, Chartrand, Okamoto, and Zhang [9] defined $\sigma(G)$ to be the smallest integer k such that G has an additive coloring using k distinct labels. They showed that $\sigma(G) \leq \chi(G)$. Note that $\sigma(G) \leq \chi_\Sigma(G)$, since with $\chi_\Sigma(G)$ we seek the smallest k such that labels are from $\{1, \ldots, k\}$, even if some integers in $\{1, \ldots, k\}$ are not used as labels, whereas $\sigma(G)$ considers the fewest distinct labels, regardless of the value of the largest label. They showed that $\sigma(C_n) = \chi(C_n)$ for all $n \geq 3$. As such, Theorems 1.5 and 1.6 then imply that $\chi_\Sigma(C_n) = \text{ch}_\Sigma(C_n) = \chi(C_n)$ for $n \geq 3$. Thus, Theorem 1.7 Part 4 is sharp in that the upper bound on ch_Σ can not be improved.

The remainder of this paper is formatted as follows. In Section 2 we introduce the notation and tools that are used throughout the remainder of the paper. We also give an overview of how we use the discharging method and the Combinatorial Nullstellensatz. In Section 3 we obtain the results of Theorem 1.7.

2. Notation and Tools

Let $N_G(v)$ be the open neighborhood of a vertex v in a graph G. For a labeling $\ell(u) : V(G) \to \mathbb{Z}$ and for $v \in V(G)$, let $S_G(v) = \sum_{u \in N_G(v)} \ell(u)$. When the context is clear we use $S(v)$ in place of $S_G(v)$. For convenience, a *j-vertex*, *j^--vertex*, or *j^+-vertex* is a vertex with degree j, at most j, or at least j, respectively. Similarly, a *j-neighbor* (respectively *j^--neighbor* or *j^+-neighbor*) of v is a j-vertex (respectively j^--vertex or j^+-vertex) adjacent to v.

For sets A and B of real numbers $A \oplus B$ is defined to be the set $\{a + b\colon a \in A, b \in B\}$. Likewise, $A \ominus B$ is defined to be the set $\{a - b : a \in A,\ b \in B\}$. When $B = \emptyset$, we define $A \oplus B = A \ominus B = A$. The following is a straightforward extension of the size of a sumset.

Proposition 1. *Let $A_1, \ldots, A_r$ be finite, nonempty sets of real numbers. We have*

$$|A_1 \oplus \cdots \oplus A_r| \geq 1 + \sum_{i=1}^{r} (|A_i| - 1).$$

Proof. Let $|A_i| = n_i$ for all i. Order the elements of each A_i so that $a_j^{(i)} < a_k^{(i)}$ when $j < k$ for all $a_j^{(i)}, a_k^{(i)} \in A_i$. Notice that

$$\begin{aligned} a_1^{(1)} + a_1^{(2)} + \cdots + a_1^{(r)} &< a_2^{(1)} + a_1^{(2)} + \cdots + a_1^{(r)} < \cdots < a_{n_1}^{(1)} + a_1^{(2)} + \cdots + a_1^{(r)} \\ &< a_{n_1}^{(1)} + a_2^{(2)} + \cdots + a_1^{(r)} < \cdots < a_{n_1}^{(1)} + a_{n_2}^{(2)} + \cdots + a_1^{(r)} \\ &< \cdots \\ &< a_{n_1}^{(1)} + a_{n_2}^{(2)} + \cdots + a_2^{(r)} < \cdots < a_{n_1}^{(1)} + a_{n_2}^{(2)} + \cdots + a_{n_r}^{(r)}. \end{aligned}$$

Since this string of inequalities is obtained by increasing the contribution of an A_i to the sum, the desired size of $A_1 \oplus \cdots \oplus A_r$ follows. ■

Note that $A \oplus (-B)$ is the same as $A \ominus B$, where $-B = \{-b : b \in B\}$. This yields the following known corollary.

Corollary 2. *Let A and B be nonempty sets of positive real numbers. We have $|A \ominus B| \geq |A| + |B| - 1$.*

Throughout, we consider when endpoints of edges need different sums to yield an additive coloring. For this reason, if we know $S(u) \neq S(v)$ for an edge uv of G, we say that uv is *satisfied*; uv is *unsatisfied* otherwise.

Our proofs rely on applying the discharging method. This proof technique assigns an initial charge to vertices and possibly faces of a graph and then distributes charge according to a list of discharging rules. The following, which we use for discharging, appears in Section 3 of [10].

Proposition 3 [10]. *If G is a planar graph with girth g, then* $\mathrm{mad}(G) < \frac{2g}{g-2}$.

A configuration is *k-reducible* if it cannot occur in a vertex minimal graph G with $\mathrm{ch}_\Sigma(G) > k$. Note that any k-reducible configuration is also $(k+1)$-reducible. The main tool we use to determine when configurations are k-reducible is the Combinatorial Nullstellensatz, which is applied to certain graph configurations.

Theorem 2.1 (Combinatorial Nullstellensatz [4]). *Let f be a polynomial of degree t in m variables over a field $\mathbb{F}$. If there is a monomial $\prod x_i^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero in $\mathbb{F}$, then f is nonzero at some point of $\prod T_i$, where each T_i is a set of $t_i + 1$ distinct values in $\mathbb{F}$.*

3. Main Result

We begin by presenting some reducible configurations for general $k \in \mathbb{N}$ that will be used in each subsection. Here and in each subsection, the reducible configurations will use the following notation. Let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a function on $V(G)$ such that $|\mathcal{L}(v)| = k$ for each $v \in V(G)$. Thus $\mathcal{L}(v)$ denotes a list of k available labels for v. In each proof we take G to be a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > k$. Then we define a proper subgraph G' of G with $V(G') \subsetneq V(G)$. By the choice of G, G' has an additive coloring ℓ such that $\ell(v) \in \mathcal{L}(v)$ for all $v \in V(G')$. This labeling of G' is then extended to an additive coloring of G by defining $\ell(v)$ for $v \in V(G) \setminus V(G')$. We discuss the details of this approach in Lemma 3.1. The remaining reducible configurations are similar in approach, so we include fewer details in the proofs.

Lemma 3.1. *The following configurations are k-reducible in the class of graphs with girth at least* 5*.*

(a) *A vertex v with $\sum_{u \in N(v)} d(u) < k$.*

(b) *A vertex v with r 1-neighbors and q 2-neighbors, possibly among others, where $Q = \{v_1, \ldots, v_q\}$ is the set of 2-neighbors of v having a $(k-1)^-$-neighbor other than v, say $v_1', \ldots, v_q'$, respectively, such that $v_1', \ldots, v_q'$ are independent and $1 + r(k-1) + \sum_{v_i \in Q}(k - d(v_i') - 1) > d(v)$.*

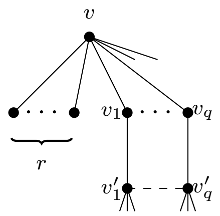

Figure 1. An illustration of the reducible configuration in Lemma 3.1(b).

Proof. Assume G is a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > k$ containing the configuration described in (a). Let $G' = G - \{v\}$. Since G is vertex minimal, $\mathrm{ch}_\Sigma(G') \le k$. Let ℓ be an additive coloring from $\mathcal{L}$ on $V(G')$. Our aim is to choose $\ell(v)$ from $\mathcal{L}(v)$ to extend the additive coloring of G' to an additive coloring of G. Note that the only unsatisfied edges of G are those incident to neighbors of v. Let e be an edge incident to a neighbor u of v. If $e = uv$, then e is satisfied when $\ell(v) \neq \sum_{w \in N(v)} \ell(w) - S_{G'}(u)$. If $e = uw$ for some $w \neq v$, then e is satisfied when $\ell(v) \neq S_{G'}(w) - S_{G'}(u)$. Thus picking $\ell(v)$ distinct from at most $\sum_{u \in N(v)} d(u)$ values ensures that all edges of G are satisfied. Since $\sum_{u \in N(v)} d(u) < k$ there exists $\ell(v)$ in $\mathcal{L}(v)$ that can be used to extend the additive coloring of G' to an additive coloring of G. Therefore $\mathrm{ch}_\Sigma(G) \le k$, a contradiction.

Now assume G is a vertex minimal graph with girth at least 5 and $\mathrm{ch}_\Sigma(G) > k$ containing the configuration described in (b). Let R be the set of r 1-neighbors of v. Let $G' = G - (R \cup Q)$. Since $\mathrm{girth}(G) \geq 5$, Q is independent. Therefore for each $i \in \{1, \ldots, q\}$ there are at least $k - d_G(v_i')$ choices for $\ell(v_i)$ that ensure all edges incident to v_i' are satisfied in G. We ensure that the remaining edges are satisfied, namely $vw \in E(G')$, vv_R with $v_R \in R$, and vv_i for $1 \leq i \leq q$. Any edge $vw \in E(G')$ is satisfied when $\sum_{x \in R \cup Q} \ell(x) \neq S_{G'}(w) - S_{G'}(v)$. Any edge vv_R with $v_R \in R$ is satisfied when $\sum_{x \in R \cup Q} \ell(x) \neq \ell(v) - S_{G'}(v)$. Any edge vv_i with $v_i \in Q$ is satisfied when $\sum_{x \in R \cup Q} \ell(x) \neq \ell(v) + \ell(v_i') - S_{G'}(v)$. Therefore, we must avoid at most $d(v)$ values for $\sum_{x \in R \cup Q} \ell(x)$ in order to satisfy all edges incident to v. Note that each vertex in R has k available labels and recall that each $v_i \in Q$ has $k - d(v_i')$ labels that avoid restricted sums arising from the other edges incident to v_i'. Proposition 1 guarantees at least $1 + r(k-1) + \sum_{v_i \in Q}(k - d(v_i') - 1)$ available values for $\sum_{x \in R \cup Q} \ell(x)$. Since, by assumption, $1 + r(k-1) + \sum_{v_i \in Q}(k - d(v_i') - 1) > d(v)$, there is at least one choice for $\ell(x)$ for each $x \in R \cup Q$ that completes an additive coloring of G. Thus $\mathrm{ch}_\Sigma(G) \leq k$, a contradiction. ■

3.1. Planar and Girth 5 implies $\mathrm{ch}_\Sigma \leq 19$

Lemma 3.2. *A configuration that is an induced cycle of the form $v_1v_2v_3v_4v_5v_1$ such that $d(v_1) \leq 17$, $d(v_2) = d(v_5) = 2$, $d(v_3) \leq 7$, and $d(v_4) \leq 7$ is 19-reducible.*

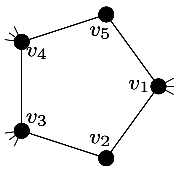

Figure 2. A reducible configuration.

Proof. Let G be a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > 19$ and let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a list assignment on $V(G)$ with $|\mathcal{L}(v)| \geq 19$ for all $v \in V(G)$. Suppose to the contrary that G contains the configuration in Figure 2. Since the most restrictions on labels occurs when $d(v_1) = 17$ and $d(v_3) = d(v_4) = 7$, we assume this is the case. Let $G' = G - \{v_2, v_5\}$. Let $\ell : V(G') \to \mathbb{R}$ be an additive coloring of G' such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G')$. The unsatisfied edges are those incident to $v_1, \ldots, v_5$. The following function has factors corresponding to the unsatisfied edges where x_2 and x_5 represent labels of v_2 and v_5, respectively.

$$f(x_2, x_5) = (S_{G'}(v_1) + x_2 + x_5 - \ell(v_1) - \ell(v_3)) \times (\ell(v_1) + \ell(v_3) - x_2 - S_{G'}(v_3))$$
$$\times (x_2 + S_{G'}(v_3) - x_5 - S_{G'}(v_4)) \times (x_5 + S_{G'}(v_4) - \ell(v_1) - \ell(v_4))$$

$$\times\,(\ell(v_1)+\ell(v_4)-x_2-x_5-S_{G'}(v_1))\times\prod_{w\in N_{G'}(v_4)-v_3}(S_{G'}(w)-S_{G'}(v_4)-x_5)$$
$$\times\prod_{w\in N_{G'}(v_1)}(S_{G'}(w)-S_{G'}(v_1)-x_2-x_5)\times\prod_{w\in N_{G'}(v_3)-v_4}(S_{G'}(w)-S_{G'}(v_3)-x_2).$$

The coefficient of $x_2^{16}x_5^{14}$ in $f(x_2,x_5)$ is the same as the coefficient of $x_2^{10}x_5^{8}$ in $-(x_2+x_5)^{17}(x_2-x_5)$, which is $\binom{17}{10}-\binom{17}{9}$. Since the coefficient of $x_2^{16}x_5^{14}$, a maximum degree monomial in $f(x_2,x_5)$, is not 0, Theorem 2.1 guarantees a choice of labels for $\ell(v_2)$ from any list of size at least $16+1=17$ and $\ell(v_5)$ from any list of size at least $14+1=15$. Therefore $\mathrm{ch}_\Sigma(G)\le 19$, a contradiction. ■

We will also require a large independent set, which is given from the following theorem.

Theorem 3.3 [19]**.** *Every planar triangle-free graph on n vertices has an independent set of size at least $\frac{n+1}{3}$.*

Theorem 3.4. *If G is a planar graph with* $\mathrm{girth}(G)\ge 5$, *then* $\mathrm{ch}_\Sigma(G)\le 19$.

Proof. Let G be a planar graph with girth at least 5 and suppose that G is vertex minimal with $\mathrm{ch}_\Sigma(G)>19$. By Proposition 3, $\mathrm{mad}(G)<10/3$. Assign each vertex v an initial charge $d(v)$, and apply the following discharging rules.

(R1) Each 1-vertex receives 7/3 charge from its neighbor.
(R2) Each 2-vertex
  (a) with two 8^+-neighbors receives 2/3 charge from each neighbor.
  (b) with a 4^--neighbor and a 15^+-neighbor receives 4/3 charge from its 15^+-neighbor.
  (c) with a 10^+-neighbor and a neighbor of degree 5, 6, or 7 receives 1 charge from its 10^+-neighbor and 1/3 charge from its other neighbor.
(R3) Each 3-vertex receives 1/3 charge from a 6^+-neighbor.

A contradiction with $\mathrm{mad}(G)<10/3$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 10/3; we show that this is the case.

By Lemma 3.1(a), each 1-vertex has a 19^+-neighbor, 2-vertices have neighbors with degree sum at least 19, and 3-vertices have at least one 6^+-neighbor. Thus, by the discharging rules, 3^--vertices have final charge 10/3. Since 4-vertices neither give nor receive charge, they have final charge 4.

Vertices of degree d with $d\in\{5,6,7\}$ give charge when incident to 3^--vertices. By the discharging rules, they give away at most $d/3$ charge. This results in a final charge of at least $d-\frac{d}{3}=\frac{2d}{3}\ge\frac{10}{3}$, since $d\ge 5$.

Vertices of degree d with $d \in \{8, 9\}$ may lose charge to 3^--vertices. By Lemma 3.1(a) each 9-vertex has at least one 3^+-neighbor. Also, each 8-vertex has at least two 3^+-neighbors or at least one 4^+-neighbor. By the discharging rules, the final charge of any 9-vertex is at least $9 - 8 \cdot \frac{2}{3} - \frac{1}{3} = \frac{10}{3}$ and the final charge of any 8-vertex is at least $\min\left\{8 - 6 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}, 8 - 7 \cdot \frac{2}{3}\right\} = \frac{10}{3}$.

Since $11 + 7 = 18 < 19$, Lemma 3.1(a) vertices of degree d with $d \in \{10, 11\}$ have no 2-neighbors with a 7^--neighbor. Thus, these vertices have final charge at least $d - \frac{2d}{3} = \frac{d}{3} \geq \frac{10}{3}$, since $d \geq 10$.

Let v have degree d where $d \in \{12, 13, 14\}$. Since $14+4 = 18$, Lemma 3.1(b) implies that v has no 2-neighbor with a 4^--neighbor. By Lemma 3.1(b) and Lemma 3.2, v has at most two 2-neighbors each having a 7^--neighbor. By the discharging rules v has final charge at least $d - 2(1) - (d-2)\left(\frac{2}{3}\right) = \frac{d-2}{3} \geq \frac{10}{3}$, since $d \geq 12$.

Similarly, by Lemma 3.1(b) and Lemma 3.2 vertices of degree 15, 16, or 17 have at most one 2-neighbor with a 4^--neighbor and at most two 2-neighbors with a 7^--neighbor. Thus these vertices give at most $1\left(\frac{4}{3}\right) + 2(1) + (d-3) \cdot \frac{2}{3}$ charge. Hence they have final charge at least $\frac{d-4}{3} \geq \frac{11}{3}$, since $d \geq 15$.

Finally, consider an 18^+-vertex v of degree d. Let r be the number of 1-neighbors of v. Let $U = \{u_1, u_2, \ldots, u_q\}$ be the set of 2-neighbors of v. For each u_i let $N(u_i) \setminus \{v\} = \{u_i'\}$. Let $T = \{u_i' \in U : d(u_i') \leq 7\}$ and let $|T| = t$. Since $G[T]$ is planar with girth at least 5, Theorem 3.3 guarantees at least $\frac{t+1}{3}$ vertices in T that form an independent set. By Lemma 3.1(b), $d \geq 18r + 11\left(\frac{t+1}{3}\right) + 1$. Thus

$$d \geq 18r + \frac{11}{3}t + \frac{14}{3}. \tag{1}$$

The final charge of v is at least $d - \frac{7}{3}r - \frac{4}{3}t - \frac{2}{3}(d-r-t)$. Hence v has final charge at least $\frac{d}{3} - \frac{5}{3}r - \frac{2}{3}t$. From (1), $\frac{d}{3} - \frac{5}{3}r - \frac{2}{3}t \geq \frac{13}{3}r + \frac{5}{9}t + \frac{14}{9}$. When $r \geq 1$ or $t \geq 4$, the final charge is at least $\frac{10}{3}$. When $r = 0$ and $t \leq 3$, the vertex v has final charge at least $d - \frac{4}{3}t - \frac{2}{3}(d-t) \geq \frac{d-6}{3} \geq \frac{12}{3}$, since $d \geq 18$. ∎

3.2. Planar and Girth 6 implies $\mathrm{ch}_\Sigma \leq 9$

Lemma 3.5. *The following configurations are* 8*-reducible in the class of graphs of girth at least* 6*.*

(a) *A* 6*-vertex* v *having six* 2*-neighbors one of which has a* 3^-*-neighbor.*

(b) *A* 7*-vertex* v *having seven* 2*-neighbors two of which have* 4^-*-neighbors.*

Proof. Let G be a vertex minimal graph of girth at least 6 with $\mathrm{ch}_\Sigma(G) > 8$ and let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a list assignment on $V(G)$ with $|\mathcal{L}(v)| \geq 8$ for all $v \in V(G)$. To the contrary suppose G contains the configuration described in (a). Let u be a 2-neighbor of v having a 3^--neighbor. Let $G' = G - \{u, v\}$. Let $\ell : V(G') \to \mathbb{R}$ be an additive coloring of G' such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G')$.

To obtain a contradiction, we extend the labeling ℓ in G' to an additive labeling in G. The only unsatisfied edges of G are those incident to neighbors of u or v. To satisfy the unsatisfied edges not incident to u or v, we avoid at most two values from $\mathcal{L}(u)$ and at most five values from $\mathcal{L}(v)$. Note that $|\mathcal{L}(u)| \geq 8$ and $|\mathcal{L}(v)| \geq 8$. Thus there are at least six labels available for u and at least three available for v. To satisfy the edges incident to u or v, $\ell(u) - \ell(v)$ must avoid at most seven values. Corollary 2 gives at least eight values for $\ell(u) - \ell(v)$ from available labels. Thus there are labels that complete an additive coloring of G. Hence $\mathrm{ch}_\Sigma(G) \leq 8$, a contradiction.

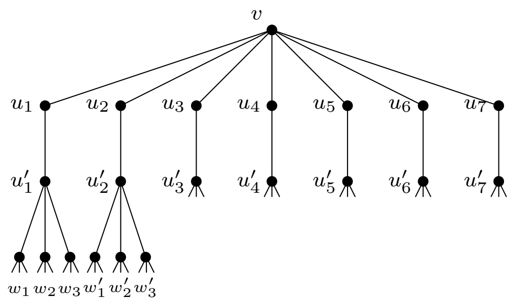

Figure 3. An 8-reducible configuration.

Now, we prove part (b). To the contrary suppose G contains the configuration described in (b). Let $u_1, \ldots, u_7$ be the 2-neighbors of v whose other neighbors are $u'_1, \ldots, u'_7$, respectively, where u'_1 and u'_2 are 4^--vertices. Since the most restrictions on labels occurs when $d(u'_1) = d(u'_2) = 4$, we assume this is the case. Note that since G has girth at least 6, $u'_1 u'_2 \notin E(G)$. Let $N(u'_1) \setminus \{u_1\} = \{w_1, w_2, w_3\}$ and $N(u'_2) \setminus \{u_2\} = \{w'_1, w'_2, w'_3\}$ (see Figure 3). Consider $G' = G - \{v, u_1, u_2\}$. The following function has factors that correspond to unsatisfied edges, where x, y, and z represent the possible values of $\ell(v)$, $\ell(u_1)$, and $\ell(u_2)$, respectively.

$$
\begin{aligned}
f(x,y,z) = \prod_{i=1}^{7} \left(y + z + \sum_{j=3}^{7} \ell(u_j) - x - \ell(u'_i) \right) \times \prod_{i=3}^{7} \big(x + \ell(u'_i) - S_{G'}(u'_i) \big) \\
\times \prod_{i=1}^{3} (y + S_{G'}(u'_1) - S_{G'}(w_i)) \times \prod_{i=1}^{3} (z + S_{G'}(u'_2) - S_{G'}(w'_i)) \\
\times \big(x + \ell(u'_1) - y - S_{G'}(u'_1) \big) \times \big(x + \ell(u'_2) - z - S_{G'}(u'_2) \big).
\end{aligned}
$$

The coefficient of $x^7 y^6 z^7$ in $f(x,y,z)$ is equal to its coefficient in $(y + z - x)^7 x^5 y^3 z^3 (x - y)(x - z)$, which is 490. By Theorem 2.1, there is a choice of labels for $\ell(v)$, $\ell(u_1)$, and $\ell(u_2)$ from lists of size at least 8 that make f nonzero. Thus these labels induce an additive coloring of G. Hence $\mathrm{ch}_\Sigma(G) \le 8$, a contradiction. ∎

Theorem 3.6. *If G is a planar graph with* $\mathrm{girth}(G) \ge 6$*, then* $\mathrm{ch}_\Sigma(G) \le 9$.

Proof. Let G be a planar graph with girth at least 6 and suppose G is vertex minimal with $\mathrm{ch}_\Sigma(G) > 9$. By Proposition 3, $\mathrm{mad}(G) < 3$. Assign each vertex v an initial charge of $d(v)$ and apply the following discharging rules.

(R1) Each 1-vertex receives a charge of 2 from its neighbor.

(R2) Each 2-vertex

- (a) with one 8^+-neighbor and one 5^--neighbor receives 1 charge from its 8^+-neighbor.
- (b) with one 7^+-neighbor and one 4^--neighbor receives 1 charge from its 7^+-neighbor.
- (c) with one 6^+-neighbor and one 3^--neighbor receives 1 charge from its 6^+-neighbor.
- (d) receives 1/2 charge from each neighbor, otherwise.

A contradiction with $\mathrm{mad}(G) < 3$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 3; we show this is the case.

By Lemma 3.1(a) each 1-vertex has a 9^+-neighbor and each 2-vertex has neighbors with degree sum at least 9. Under the discharging rules, 1-vertices and 2-vertices gain charge 2 and 1, respectively, and 3-vertices neither gain nor lose charge. Thus, 3^--vertices have final charge 3.

By Lemma 3.1(b) each 4-vertex v has no 1-neighbor and has at most one 2-neighbor whose other neighbor is a 6^--vertex. Therefore each 4-vertex has final charge at least $4 - \frac{1}{2}$. Similarly, each 5-vertex has no 1-neighbor and has at most four 2-neighbors having another 7^--neighbor. Therefore each 5-vertex has final charge at least $5 - 4\left(\frac{1}{2}\right)$, as desired.

If v is a 6-vertex, then by Lemma 3.1(a), v has no 1-neighbor. By Lemma 3.1(b), v has at most one 2-neighbor with a 3^--neighbor. Moreover, by Lemma 3.5, if v has six 2-neighbors, none of them has a 3^--neighbor. Thus v has either six 2-neighbors each with no 3^--neighbor, or at most five 2-neighbors with at most one with a 3^--neighbor. Hence v has charge at least $6 - \max\left\{6\left(\frac{1}{2}\right), 1(1) + 4\left(\frac{1}{2}\right)\right\}$, which is 3 as desired.

Similarly by Lemma 3.1(a), a 7-vertex v has no 1-neighbor. By Lemma 3.1(b), v has at most two 2-neighbors with a 4^--neighbor. Moreover, by Lemma 3.5, if v has seven 2-neighbors, at most one of them has a 4^--neighbor. Thus v has seven 2-neighbors each with no 4^--neighbor, seven 2-neighbors with at most one with a 4^--neighbor, or at most six 2-neighbors with at most two with a 4^--neighbor.

Hence v has charge at least $7 - \max\left\{7\left(\frac{1}{2}\right), 1(1) + 6\left(\frac{1}{2}\right), 2(1) + 4\left(\frac{1}{2}\right)\right\}$, which is at least 3 as desired.

Finally, if v is a d-vertex with $d \geq 8$, then by Lemma 3.1(b) we have

$$d \geq 8r + 3q + 1, \tag{2}$$

where r is the number of 1-neighbors and q is the number of 2-neighbors having a 5^{-}-neighbor. The final charge on v is at least $d - 2r - q - \frac{1}{2}(d - r - q) = \frac{d}{2} - \frac{3}{2}r - \frac{1}{2}q$. Thus by (2) v has final charge at least $\frac{1}{2}(8r + 3q + 1) - \frac{3}{2}r - \frac{1}{2}q = \frac{5}{2}r + q + \frac{1}{2}$. When $r \geq 1$ or $q \geq 3$, this final charge is at least 3. If $r = 0$ and $q \leq 2$ then v has final charge at least $d - 2 - \frac{1}{2}(d - 2) = \frac{d-2}{2} \geq 3$, since $d \geq 8$. ■

3.3. Planar and Girth 7 implies $\mathrm{ch}_{\Sigma} \leq 8$

Theorem 3.7. *If G is a planar graph with* $\mathrm{girth}(G) \geq 7$*, then* $\mathrm{ch}_{\Sigma}(G) \leq 8$.

Proof. Let G be a planar graph with girth at least 7 and suppose G is a vertex minimal planar graph with $\mathrm{ch}_{\Sigma}(G) > 9$. By Proposition 3, $\mathrm{mad}(G) < 14/5$. Assign each vertex v an initial charge of $d(v)$ and apply the following discharging rules.

(R1) Each 1-vertex receives 9/5 charge from its neighbor.

(R2) Each 2-vertex
- (a) with one 3^{-}-neighbor and one 6^{+}-neighbor receives 4/5 charge from its 6^{+}-neighbor.
- (b) with one 3-neighbor and one 5-neighbor receives 1/5 and 3/5 charge, respectively.
- (c) with two 4-neighbors receives 2/5 charge from each neighbor.
- (d) with one 4-neighbor and one 5^{+}-neighbor receives 1/5 and 3/5 charge, respectively.
- (e) with two 5^{+}-neighbors receives 2/5 charge from each neighbor.

A contradiction with $\mathrm{mad}(G) < 14/5$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 14/5; we show this is the case.

By Lemma 3.1(a) each 1-vertex has an 8^{+}-neighbor and each 2-vertex has neighbors with degree sum at least 8. Under the discharging rules, 1-vertices and 2-vertices gain 9/5 and 4/5 charge, respectively. If v is a 3-vertex, then by Lemma 3.1(b), v has at most one 2-neighbor with a 5-neighbor. Thus v gives at most 1/5 charge. Hence, 3^{-}-vertices have final charge at least 14/5.

If v is a 4-vertex, then by Lemma 3.1(b) v has at most one 2-neighbor with a 4^{-}-neighbor other than v. Thus v has final charge at least $4 - 1\left(\frac{2}{5}\right) - 3\left(\frac{1}{5}\right) = 3$.

If v is a 5-vertex, then by Lemma 3.1(b) v has at most one 2-neighbor with a 4^--neighbor. Thus v has final charge at least $5 - 1\left(\frac{3}{5}\right) - 4\left(\frac{2}{5}\right) \geq \frac{14}{5}$.

If v is a 6-vertex, then by Lemma 3.1(b) v has at most one 2-neighbor with a 4^--neighbor. Thus v has final charge at least $6 - 1\left(\frac{4}{5}\right) - 5\left(\frac{2}{5}\right) = \frac{16}{5}$.

If v is a 7-vertex, then by Lemma 3.1(b) v has at most one 2-neighbor with a 3^--neighbor, and has at most two 2-neighbors with a 4^--neighbor (including the possible 3^--neighbor). Thus v has final charge at least $7 - 1\left(\frac{4}{5}\right) - 1\left(\frac{3}{5}\right) - 5\left(\frac{2}{5}\right) = \frac{18}{5}$.

If v is an 8-vertex, then by Lemma 3.1(b) v has at most one 1-neighbor, at most one 2-neighbors with a 3^--neighbor, and at most two 2-neighbors with a 4^--neighbor. Moreover, if v has a 1-neighbor, then v does not have a 2-neighbor with a 4^--neighbor. Since the discharging rules allocate charge to neighbors with these constraints, v has final charge at least $8 - \max\left\{1\left(\frac{9}{5}\right) + 7\left(\frac{2}{5}\right), 2\left(\frac{4}{5}\right) + 6\left(\frac{2}{5}\right)\right\} = \frac{17}{5}$.

If v is a d-vertex with $d \geq 9$, then by Lemma 3.1(b) v has at most $\frac{d}{8}$ 1-neighbors, at most $\frac{d}{4}$ neighbors that are either a 1-vertex or a 2-vertex with a 3^--neighbor, and at most $\frac{d}{3}$ neighbors that are either a 1-vertex or a 2-vertex with a 4^--neighbor. Since v gives more charge to neighbors of low degree, we assume v has as many low degree neighbors as possible. Hence v has final charge at least $d - \frac{d}{8}\left(\frac{9}{5}\right) - \left(\frac{d}{4} - \frac{d}{8}\right)\left(\frac{4}{5}\right) - \left(\frac{d}{3} - \frac{d}{4}\right)\left(\frac{3}{5}\right) - \left(d - \frac{d}{3}\right)\left(\frac{2}{5}\right) = \frac{43}{120}d$, which is at least 3 since $d \geq 9$. Therefore, all vertices have final charge at least 14/5 and we obtain a contradiction. ■

3.4. Planar and Girth 26 implies $\mathbf{ch}_\Sigma \leq 3$

Lemma 3.8. *Let $P(t_2, \ldots, t_{n-1})$ be the path $v_1 \cdots v_n$ such that for each i in $\{2, \ldots, n-1\}$ the vertex v_i has t_i 1-neighbors and $d(v_i) = 2 + t_i$. The configurations $P(1,0,1)$, $P(1,1,1)$, $P(1,1,0,0)$, $P(0,1,0,0)$, $P(1,0,0,0)$, and $P(0,0,0,0,0)$ are 3-reducible.*

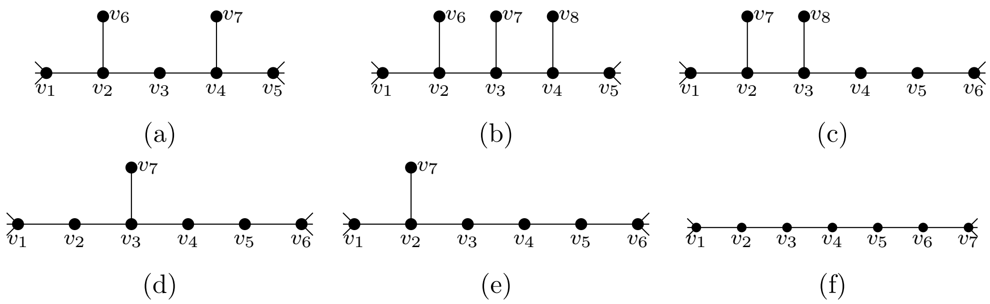

Figure 4. Some 3-reducible configurations.

Proof. Let G be a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > 3$ and let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a list assignment on $V(G)$ with $|\mathcal{L}(v)| \geq 3$ for all $v \in V(G)$. To the contrary suppose G contains $P(1,0,1)$, see Figure 4(a). Let v_6 and v_7 be the neighbors of v_2 and v_4, respectively. Consider $G' = G - \{v_3, v_6, v_7\}$. Let $\ell\colon V(G') \to \mathbb{R}$ be an additive coloring of G' such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G')$. The only unsatisfied edges of G are those incident to v_2 and v_4. The following function has factors that correspond to unsatisfied edges, where x_3, x_6, x_7 represent the possible values of $\ell(v_3), \ell(v_6), \ell(v_7)$, respectively.

$$\begin{aligned}
&f(x_3, x_6, x_7)\\
&= (S_{G'}(v_1) - \ell(v_1) - x_3 - x_6) \times (\ell(v_1) + x_3 + x_6 - \ell(v_2))\\
&\times (\ell(v_1) + x_3 + x_6 - \ell(v_2) - \ell(v_4)) \times (\ell(v_2) + \ell(v_4) - \ell(v_5) - x_3 - x_7)\\
&\times (\ell(v_5) + x_3 + x_7 - \ell(v_4)) \times (\ell(v_5) + x_3 + x_7 - S_{G'}(v_5)).
\end{aligned}$$

The coefficient of $x_3^2x_6^2x_7^2$ in $f(x_3, x_6, x_7)$ is 9. By Theorem 2.1 there is a choice of labels for $\ell(v_3)$, $\ell(v_6)$, and $\ell(v_7)$ from lists of size at least 3 that make f nonzero. Thus these labels induce an additive coloring of G. Hence $\mathrm{ch}_\Sigma(G) \le 3$, a contradiction.

For the following, we simply present the proper subgraph G', the function f derived from the configuration, the monomial, and its coefficient. In each function f, x_i corresponds to the label of v_i.

Suppose G contains $P(1,1,1)$, see Figure 4(b). Let $G' = G - \{v_3, v_6, v_7, v_8\}$.

$$\begin{aligned}
&f(x_3, x_6, x_7, x_8)\\
&= (S_{G'}(v_1) - \ell(v_1) - x_3 - x_6) \times (\ell(v_1) + x_3 + x_6 - \ell(v_2))\\
&\times (\ell(v_1) + x_3 + x_6 - \ell(v_2) - x_7 - \ell(v_4)) \times (\ell(v_2) + \ell(v_4) + x_7 - x_3)\\
&\times (\ell(v_2) + \ell(v_4) + x_7 - \ell(v_5) - x_3 - x_8) \times (\ell(v_5) + x_3 + x_8 - \ell(v_4))\\
&\times (\ell(v_5) + x_3 + x_8 - S_{G'}(v_5)).
\end{aligned}$$

The coefficient of $x_3^2x_6^2x_7x_8^2$ is 15.

Suppose G contains $P(1,1,0,0)$, see Figure 4(c). Let $G' = G - \{v_3, v_4, v_7, v_8\}$.

$$\begin{aligned}
f(x_3, x_4, x_7, x_8) &= (x_3 + x_7 + \ell(v_1) - S_{G'}(v_1)) \times (x_3 + x_7 + \ell(v_1) - \ell(v_2))\\
&\times (x_4 + x_8 + \ell(v_2) - x_3 - x_7 - \ell(v_1)) \times (x_4 + x_8 + \ell(v_2) - x_3)\\
&\times (x_4 + x_8 + \ell(v_2) - x_3 - \ell(v_5)) \times (x_3 + \ell(v_5) - x_4 - \ell(v_6))\\
&\times (x_4 + \ell(v_6) - S_{G'}(v_6)).
\end{aligned}$$

The coefficient of $x_3x_4^2x_7^2x_8^2$ is 8.

Suppose G contains $P(0,1,0,0)$, see Figure 4(d). Let $G' = G - \{v_3, v_4, v_7\}$.

$$\begin{aligned}
f(x_3, x_4, x_7) &= (S_{G'}(v_1) - \ell(v_1) - x_3) \times (\ell(v_1) + x_3 - \ell(v_2) - x_7 - x_4)\\
&\times (\ell(v_2) + x_7 + x_4 - x_3) \times (\ell(v_2) + x_7 + x_4 - x_3 - \ell(v_5))\\
&\times (x_3 + \ell(v_5) - x_4 - \ell(v_6)) \times (x_4 + \ell(v_6) - S_{G'}(v_6)).
\end{aligned}$$

The coefficient of $x_3^2x_4^2x_7^2$ is 6.

Suppose G contains $P(1,0,0,0)$, see Figure 4(e). Let $G' = G - \{v_3, v_4, v_7\}$.

$$\begin{aligned}f(x_3,x_4,x_7) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3 - x_7) \times (\ell(v_2) - \ell(v_1) - x_3 - x_7) \\ & \times (\ell(v_2) + x_4 - \ell(v_1) - x_3 - x_7) \times (\ell(v_2) + x_4 - x_3 - \ell(v_5)) \\ & \times (\ell(v_6) + x_4 - x_3 - \ell(v_5)) \times (\ell(v_6) + x_4 - S_{G'}(v_6)).\end{aligned}$$

The coefficient of $x_3^2x_4^2x_7^2$ is 7.

Suppose G contains $P(0,0,0,0,0)$, see Figure 4(f). Let $G' = G - \{v_3, v_4, v_5\}$.

$$\begin{aligned}f(x_3,x_4,x_5) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3) \times (\ell(v_1) + x_3 - \ell(v_2) - x_4) \\ & \times (\ell(v_2) + x_4 - x_3 - x_5) \times (x_3 + x_5 - x_4 - \ell(v_6)) \\ & \times (x_4 + \ell(v_6) - x_5 - \ell(v_7)) \times (x_5 + \ell(v_7) - S_{G'}(v_7)).\end{aligned}$$

The coefficient of $x_3^2x_4^2x_5^2$ is -7. Theorem 2.1 implies that these configurations are 3-reducible. ∎

We call a d-vertex *lonely* if it is incident to exactly one face of G. We say that a non-lonely 3^+-vertex v is *unique* to a face f of G if it is incident to a cut-edge uv such that $d(u) > 1$ and uv is also incident to f. In the graph pictured in Figure 5, both v and w are unique to f, but neither g nor h. This graph is a sharpness example for Lemma 3.9 because $e_c = 3$ and $s = t = 2$.

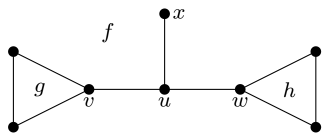

Figure 5. A planar graph with lonely vertices is u and v.

Lemma 3.9. *Let G be a planar graph and $f \in F(G)$ such that f is incident to e_c cut-edges, s lonely vertices, and t 3^+-vertices that are unique to f. We have $s + \frac{t}{2} \leq e_c$.*

Proof. We apply induction on e_c. If $e_c = 0$, then $s = t = 0$ and the inequality holds. In the following two cases, given some face f incident to cut-edge uv, let G' be the graph obtained by contracting the edge uv to a vertex w. Let f' be the face in G' corresponding to f. Let s' and t' be the number of lonely vertices in f' and the number of 3^+-vertices unique to f', respectively.

Case 1. u or v is lonely. Without loss of generality assume u is lonely. If v is also lonely, then w is lonely and therefore $s' = s - 1$. If v is not lonely, then w is not lonely and still $s' = s - 1$. The number of vertices unique to f are not affected by the contraction, thus $t' = t$. Since f' has $e_c - 1$ cut-edges, by the induction hypothesis $s' + \frac{t'}{2} \leq e_c - 1$. Therefore, $s + \frac{t}{2} \leq e_c$.

Case 2. u and v are unique to f. Since u and v are not lonely, w is not lonely and $s' = s$. After contracting uv, either w is unique to f and $t' = t - 1$ or w is not unique to f and $t' = t - 2$, which yields $t' + 1 \leq t \leq t' + 2$. By the induction hypothesis, $s' + \frac{t'}{2} \leq e_c - 1$. Since $t \leq t' + 2$, we have $s + \frac{t}{2} \leq e_c$, as desired. ■

The following follows from Theorem 1.5.

Corollary 4 [11]**.** *If G is a bipartite planar graph, then* $\mathrm{ch}_\Sigma(G) \leq 3$*.*

The following appears as Proposition 3.1 in [10]: given a planar graph G,

$$\sum_{f \in F(G)} (l(f) - 4) + \sum_{v \in V(G)} (d(v) - 4) = -8. \tag{3}$$

Theorem 3.10. *If G is a planar graph with* $\mathrm{girth}(G) \geq 26$*, then* $\mathrm{ch}_\Sigma(G) \leq 3$*.*

Proof. Let G be planar with girth at least 26 and suppose G is vertex minimal with $\mathrm{ch}_\Sigma(G) > 3$. Assign each vertex v an initial charge $d(v)$, each face f an initial charge $l(f)$, and apply the following discharging rules.

(R1) Each 1-vertex receives 2 charges from its incident face and 1 charge from its neighbor.

(R2) Each 2-vertex receives 2 charges from its incident face if it is lonely; it receives 1 from each incident face otherwise.

(R3) Each 3-vertex with a 1-neighbor and
- (a) incident to two faces receives 1 charge from each incident face.
- (b) incident to one face receives 2 charges from its face.

(R4) Each 3-vertex without a 1-neighbor and
- (a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.
- (b) incident to two faces receives $\frac{1}{2}$ charge from each incident face.
- (c) incident to one face receives 1 charge from its face.

(R5) Each 4-vertex that has a 1-neighbor and is
- (a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.
- (b) lonely or unique to some face f receives 1 charge from f.

(R6) Each 5-vertex that has two 1-neighbors and is
- (a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.
- (b) lonely or unique to some face f receives 1 charge from f.

A contradiction with (3) occurs if the discharging rules reallocate charge so that every vertex and face has charge at least 4; we show this is the case.

By Lemma 3.1(a) a 1-vertex has a 3^+-neighbor. By Lemma 3.1(b) a 4^--vertex has at most one 1-neighbor, a 5-vertex has at most two 1-neighbors, and in general a d-vertex has at most $\frac{d-1}{2}$ neighbors of degree 1. Since vertices only give charge to 1-neighbors, 6^+-vertices have final charge at least 4. Note that according to our discharging rules, if v is a d-vertex having at least one 1-neighbor with $d \in \{3, 4, 5\}$, then v receives enough charge from the faces so that its final charge is at least 4. Thus all vertices have final charge at least 4 under the discharging rules.

We turn our attention to the final charge of faces. By Corollary 4 and the choice of G, G is connected and the boundary of each face has a subset that forms a cycle. Therefore, each face has length at least 26. Let R_f be the set of vertices incident to a face f that are either a 2-vertex, or a 3-vertex that is not lonely and has one 1-neighbor. Let f be a face with s lonely vertices, t unique vertices, and r vertices in R_f. By Lemma 3.9 f has at least $s + \frac{t}{2}$ cut edges. Thus,

$$l(f) \geq 26 + 2s + t. \tag{4}$$

The reducible configurations in Lemma 3.8 imply that there are at most four consecutive vertices from R_f in any cycle of f. Thus

$$r \leq \left\lfloor \frac{4}{5}(l(f) - 2s - t) \right\rfloor. \tag{5}$$

By the discharging rules, f has final charge at least

$$l(f) - 2s - t - r - \frac{1}{3}(l(f) - 2s - t - r) = \frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}r.$$

By (5),

$$\frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}r \geq \frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}\left\lfloor \frac{4}{5}(l(f) - 2s - t) \right\rfloor. \tag{6}$$

Therefore the final charge of f is at least

$$\frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}\left(\frac{4}{5}(l(f) - 2s - t)\right) = \frac{2}{15}(l(f) - 2s - t),$$

which is at least 4 when $l(f) - 2s - t \geq 30$. When $l(f) - 2s - t \in \{26, \ldots, 29\}$, (6) gives final charge at least 4. ■

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