Weak Dynamic Coloring of Planar Graphs

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Abstract

The *k-weak-dynamic number* of a graph G is the smallest number of colors we need to color the vertices of G in such a way that each vertex v of degree $d(v)$ sees at least $\min\{k, d(v)\}$ colors on its neighborhood. We use reducible configurations and list coloring of graphs to prove that all planar graphs have 3-weak-dynamic number at most 6.

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1 Introduction

A *proper coloring* of G is a vertex coloring of G in which adjacent vertices receive different colors. The *chromatic number* of G, written as $\chi(G)$, is the smallest number of colors needed to find a proper coloring of G. For notation and definitions not defined here we refer the reader to [14].

A *k-dynamic coloring* of a graph G is a proper coloring of G in such a way that each vertex sees at least $\min\{\mathrm{d(v)}, \mathrm{k}\}$ colors in its neighborhood. The *k-dynamic chromatic number* of a graph G, written as $\chi_k(G)$, is the smallest number of colors needed to find an k-dynamic coloring of G. Dynamic coloring of graphs was first introduced by Montgomery in [11].

Montgomery [11] conjectured that $\chi_2(G) \le \chi(G) + 2$, for all regular graphs G. Montgomery's conjecture was shown to be true for some families of graphs including bipartite regular graphs [1], claw-free regular graphs [11], and regular graphs with diameter at most 2 and chromatic number at least 4 [2]. For all integers k, Alishahi [2] provided a regular graph G with $\chi_2(G) \ge \chi(G) + 1$ and $\chi(G) = k$. In [3], Alishahi proved that $\chi_2(G) \le 2\chi(G)$ for all regular graphs G. Later Bowler et al. [6] disproved the Montgomery's conjecture by showing that Alishahi's bound is best possible. For all integers n with $n \ge 2$, they found a regular graph G with $\chi(G) = n$ but $\chi_2(G) = 2\chi(G)$. Other upper bounds have also been determined for the k-dynamic chromatic number of regular graphs and general graphs. See for example [3, 7, 9, 12].

In this paper we look at a weaker form of dynamic coloring in which we do not look at the constraint that the coloring must be proper. We refer to this type of coloring as a *weak-dynamic coloring*. Therefore a *k-weak-dynamic coloring* of a graph G is a coloring of the vertices of G in such a way that each vertex v sees at least $\min\{\mathrm{d(v)}, \mathrm{k}\}$ colors in its neighborhood. We define *k-weak-dynamic number* of G, written as $wd_k(G)$, to be the smallest number of colors needed to obtain a k-weak-dynamic coloring of G.

By an observation in [9] we have $\chi_k(G) \le \chi(G) wd_k(G)$, because we can associate to each vertex of G an ordered pair of colors in which the first color comes from a proper coloring of G and the second color comes from a k-weak-dynamic coloring of G, to obtain a k-dynamic coloring of G.

A *proper coloring* of a hypergraph is a coloring of its vertices in such a way that each hyperedge sees at least two different colors. For a graph G, let H be the hypergraph with vertex set $V(G)$ whose edges are

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the vertex neighborhoods in G. When $\delta(G) \geq 2$, any 2-weak-dynamic coloring of G corresponds to a proper coloring of H and vice versa.

In this paper we study weak-dynamic coloring of planar graphs. Kim et al. [10] proved that $\chi_2(G) \leq 4$ for all planar graphs G with no C_5-component. Note also that we can find a 2-weak-dynamic coloring of C_5 using only 3 colors. Therefore the inequality $wd_2(G) \leq \chi_2(G)$ implies that all planar graphs have 2-weak-dynamic coloring at most 4. We also know that the upper bound 4 for the 2-weak-dynamic coloring of planar graphs is best possible, as $wd_2(G) = 4$ when G is a subdivision of K_4. Our aim in this paper is to obtain an upper bound for $wd_3(G)$ when G is a planar graph. We prove the following theorem.

Theorem 1. *Any planar graph G satisfies $wd_3(G) \leq 6$.*

In order to prove Theorem 1, we first study an edge-minimal counterexample G to the statement of the theorem. In Section 2 we provide some tools we need during our proofs. In Section 3 we determine some configurations that do not exist in G; we call these *reducible configurations*. In Section 4 we use the reducible configurations we obtain in Section 3 and the the tools we introduce in Section 2 to obtain a 3-weak-dynamic coloring of G using 6 colors, which gives us a contradiction showing that no counterexample exists.

2 Preliminary Tools

A d-vertex in G is a vertex of degree d in G. A d^+-vertex in G is a vertex of degree at least d in G and a d^--vertex in G is a vertex of degree at most d in G. A d-neighbor of a vertex v in G is a neighbor of v having degree d. Similarly, d^+-neighbors of v have degree at least d, and d^--neighbors of v have degree at most d. For a vertex v, $N_G(v)$ (or simply $N(v)$) is the set of neighbors of v in G. We define $N^2(v)$ to be the set of vertices in G having a common neighbor with v. Let c be a vertex coloring of G and $A \subseteq V(G)$. We define $c(A)$ to be the set of colors on vertices in A.

During the proof of Theorem 1, we correspond an edge-minimal counterexample graph G to an auxiliary graph H having the same vertex set as G but with different set of edges. We build H in such a way that any proper coloring of H corresponds to a 3-weak-dynamic coloring of G. Hence for the rest of the proof, our aim would be to find a proper coloring of H using 6 colors. To fulfill the aim we use the following results on proper coloring of graphs and on planar graphs.

Theorem 2 (Four-Color Theorem, Appel and Haken [4])**.** *Any planar graph has chromatic number at most 4.*

Theorem 3 (Wagner's Theorem, Wagner [13])**.** *A graph G is planar if and only if $K_{3,3}$ and K_5 are not minors of G.*

For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A *list coloring* of G is a proper coloring f such that $f(v) \in L(v)$ for each vertex v of G. We say that G is *L-choosable* if it has a list coloring under L. We say that G is *degree-choosable* if G has a list coloring for all lists L with $|L(v)| = d(v)$. A graph G is *2-connected* if it is connected and the removal of any vertex from G leaves it connected. A *block* of G is a maximal 2-connected subgraph of G or a cut-edge. Not all graphs are degree-choosable. For example, odd cycles and complete graphs are not degree choosable. The following result classifies all graphs G that are degree-choosable.

Theorem 4 (Borodin [5] and Erdős, Rubin, and Taylor [8])**.** *Let G be a connected graph having a block that is not an odd cycle nor a complete graph. The graph G is degree-choosable.*

Theorem 4 implies the following Corollary.

Corollary 1. *Let G be a connected graph and L be a list assignment on the vertices $x \in G$ such that $|L(x)| \geq d(x)$ for all x. If there each vertex $v \in V(G)$ such that $|L(v)| > d(v)$, then G is L-choosable.*

Proof. Add a vertex u, an edge uv to G, and add a pendant even cycle C to u in this graph. Give all vertices of C a list of size 3 and keep the list L on other vertices of G. Let H be the resulting graph and L' be the list we defined on vertices of H. Since C is a block of H, by Theorem 4 the graph H is L'-choosable, which implies that G is L-choosable. $\square$

The following propositions are known results on proper list coloring of complete graphs and odd cycles.

Proposition 1. *Let L be a list assignment on the vertices of the complete graph K_n with vertex set $\{v_1, \ldots, v_n\}$ in such a way that $|L(v_i)| = n-1$ for each i and $L(v_1) \neq L(v_k)$. The graph K_n is L-choosable.*

Proof. First color v_1 by a color in $L(v_1) - L(v_n)$. Now choose appropriate colors for vertices $v_2, \ldots, v_{n-1}$ from their lists respectively in such a way that adjacent vertices get different colors. At each step the vertex v_i must have a color different from the color of at most $n-2$ other vertices. Having $|L(v_i)| = n-1$, we are able to choose these colorings. Finally since the color of v_1 does not belong to $L(v_n)$, it is enough to choose a color for v_n to be a color in $L(v_n)$ and different from the colors of $v_2, \ldots, v_{n-1}$ to obtain a proper coloring of K_n. $\square$

Proposition 2. *Let L be a list assignment on the vertices of an odd cycle C with vertices $v_1, \ldots, v_k$ so that $|L(v_i)| = 2$ for each $i \in [k]$ and $L(v_1) \neq L(v_k)$. The cycle C is L-choosable.*

Proof. First color v_1 by a color in $L(v_1) - L(v_k)$. Now choose appropriate colors for vertices $v_2, \ldots, v_{k-1}$ from their lists respectively in such a way that adjacent vertices get different colors. At each step the vertex v_i must have a color different from the color of v_{i-1}. Having $|L(v_i)| = 2$, we are able to choose these colorings. Finally choose a color for v_k to be a color in $L(v_k)$ and different from the color of v_{k-1} to obtain a proper coloring of C. $\square$

The following Proposition is an excercie in [14].

Proposition 3. *Let W be a closed walk of a graph G in such a way that no edge is repeated immediately in W. The graph G contains a cycle.*

Proof. We prove the assertion by applying induction on the length of W. Note that such a closed walk W cannot have length 1 or 2. If W has length 3, then it is a triangle, which is a cycle, as desired. Now suppose W is a walk of length at least 4 in which no edge is repeated immediately. If there is no vertex repetition other than the first vertex, then W is a cycle, as desired. Hence suppose there is some other vertex repetition. Let W' be the portion of W between the instances of such a repetition. In case we have several options for W', we choose one to be the shortest such portion. The walk W' is a shorter closed walk than W and has the property that no edge is repeated immediately, since W has this property. By the induction hypothesis, the subgraph of G over the edges of W' has a cycle, and thus G contains a cycle. $\square$

3 Reducible Configurations

To prove Theorem 1 we show that no counterexample exists to the statement of the theorem. Therefore we start by studying an edge-minimal counterexamples G of the theorem. If there are several such counterexamples, we choose G to be a graph with the smallest number of vertices.

During the proofs of the lemmas in this section, we look at a particular configuration that exists in G. We use deletion of edges and vertices, and sometimes contracting edges to obtain a new graph H with smaller number of edges than G. As a result, the graph H is not a counterexample any more. Hence $wd_3(H) \leq 6$. To obtain a contradiction, we use a 3-weak-dynamic coloring of H to find a 3-weak-dynamic coloring of G using 6 colors.

In a partial coloring of the vertices of a graph G, once a vertex has satisfied the requirements for a 3-weak-dynamic coloring (it sees at least three different colors in its neighborhood) we say the vertex is *satisfied*.

In the following we determine a set of reducible configurations via different lemmas.

Lemma 1. *The edge-minimal graph G with $wd_3(G) > 6$ satisfies $\delta(G) \geq 2$. Moreover G has no 2-vertex with a 3^--neighbor.*

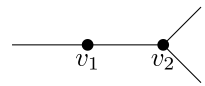


Figure 1: A 2-vertex adjacent to a 3-vertex.

Proof. By the choice of G the graph G is connected. Therefore it has no isolated vertex. If G has a vertex u of degree 1, then $wd_3(G-u) \leq 6$, as $G-u$ has fewer edges than G. Therefore there exists a 3-weak-dynamic coloring of $G-u$ with colors $\{1,\dots,6\}$. Extend this coloring by giving u a color in $\{1,\dots,6\}$ that is different from two colors in the second neighborhood of u. This new coloring is a 3-weak-dynamic coloring of G, a contradiction. Hence $\delta(G) \geq 2$.

Now we prove that G has no 2-vertex v_1 having a 3^--neighbor v_2. We prove $d(v_2) = 3$ gives us a contradiction. The proof of the case that $d(v_2) = 2$ is similar. Hence we suppose $d(v_2) = 3$. Let $H = G - \{v_1v_2\}$. Since H has fewer edges than G, by the choice of G we have $wd_3(H) \leq 6$. Therefore, there exists $c : V(H) \to \{1,\dots,6\}$ that is a 3-weak-dynamic coloring of H. We recolor v_1 and v_2 in c to obtain a 3-weak-dynamic coloring of G.

Let u_1 be the other neighbor of v_1 in G and let u_2 and u_3 be the other neighbors of v_2 in G. Choose a color in $\{1,\dots,6\}$ for v_1 that satisfies v_2 and u_1. Satisfying v_2 and u_1 requires at most four restrictions. Therefore a desired color for v_1 exists. Similarly, choose a color in $\{1,\dots,6\}$ for v_2 to be different from $c(u_1)$ and to satisfy u_2 and u_3. We have at most five restrictions for the coloring of v_2. With six available colors, a desired coloring for v_2 exists. Hence this new coloring is a 3-weak-dynamic coloring of G with six colors, which is a contradiction. $\square$

Lemma 2. *The edge-minimal graph G with $wd_3(G) < 6$ has no pair of adjacent vertices of degree at least 4.*

Proof. Suppose $uv \in E(G)$ with $d(u), d(v) \geq 4$. By the choice of G, we have $wd_3(G - uv) \leq 6$. But any 3-weak-dynamic coloring of $G - uv$ is also a 3-weak-dynamic coloring of G, so we obtain a contradiction. $\square$

Lemma 3. *The edge-minimal graph G with $wd_3(G) > 6$ does not contain distinct vertices $v_1, v_2, v_3, v_4, v_5, v_6$ such that $v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_6 \in E(G)$, $d(v_1) \geq 4$, $d(v_4) \geq 4$ and $d(v_2) = d(v_3) = d(v_5) = d(v_6) = 3$*

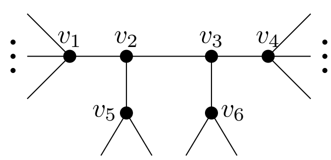


Figure 2: Adjacent 3-vertices with 3-neighbors and 4^+-neighbors.

Proof. On the contrary suppose G contains this configuration. Let $H = G - \{v_2, v_3\}$. Since H has fewer edges than G, we have $wd_3(H) \leq 6$. Thus there exists $c : V(H) \to \{1,\dots,6\}$ that is a 3-weak-dynamic coloring of H. We use c to find a 3-weak-dynamic coloring of G. To obtain this new coloring, we first recolor $c(v_5)$ and $c(v_6)$ and then choose appropriate colors for v_2 and v_3.

Let $N(v_5) = \{v_2, v_5', v_5''\}$ and $N(v_6) = \{v_3, v_6', v_6''\}$. By Lemma 1, the vertices v_5', v_5'', v_6', v_6'' have degree at least 3 in G. We first redefine $c(v_5)$ to be a color in $\{1,\dots,6\}$ and different from $c(v_1)$, different from two distinct colors on $N(v_5')$, and different from two distinct colors on $N(v_5'')$. Since we require at most five restrictions for v_5, such a coloring for v_5 exists. Next, we redefine $c(v_6)$ to to be a color in $\{1,\dots,6\}$ and different from $c(v_4)$, different from two distinct colors on $N(v_6')$, and different from two distinct colors on $N(v_6'')$. Since we require at most five restrictions for v_6, such a coloring for v_6 exists. We have not colored v_2 and v_3 yet, but we know that vertices v_1 and v_4 are already satisfied, because they have degree at least 3 in H and they are satisfied in H.

We then choose $c(v_2)$ to be a color in $\{1,\ldots,6\}$ different from $c(v_4), c(v_6), c(v'_5), c(v''_5)$. Since we have four restrictions for $c(v_2)$, such a coloring for v_2 exists. Last, we choose $c(v_3)$ to differ from $c(v_1), c(v_5), c(v'_6), c(v''_6)$. Therefore we obtain a 3-weak-dynamic coloring of G using six colors, which is a contradiction. $\square$

Lemma 4. *The edge-minimal graph G with $wd_3(G) > 6$ does not contain a 3-face with vertices v_1, v_2, v_3 adjacent to a 3-face with vertices v_1, v_3, v_4, where $d(v_1) = d(v_3) = 3$.*

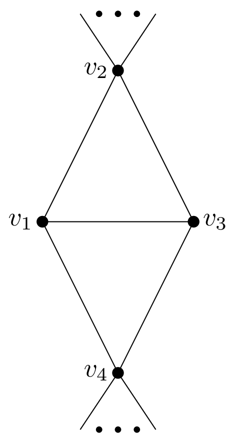


Figure 3: Two adjacent triangles.

Proof. On the contrary suppose G contains this configuration. Contract the edge v_1v_3 into a single vertex $v_{1,3}$ and let H be the resulting graph. Since H has fewer edges than G, it follows that $wd_3(H) \leq 6$. Therefore there exists $c : V(H) \to \{1,\ldots,6\}$ that is a 3-weak-dynamic coloring of H. To obtain a contradiction, we use c to find a 3-weak-dynamic coloring of G. Note that the neighbors of the vertex $v_{1,3}$ in H are v_2 and v_4, therefore we know $c(v_2) \neq c(v_4)$.

By Lemma 1, we have $d_G(v_2) \geq 3$ and $d_G(v_4) \geq 3$. First suppose that $d_G(v_2) \geq 4$ and $d_G(v_4) \geq 4$. In this case each of the vertices v_2 and v_4 has degree at least 3 in H. Hence v_2 sees at least three different colors on its neighborhood in H. As a result, v_2 sees at least two different colors on $N_H(v_2) - \{v_{1,3}\}$. Let's call these two colors c_1 and c_2. Similarly, suppose c_3 and c_4 are two different colors that appear on $N_H(v_4) - \{v_{1,3}\}$. We use the coloring of c over $V(H) - \{v_{1,3}\}$ and then extend it to a 3-weak-dynamic coloring of G.

Choose $c(v_1)$ to be a color in $\{1,\ldots,6\} - \{c(v_2), c(v_4), c_1, c_2\}$. Then choose $c(v_3)$ to be a color in $\{1,\ldots,6\} - \{c(v_2), c(v_4), c_3, c_4\}$. The coloring v_1 is in such a way that the vertex v_2 gets satisfied and the coloring of v_3 is picked in such a way that v_4 becomes satisfied. Since the neighbors of v_1 get different colors and the neighbors of v_3 get different colors, this extension is indeed a 3-weak-dynamic coloring of G.

Now suppose that $d_G(v_2) = 3$. Let c_1 be the color of the neighbor of v_2 in H that is different from $v_{1,3}$. We use the coloring of c over $V(H) - \{v_{1,3}\}$ and then extend it to a 3-weak-dynamic coloring of G.

Let c_2 and c_3 be colors on $N_H(v_4) - v_{1,3}$. We choose c_2 to be different from c_3, when $d_G(v_4) \geq 4$. Otherwise $c_2 = c_3$. Now choose $c(v_3)$ to be a color in $\{1,\ldots,6\} - \{c(v_2), c(v_4), c_1, c_3, c_4\}$. Then choose $c(v_1)$ to be a color in $\{1,\ldots,6\} - \{c(v_2), c(v_3), c(v_4), c_1, c_3\}$. These assignments satisfy the vertices v_2 and v_4. Since the neighbors of v_1 get different colors and the neighbors of v_3 get different colors, this extension is a 3-weak-dynamic coloring of G. $\square$

Lemma 5. *The edge-minimal graph G with $wd_3(G) > 6$ does not contain a triangle with vertices v_1, v_2, v_3, where $d(v_1) = d(v_2) = d(v_3) = 3$.*

Proof. On the contrary suppose G contains this configuration. For each i, let $N_G(v_i) - \{v_1, v_2, v_3\} = \{v'_i\}$. By Lemma 4 the vertices v'_1, v'_2, v'_3 are distinct. Let $H = G - \{v_1, v_2, v_3\}$. Since H has fewer edges than G, we have $wd_3(H) \leq 6$. Thus there exists $c : V(H) \to \{1,\ldots,6\}$ that is a 3-weak-dynamic coloring of H. We use c to find a 3-weak-dynamic coloring of G. By Lemma 1 we have $d_G(v'_1) \geq 3$, $d_G(v'_2) \geq 3$, and $d_G(v'_3) \geq 3$. We consider two cases.

Case 1: $d_G(v_1') = d_G(v_2') = d_G(v_3') = 3$. Let $N_G(v_i') - \{v_i\} = \{w_i, w_i'\}$. We recolor v_1', v_2', v_3' and find appropriate colors for v_1, v_2, v_3. We will call the set of vertices that we plan to color or recolor S. Thus, $S = \{v_1, v_2, v_3, v_1', v_2', v_3'\}$.

Now we study the restrictions we must consider for the coloring on S to make sure that a 3-weak-dynamic coloring of G is obtained. We must choose $c(v_1')$ to be a color different from $c(v_2)$, $c(v_3)$, as well as two distinct colors in $N_G(w_1) - \{v_1'\}$, and also two distinct colors in $N_G(w_1') - \{v_1'\}$. Similarly, $c(v_2')$ must be a color different from $c(v_1)$, $c(v_3)$, and at most four other colors from vertices outside of S, and $c(v_3')$ must be a color different from $c(v_1)$, $c(v_2)$, and at most four other colors from vertices outside of S.

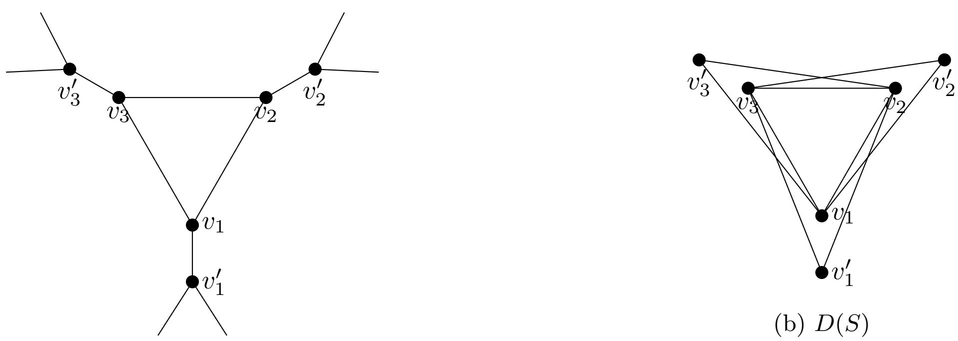
We must also choose $c(v_1)$ to differ from $c(v_2), c(v_3), c(v_2'), c(v_3')$ and also different from $c(w_1)$ and $c(w_1')$. Similarly, $c(v_2)$ must be different from $c(v_1), c(v_3), c(v_1'), c(v_3')$ and also different from $c(w_2), c(w_2')$, and $c(v_3)$ must be different from $c(v_1), c(v_2), c(v_1'), c(v_2'), c(w_3), c(w_3')$.

For each vertex u in S let $R(u)$ be the set of those colors we need to avoid for $c(u)$ that come from vertices outside S. By the above argument we have $|R(u)| \leq 2$ when $u = v_i$ and $|R(u)| \leq 4$ when $u = v_i'$ for each i. For each vertex u in S define $L(u) = \{1, \ldots, 6\} - R(u)$.

Now we form a graph D that represents the dependencies among the vertices of S. D has vertex set S. Two vertices of S are adjacent in D if we require them to have different colors.

First suppose that no pair of vertices in $\{v_1', v_2', v_3'\}$ have a common neighbor. See Figure 4. In this case, in D each v_i has degree 4 and each v_i' has degree 2. Consider the list of colors $L(u)$ we defined on each vertex u of S. Each vertex u has a list of size at least its degree in D. Note that D has one component which is 2-connected and it is not an odd cycle or a complete graph. Therefore by Theorem 4 the graph D is L-choosable. Such a coloring for the vertices of S extends c over $H - S$ to a 3-weak-dynamic coloring of G.

If one or the three pair of vertices in $\{v_1', v_2', v_3'\}$ have common neighbors in G, then in D we will have one, two, or the three edges $v_1'v_2', v_1'v_3', v_2'v_3'$ present, while still each vertex has a list of size at least its degree. Similar to the above argument, Theorem 4 implies that D is L-choosable, as desired.



(a) G

(b) $D(S)$

Figure 4: A triangle with all 3-vertices.

Case 2: $d_G(v_1') \geq 4$.

Since $d_G(v_1') \geq 4$, we have $d_H(v_1') \geq 3$. Hence under the coloring c in H, the vertex v_1' sees at least three different colors on its neighborhood. Therefore when trying to extend the coloring c to a 3-weak-dynamic coloring of G, the vertex v_1' is already satisfied. In this case we keep the colors on all vertices of H. We then choose $c(v_1)$, $c(v_2)$, and $c(v_3)$ to extend c to a 3-weak-dynamic coloring of G.

First choose $c(v_2)$ to be a color in $\{1, \ldots, 6\}$ that is different from $c(v_1')$, $c(v_3')$, and different from two distinct colors on vertices in $N_G(v_2') - \{v_2\}$. We then choose $c(v_3)$ to be a color in $\{1, \ldots, 6\}$, different from $c(v_2)$,$c(v'_1)$, and $c(v'_2)$, and different from two distinct colors on vertices in $N_G(v'_3)-\{v_3\}$. Finally, considering the fact that v'_1 is already satisfied, we choose $c(v_1)$ to be a color in $\{1,\ldots,6\}$ and different from $c(v_2)$, $c(v_3)$, $c(v'_1)$, and $c(v'_2)$. It is easy to see that this extension provides a 3-weak-dynamic coloring of G, which is a contradiction.

□

Lemma 6. *The edge-minimal graph G with $wd_3(G) > 6$ contains no triangle with vertices v_1, v_2, v_3 adjacent to a triangle with vertices v_1, v_3, v_4 such that $d(v_2) = d(v_3) = d(v_4) = 3$ and $d(v_1) \geq 4$.*

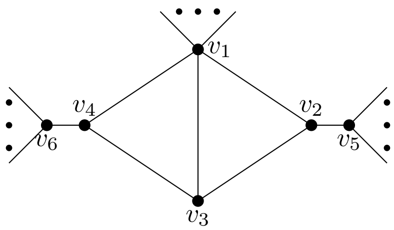


Figure 5: Two adjacent triangles.

Proof. On the contrary suppose G contains this configuration. Let $N_G(v_2) = \{v_1, v_3, v_5\}$ and $N_G(v_4) = \{v_1, v_3, v_6\}$. Let $H = G - \{v_3\}$. Since H has fewer edges than G, we have $wd_3(H) \leq 6$. Therefore, there exists $c : V(H) \to \{1,\ldots,6\}$ that is a 3-weak-dynamic coloring of H. To find a 3-weak-dynamic coloring of G, we recolor vertices v_2 and v_4 and find an appropriate color for v_3.

Let c_1 and c_2 be two different colors on $N_H(v_5) - \{v_2\}$, let c_3 and c_4 be two different colors on $N_H(v_6) - \{v_4\}$, and let c_5 be a color on $N_H(v_1) - \{v_2, v_4\}$.

We first recolor v_2 to be a color in $\{1,\ldots,6\} - \{c(v_1), c_1, c_2, c_5\}$. Now choose $c(v_3)$ to be a color in $\{1,\ldots,6\} - \{c(v_1), c(v_2), c(v_5), c(v_6), c_5\}$. Note that the vertex v_1 becomes satisfied at this stage. Finally recolor v_4 to be a color in $\{1,\ldots,6\} - \{c(v_1), c(v_2), c_3, c_4\}$. Since each of the vertices v_1, v_2, v_3, v_4, v_5, and v_6 become satisfied with these assignments of colors and since c satisfies all other vertices of H, we obtain a 3-weak-dynamic coloring of G. □

Lemma 7. *The edge-minimal graph G does not contain a triangle with vertices v_1, v_2, v_3, where $d(v_1) = d(v_2) = 3$ and $d(v_3) = 4$ such that each of v_1 and v_2 has only one 4^+-neighbor.*

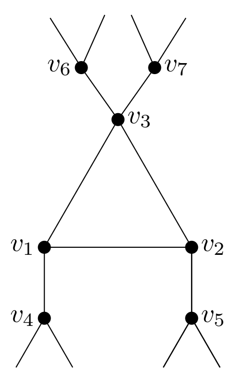


Figure 6: A triangle with a vertex of degree 4.

Proof. On the contrary suppose, G contains this configuration. Let $N_G(v_1) - \{v_2, v_3\} = \{v_4\}$, $N_G(v_2) - \{v_1, v_3\} = \{v_5\}$, and $N_G(v_3) - \{v_1, v_2\} = \{v_6, v_7\}$. Since each of v_1 and v_2 has only one 4^+-neighbor, Lemma

1 implies that $d_G(v_4) = d_G(v_5) = 3$. Moreover Lemma 2 implies that $d_G(v_6) \leq 3$ and $d_G(v_7) \leq 3$. We may suppose that $d_G(v_6) = d_G(v_7) = 3$, because degree 3 vertices provide more restrictions on the coloring.

Contract the edge v_1v_2 to a single vertex $v_{1,2}$ and let H be the resulting graph. Since H has fewer edges than G, we have $wd_3(H) \leq 6$. Therefore there is $c : V(H) \to \{1, ..., 6\}$ that is a 3-weak-dynamic coloring of H. We aim to reach a contradiction by using c to extend the coloring of H to G. Let c_1 and c_2 be two distinct colors in $c(N_H(v_4) - \{v_{1,2}\})$, and let c_3 and c_4 be two distinct colors in $c(N_H(v_5) - \{v_{1,2}\})$. Note that $c(v_6) \neq c(v_7)$, because v_3 has degree 3 in H.

We consider three cases.

Case 1: $|\{c_1, c_2, c(v_6), c(v_7), c(v_3), c(v_5)\}| < 6$.

In this case, we keep the coloring of c over all vertices of $V(H) - \{v_{1,2}\}$. Choose $c(v_1)$ to be a color in $\{1, \ldots, 6\} - \{c_1, c_2, c(v_6), c(v_7), c(v_3), c(v_5)\}$ that satisfies v_2, v_3, and v_4. Then assign v_2 a color in $\{1, \ldots, 6\} - \{c_3, c_4, c(v_3), c(v_4)\}$ that satisfy v_1 and v_5. Therefore we obtain a 3-weak-dynamic coloring of G with at most six colors.

Case 2: $|\{c_3, c_4, c(v_6), c(v_7), c(v_3), c(v_4)\}| < 6$.

In this case, we keep the coloring of c over all vertices of $V(H) - \{v_{1,2}\}$. Choose $c(v_2)$ to be a color in $\{1, \ldots, 6\} - \{c_3, c_4, c(v_6), c(v_7), c(v_3), c(v_4)\}$, satisfying v_1, v_3, and v_5. Then assign v_1 a color in $\{1, \ldots, 6\} - \{c_1, c_2, c(v_3), c(v_5)\}$ to satisfy v_2 and v_4. Therefore we obtain a 3-weak-dynamic coloring of G with at most six colors.

Case 3: $\{c_1, c_2, c(v_6), c(v_7), c(v_3), c(v_5)\} = \{c_3, c_4, c(v_6), c(v_7), c(v_3), c(v_4)\} = \{1, \ldots, 6\}$.

Therefore we have $\{c_1, c_2, c(v_5)\} = \{c_3, c_4, c(v_4)\}$. Since v_4 and v_5 have a common 3-neighbor in H, we have $c(v_4) \neq c(v_5)$. Hence we may suppose that $c(v_4) = c_1$, $c(v_5) = c_3$, and $c_2 = c_4$. As a result, we may suppose that $c_1 = c(v_4) = 1$, $c_2 = c_4 = 2$, $c_3 = c(v_5) = 3$, $c(v_3) = 4$, $c(v_6) = 5$, and $c(v_7) = 6$.

Let $N_G(v_4) = \{v_8, v_9\}$ and let $N_G(v_5) = \{v_{10}, v_{11}\}$. Let c_7 and c_8 be two distinct colors on the neighborhood of v_8, and let c_9 and c_{10} be two distinct colors on the neighborhood of v_9. Now recolor v_4 to be a color in $\{1, \ldots, 6\}$ different from its current color (color 1) and different from $\{c_7, c_8, c_9, c_{10}\}$. If the new color of v_4 is not 4, then choose $c(v_2)$ to be equal to 1 to satisfy v_1, v_3, v_5. Then assign v_1 a color in $\{1, \ldots, 6\} - \{1, 2, 3, 4\}$ to satisfy v_2 and v_4. Therefore we obtain a 3-weak-dynamic coloring of G with at most six colors.

Hence we may suppose we have recolored v_4 and the new color is 4, i.e. $c(v_4) = 4$. By a similar argument as above, we may also recolor v_5 and we can suppose that the new color on v_5 is 4 too. Now recolor v_3 to be a color different from 4, different from two distinct colors in $c(N_G(v_6) - \{v_3\})$, and different from two distinct colors in $c(N_G(v_7) - \{v_3\})$. Now consider the new coloring on v_3, v_4, and v_5.

If $c(v_3) \neq 3$, then let $c(v_1) = 3$ and choose $c(v_2)$ to be a color in $\{1, 5, 6\} - \{c(v_3)\}$. If $c(v_3) = 3$, then let $c(v_1) = 5$ and $c(v_2) = 1$. In the both cases, c provides a 3-weak-dynamic coloring of G, which is a contradiction.

$\square$

Lemma 8. *The edge minimal graph G does not contain a triangle with vertices v_1, v_2, v_3, such that $d(v_1) = d(v_2) = 3$, $d(v_3) \geq 5$, and each of v_1 and v_2 has only one 4^+-neighbor.*

Proof. On the contrary suppose G contains this configuration. Let $H = G - \{v_1, v_2\}$. Since H has fewer edges than G, we have $wd_3(H) \leq 6$. Therefore there exists $c : V(H) \to \{1, ..., 6\}$ that is a 3-weak-dynamic coloring of H. Let $N_G(v_1) = \{v_2, v_3, v_4\}$ and $N_G(v_2) = \{v_1, v_3, v_5\}$. Since $d_G(v_4) \leq 3$ and $d_G(v_5) \leq 3$, Lemma 1 implies that $d(v_4) = d(v_5) = 3$. Fix the coloring c over the vertices $V(G) - \{v_1, v_2, v_4, v_5\}$. We recolor v_4 and v_5 and then find appropriate colors for v_1 and v_2 to obtain a 3-weak-dynamic coloring of G.

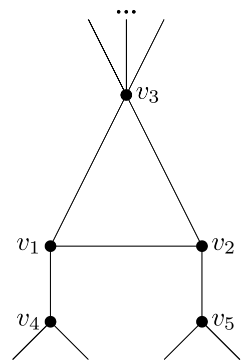


Figure 7: A triangle with a vertex of degree at least 5.

Note that v_3 was satisfied by the coloring of H since $d_H(v_3) \geq 3$. Therefore, when we color v_1 and v_2, the neighbors of v_3 do not create any dependencies for them.

We begin by recoloring v_4 and v_5. We have $d(v_4) = d(v_5) = 3$ and therefore, by the coloring of H, we know that v_4 must avoid two colors from the neighborhood of each vertex in $N(v_4) - \{v_1\}$. Additionally v_4 must avoid $c(v_3)$. Therefore we have only five dependencies on v_4 and we are able to choose an appropriate color for v_4 in $\{1, \ldots, 6\}$. Similarly we have that v_5 must avoid at most five colors. Therefore we can recolor v_5 as well.

Now choose $c(v_1)$ to be a color in $\{1, \ldots, 6\}$, different from $c(v_3)$ and $c(v_5)$, and also different from the colors of the two vertices in $N_G(v_4) - \{v_1\}$. Finally choose $c(v_2)$ to be a color in $\{1, \ldots, 6\}$, different from $c(v_3)$ and $c(v_4)$, and also different from the colors of the two vertices in $N_G(v_5) - \{v_2\}$. This new coloring is a 3-weak-dynamic coloring of G, a contradiction. □

Lemma 9. *The edge-minimal graph G with $wd_3(G) > 6$ contains no cycle C with vertices $v_1, \ldots, v_k$ such that $d(v_1) = \ldots = d(v_k) = 3$, and*

1. *when k is odd, a vertex in $\{v_1, \ldots, v_k\}$ has no 4^+-neighbor, and*
2. *when k is even, a vertex in $\{v_1, v_3, \ldots, v_{k-1}\}$ and a vertex in $\{v_2, v_4, \ldots, v_k\}$ both have no 4^+-neighbor.*

Proof. On the contrary, suppose G contains such a configuration C. We may choose C to be the shortest such configuration. Hence C has no chord. For each i, let v'_i be the neighbor of v_i outside C. Note $v'_1, \ldots, v'_k$ are not necessarily distinct vertices, but they are distinct from $v_1, \ldots, v_k$ because C has no chord. Let $H = G - \{v_1, \ldots, v_k\}$. Since H has fewer edges than G, we have $wd_3(H) \leq 6$. Thus there exists $c : V(H) \to \{1, \ldots, 6\}$ that is a 3-weak-dynamic coloring of H. To obtain a contradiction, we use c to find a 3-weak-dynamic coloring of G.

By Lemma 1 all the vertices $v'_1, \ldots, v'_k$ have degree at least 3 in G. By the structure of C, not all vertices in $\{v'_1, \ldots, v'_k\}$ have degree at least 4. Hence we may suppose that when k is odd, $d(v'_1) = 3$, and when k is even, $d(v'_1) = d(v'_2) = 3$. The proof of the remaining cases is very similar.

Let $S = \{v_1, \ldots, v_k\}$. We aim to extend the coloring c to a 3-weak-dynamic coloring of G by choosing appropriate colors for the vertices in S. Now we study the restrictions we must consider for the coloring on S to make sure that a 3-weak-dynamic coloring of G is obtained. Let $i \in \{1, \ldots, k\}$. If v'_i appears only once in the multiset $\{v'_1, \ldots, v'_k\}$, then we choose $c(v_i)$ to be different from $c(v_{i+2}), c(v_{i-2}), c(v'_{i+1}), c(v'_{i-1})$ as well as at most two distinct colors in $N_H(v'_i)$.

If v'_i appears twice in the multiset $\{v'_1, \ldots, v'_k\}$, then in G the vertex v'_i is adjacent to two vertices of C. As a result we choose the color of v_i to be different from a color in $N_H(v'_i)$ and different from $c(v_{i+2}), c(v_{i-2}), c(v'_{i+1}), c(v'_{i-1})$, and different from the color of an additional vertex in C (the vertex v_j such that $v'_i = v'_j$).

For any vertex x that appears at least three times in the multiset $\{v'_1, \ldots, v'_k\}$, choose S_x to consist of three indices j_1, j_2, j_3 such that $x = v'_{j_1} = v'_{j_2} = v'_{j_3}$. Then if we choose the colors of the vertices $v_{j_1}, v_{j_2}, v_{j_3}$

to be different, the vertex x becomes satisfied in G. Therefore if v'_i appears three or more times in the multiset $\{v'_1, \ldots, v'_k\}$, then we choose the color of v_i to be different from $c(v_{i+1}), c(v_{i-2}), c(v'_{i+1}), c(v'_{i-1})$ and moreover if $i \in S_{v'_i}$ choose $c(v_i)$ to be also different from the color of two other vertices in C (the two vertices other than v_i whose indices belong to $S_{v'_i}$). Note that by the way we aim to choose colors for the vertices $v_1, \ldots, v_k$, if this extension exists, all the vertices $v_1, \ldots, v_k, v'_1, \ldots, v'_k$ become satisfied.

Now we form a graph D that represents the dependencies among the vertices of S. The graph D has vertex set S, and two vertices of S are adjacent in D if we require their colors to be different. For each vertex w in S, let $R(w)$ be the set of those colors we need to avoid for $c(w)$ that come from vertices outside of S. Define $L(w) = \{1, \ldots, 6\} - R(w)$. By the above argument each vertex of S has at most six restrictions, hence $|L(w)|$ is at least the degree of w in D for all $w \in S$. It is enough to show that D is L-choosable, because then the coloring of vertices of D can be used on the corresponding vertices in G to extend c to a 3-weak-dynamic coloring of G.

In D each vertex v_i is adjacent to v_{i-2}, v_{i+2}. When v'_i appears more than once in the multiset $\{v'_1, \ldots, v'_k\}$, the vertex v_i might have other neighbors in D as well. As a result when k is odd, D has one component which is Hamiltonian, and when k is even, D has at most two components.

By Lemma 5, we have $k \neq 3$. When $k = 4$ each of the vertices $v_1, \ldots, v_4$ has at most five restrictions, which makes their lists larger than their degrees. By Corollary 1, D is L-choosable in this case. Hence suppose $k \geq 5$.

First suppose that D is 2-connected. If D is not a complete graph, an odd cycle, if D has a vertex u with $|L(u)| > d_D(u)$, or if not all vertices of D have the same lists, then by Theorem 4, Corollary 1, Proposition 1, and Proposition 2 the graph D is L-choosable, as desired. Hence suppose D is an odd cycle or a complete graph, all its lists are the same, and have size equal to the degrees of the vertices in D. Recall that vertex v'_1 has degree 3 in G. Thus the degree of v'_1 in H is at most 2. Therefore we can recolor v'_1 in H by another color in such a way that the coloring on H stays 3-weak-dynamic. Let c^* be the new 3-weak-dynamic coloring of H. Now repeat the above argument over the coloring c^* of H.

Since $|L(v_i)| = d_D(v_i)$ for all i, we have $v'_{i+1} \neq v'_{i+1}$ (otherwise v_i has at most five restrictions). Moreover the choice of C and Lemma 5 imply that v'_1 appears at most once in the multiset $\{v'_1, \ldots, v'_k\}$. Hence by moving from the coloring c to the coloring c^*, the lists of the vertices v_2 and v_k change to another list, while the lists on other vertices stay as before. Therefore not all the lists are the same now. As a result, by Corollary 1 and Propositions 1 and 2, the graph D is L-choosable, as desired.

Recall that when k is odd, D is Hamiltonian. Hence for the case that k is odd, or k is even but D is 2-connected, the above argument shows that D is L-choosable. Now suppose that k is even and D is not 2-connected. The graph D contains at most two components.

If D has exactly two components C_1 and C_2, then vertices v_1 and v_2 belong to different components of D, because we know that $v_1 v_3 \ldots v_{k-1} v_1$ and $v_2 v_4 \ldots v_k v_2$ are cycles in D. Moreover each of the components is 2-connected, because they are Himiltonian. Since v'_1 and v'_2 have degree at most 2 in H, a similar argument as the one we applied above can be applied here independently for C_1 and C_2 to extend the coloring c (and change it if necessary) to a 3-weak-dynamic coloring of G.

Hence suppose D is connected, but is not 2-connected. Therefore D has two blocks, one with vertices of odd indices, say B_1, and one with vertices of even indices, say B_2. Therefore D has a cut-vertex v. We may suppose that v belongs to B_1.

Now choose colors for vertices of B_2 from their lists in such a way that a proper coloring for B_2 is obtained. This is possible because all vertices of B_2 have lists of size at least their degrees and at least one vertex of B_2 (the neighbor(s) of v in B_2) has a list of size one more than its degree in B_2. Note that v is the only vertex of B_1 that has a neighbor in B_2, since otherwise v cannot be a cut-vertex of D. Now redefine $L(v)$ by removing from it the colors that are already picked for the neighbor(s) of v in B_2. Now consider the new list assignment L over the vertices of B_1. Each vertex has a list of size at least its degree in B_1, and B_1 is 2-connected. If B_1 is not a complete graph or odd cycle (Theorem 4), if B_1 is a complete graph or odd cycle but the lists on its vertices are not identical (Corollary 1), or if B_1 is a complete graph or odd cycle but it has a vertex u with $|L(u)| > d_{B_1}(u)$ (Propositions 1 and 2), then B_1 is L-choosable, as desired.

Hence suppose B_1 is a complete graph or odd cycle, and the lists on the vertices of B_1 are identical and have size equal to the degrees of vertices in B_1. Recall that we supposed $d_G(v'_2) = 3$. Hence in H the vertex v'_2 has degree at most 2. Therefore we can recolor this vertex using a color in $\{1, \ldots, 6\}$ by a different color in such a way that the new coloring c^* is still a 3-weak-dynamic coloring of H. Now repeat the same process as above on defining a list L' on the vertices of D, but using coloring c^* in place of color c.

The vertex v'_2 appears only once in the multiset $\{v'_1, \ldots, v'_k\}$, because if $v'_2 = v'_4$ or $v'_2 = v'_k$, then the vertex v_3 or the vertex v_{k-1} have lists of size larger than their degrees in D, which is not accepted. If $v'_2 = v'_j$ for some $j \notin \{4, k-1\}$, then a configuration smaller than C exists in G, which is also not accepted by the choice of C.

Note that the only difference between colorings c and c^* is on the color of vertex v'_2. By the argument in the above paragraph, only the list of vertices v_1 and v_3 are affected by the color of the vertex v'_2. Hence the only difference between L and L' is on the lists of vertices v_1 and v_3. Therefore the vertices of B_2 get the same colors as before, because for these vertices L and L' are the same. Now redefine $L'(v)$ by removing from it the color of neighbors of v in B_2. Now we try to color the vertices of B_1 using the list assignment L'. But exactly two vertices of B_1 (the vertices v_1 and v_3) have different lists than before. Moreover $k \geq 5$ implies that B_1 has at least three vertices. Therefore not all lists on the vertices of B_1 are now the same. Hence by Corollary 1, Proposition 1, and Proposition 2, B_1 is L'-choosable, as desired. □

Lemma 10. *The edge-minimal graph G with $wd_3(G) > 6$ contains no cycle C with vertices $v_1, \ldots, v_k$ such that $d(v_1) = \ldots = d(v_k) = 3$.*

Proof. On the contrary suppose G contains such a configuration C. We may choose C to be the shortest cycle in G that forms this configuration. Therefore C has no chord. For each i, let v'_i be the neighbor of v_i outside C. Hence, while $v'_1, \ldots v'_k$ are not necessarily distinct vertices, by the choice of C they are distinct from $v_1, \ldots, v_k$. By Lemmas 5, 8, and 9, we have $v'_i \neq v'_{i+1}$ for all i. By Lemma 3, $d(v'_i) \geq 4$ and $d(v'_{i+1}) \geq 4$ do not simultaneously happen for all i. Therefore by Lemma 9, k is even. Moreover by Lemma 9, all vertices in $\{v'_1, v'_3, \ldots, v'_{k-1}\}$ or all vertices in $\{v'_2, v'_4, \ldots, v'_k\}$ have degree at least 4 in G. By symmetry, suppose all vertices in $\{v'_1, v'_3, \ldots, v'_{k-1}\}$ have degree at least 4 in G. As a result by Lemmas 1 and 3, all vertices in $\{v'_2, v'_4, \ldots, v'_k\}$ have degree 3 in G.

Let $H = G - \{v_1, \ldots, v_k\}$. Let H' be the graph obtained from H by identifying vertices v'_1 and v'_3 in H into a single vertex $v'_{1,3}$. Note that H' is still planar and has fewer edges than G. Therefore we have $wd_3(H') \leq 6$. Thus there exists $c : V(H') \to \{1, \ldots, 6\}$ that is a 3-weak-dynamic coloring of H. Now give each vertex v in H the color its corresponding vertex in H' has. Also give vertices v'_1 and v'_3 in H the color of the vertex $v'_{1,3}$ in H'. In the current coloring of H all the vertices of H are satisfied (with respect to 3-weak-dynamic coloring property) except for possibly vertices v'_1 and v'_3.

If v'_1 sees only one color on its neighborhood in H, then choose a neighbor x of v'_1 (which we know has degree at most 3 by Lemma 1). We can recolor x by a different color in $\{1, \ldots, 6\}$ in such a way that its neighbors in $N_H(x) - \{v'_1, v'_3\}$ stay satisfied. Similarly, we can recolor a neighbor of v'_3 in H, when v'_3 sees only one color on its neighborhood in H. Let c^* be the resulting coloring on H. We extend c^* to a 3-weak-dynamic coloring of G by finding appropriate colors for $v_1, \ldots, v_k$. We will call the set of vertices that we want to color S. Thus, $S = \{v_1, \ldots, v_k\}$. Now we study the restrictions we must consider for the coloring on S to make sure that a 3-weak-dynamic coloring of G is obtained.

For each odd i with $i \notin \{1, 3\}$, if v'_i appears only once in the multiset $\{v'_1, \ldots, v'_k\}$, then v'_i is already satisfied in H. Therefore it is enough to choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, and $c(v'_{i-1})$. For such an i, if v'_i appears twice in $\{v'_1, \ldots, v'_k\}$, then we choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, $c(v'_{i-1})$, and different form two colors in $N_H(v'_i)$.

For any vertex x that appears at least three times in the multiset $\{v'_1, \ldots, v'_k\}$, choose S_x to be a set containing three indices j_1, j_2, j_3 such that $x = v'_{j_1} = v'_{j_2} = v'_{j_3}$. Thus if we choose the colors of the vertices $v_{j_1}, v_{j_2}, v_{j_3}$ to be different, the vertex x becomes satisfied in G. Therefore, for the case that i is odd and $i \notin \{1, 3\}$, if v'_i appears three or more times in the multiset $\{v'_1, \ldots, v'_k\}$, then we choose the color of v_i to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, and $c(v'_{i-1})$. If moreover $i \in S_{v'_i}$, then choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, and $c(v'_{i-1})$ and different from the color of two other vertices in C (the two vertices other than v_i whose indices belong to $S_{v'_i}$).

Now suppose $i \in \{1,3\}$. Note that the vertices v'_1 and v'_3 might not be satisfied in H. If v'_i appears only once in $\{v'_1, \ldots, v'_k\}$, then choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, $c(v'_{i-1})$, and also different from two colors in $N_H(v'_i)$. If v'_i appears twice in $\{v'_1, \ldots, v'_k\}$, then we choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, $c(v'_{i-1})$, and different form two colors in $N_H(v'_i)$. And if v'_i appears three or more times in the multiset $\{v'_1, \ldots, v'_k\}$, then we choose the color of v_i to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, $c(v'_{i-1})$ and when $i \in S_{v'_i}$ choose $c(v_i)$ to be also different from the color of two other vertices in C (the two vertices other than v_i whose indices belong to $S_{v'_i}$).

For each even i, the vertex v'_i appears at most twice in the multiset $\{v'_1, \ldots, v'_k\}$, since otherwise a configuration smaller than C exists in G. In fact when $k \neq 4$, the vertex v'_i appears at most once in the multiset $\{v'_1, \ldots, v'_k\}$, by the same reason. If v'_i appears only once in $\{v'_1, \ldots, v'_k\}$, then choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$,$c(v'_{i-1})$, and also different from two colors in $N_H(v'_i)$. If v'_i appears twice in $\{v'_1, \ldots, v'_k\}$, i.e. if $k = 4$ and $v'_2 = v'_4$, then we choose $c(v_i)$ to be different from $c(v_{i+2})$, $c(v_{i-2})$, $c(v'_{i+1})$, $c(v'_{i-1})$, and different from the color of the vertex in $N_H(v'_i)$.

Now we form a graph D that represents the dependencies among the vertices of S. The graph D has vertex set S and two vertices of S are adjacent in D if we require their colors to be different. For each vertex w in S, let $R(w)$ be the set of those colors we need to avoid for $c(w)$ that come from vertices outside S. Define $L(w) = \{1, \ldots, 6\} - R(w)$. By the above argument, each vertex of S has a total of at most six restrictions. Moreover vertices of indices in $\{5, 7, \ldots, k-1\}$ have four restrictions. Since $c^*(v'_1) = c^*(v'_3)$, the vertex v_2 has at most five restrictions, and finally when $k = 4$, all the vertices of S have at most five restrictions, because v_{i+2} and v_{i-2} are the same vertices in this case.

Hence $|L(w)|$ is at least the degree of w in D for all $w \in S$, and $|L(w)|$ has size more than the degree of w in D when $w \in \{v_2, v_5, v_7, \ldots, v_{k-1}\}$. Therefore it is enough to show that D is L-choosable, because in this case the proper coloring we obtain for D would be an extension of c^* to a 3-weak-dynamic coloring of G.

Recall that k is even. If $k = 4$, then since the lists on all vertices have size larger than their degrees in D the graph D is L-choosable by Corollary 1. Thus suppose $k \geq 6$. Since k is even and $k \geq 6$, the graph D contains at most two components and for the case that it contains exactly two components, the vertices v_5 and v_2 belong to different components of D. Therefore all components of D have vertices with lists larger than their degrees in D, which implies that D is L-choosable by Corollary 1. □

4 Proof of Theorem 1

Proof. Let G be an edge-minimal planar graph with $wd_3(G) > 6$. By Lemma 2, the 4^+-vertices of G form an independent set in G. Let A_4 be the set of vertices of degree at least 4 in G. Let A^*_3 be the set of vertices v of degree 3 in G having neighbors u_1, u_2, u_3 that satisfy the following properties:

- $d(u_1) = d(u_2) = 3$;
- each of u_1 and u_2 has two 4^+-neighbors;
- all neighbors of u_3 have degree 3.

For each vertex w of G, choose $N^*(w)$ to be $\min\{\mathrm{d(w)}, 3\}$ vertices on $N(w)$ in such a way that $|N(w) \cap A^*_3|$ is as small as possible. In case we have several options to choose $N^*(w)$ under this condition, we choose a set whose induced subgraph in G has the maximum number of edges.

Let G' be an auxiliary graph of G having the same vertex set as G. For each vertex v in G, make the vertices in $N^*(v)$ pairwise adjacent in G'. Note that by the structure of G', any proper coloring of G' corresponds to a 3-weak-dynamic coloring of G. Thus it is enough to prove that $\chi(G') \leq 6$.

Successively remove vertices v in $V(G) - (A_4 \cup A^*_3)$ from G and instead make all vertices in $N_G(v) \cap (A_4 \cup A^*_3)$ pairwise adjacent. Let H be the resulting graph. Each of these operations preserves planarity, because it corresponds to adding cords to two or three faces of a planar graph and then removing a vertex. Also note that none of the edges added via this type of operation intersect, because their corresponding cords in G are non-intersecting. Therefore H is planar.

If u and v are 4^+-vertices in G having a common neighbor w, then by the structure of A_3^* and by Lemma 2 we have $w \in V(G) - (A_3^* \cup A_4)$. Similarly, if $u \in A_4$ and $v \in A_3^*$ have a common neighbor w in G, then $w \in V(G) - (A_3^* \cup A_4)$. Hence H contains all the edges of G' having at least one endpoint in A_4.

Since H is planar, by the Four Color Theorem there exists a proper coloring $c : V(H) \to \{1, 2, 3, 4\}$. For any vertex $v \in A_4$, define $c^*(v) = c(v)$. Since $G'[A_4] \subseteq H$, the coloring c^* is a proper coloring of $G'[A_4]$. To finish the proof we aim to extend c^* to a proper coloring of G' using colors in $\{1, \ldots, 6\}$.

For each v in $V(G')$, let $N_4(v) = N_{G'}(v) \cap A_4$. For each vertex v in $V(G') - A_4$, we define $L(v) = \{1, \ldots, 6\} - c^*(N_4(v))$. Note that all vertices in $V(G') - A_4$ have degree at most 3 in G, and that by the choice of N^*, each 3-vertex of G has degree at most 6 in G'. We already have a proper coloring of $G'[A_4]$ using four colors $\{1, 2, 3, 4\}$. We aim to extend this coloring to a proper coloring of G'. Hence let $G'' = G' - A_4$. Note that if G'' is L-choosable, then we obtain an extention of the proper coloring of $G'[A_4]$ to a proper coloring of G' using colors $\{1, \ldots, 6\}$. Therefore for the remaining of the proof our aim is to prove that G'' is L-choosable.

Since $d_{G'}(v) \leq 6$ for each vertex v in $V(G') - A_4$, we have $|L(v)| \geq d_{G''}(v)$. If any component of G'' has a vertex whose list size is greater than its degree, or if it has a block that is not a clique or odd cycle, then by Theorem 4 and Corollary 1 G'' is L-choosable, as desired. Therefore let C^* be a component of G'' whose vertices have list size equal to their degrees in G'' and whose blocks are complete graphs or odd cycles.

If $d_{G'}(v) \leq 5$, then $|L(v)| > d_{G''}(v)$. Hence C^* does not contain such a vertex v. This simple observation implies that:

- C^* contains no vertex u whose degree is 2 in G;
- C^* contains no vertex u such that u has a 2-neighbor in G;
- C^* contains no vertex u that is inside a 4-cycle in G;
- C^* does not contain a vertex u such that u is a 3-vertex of G, it has a 4^+-neighbor u' in G, and $u \notin N^*(u')$.

Also note that

- C^* contains no vertex u of A_3^*,

because otherwise using the fact that c is a proper coloring of H using only 4 colors, we know that the four vertices in $N_{G'}(u) \cap A_4$ have at most three distinct colors under c. As a result, $|L(v)| \geq 3$ while $d_{G''}(v) \leq 2$.

Let B be a pendant block of C^*. By the choice of C^* the block B is a complete graph or an odd cycle. Note that since each vertex of A_4 has a color in $\{1, 2, 3, 4\}$, each vertex of G'' gets a list of size at least 2. Therefore no vertex in B has degree 1. Hence B contains at least three vertices.

We consider three cases.

Case 1: B is an odd cycle.

Let the cycle B be $u_1, u_2, \ldots, u_r$. Therefore for each pair of vertices u_i and u_{i+1}, there exists a vertex v_i in G such that u_i and u_{i+1} are neighbors of v_i in G. Therefore $u_1v_1, v_1u_2, u_2v_2, v_2v_3, \ldots, u_rv_r, v_ru_1$ are all edges in G.

Let $r \geq 5$. For each i, if v_i has degree at least 4 in G, then by the construction of G' and since all neighbors of 4^+-vertices in G are 3^--vertices, u_i would be inside a triangle in B. Hence all vertices $v_1, \ldots, v_r$ have degree 3 in G. If $r \geq 4$ and $v_i = v_{i+1}$ for some i, then $N^*(v_i) = \{u_i, u_{i+1}, u_{i+2}\}$. As a result, the vertex u_i has neighbors $u_{i-1}, u_{i+1}, u_{i+2}$ in B. This is a contradiction since B is a cycle. Otherwise, recall that $u_1, \ldots, u_r$ are distinct vertices. Note that $u_1v_1u_2v_2 \ldots u_rv_ru_1$ is a closed walk in G. Since u_is are distinct and since $v_i \neq v_{i+1}$ for all i, no edge is repeated immediately in the closed walk.

As a result of Proposition 3, there exists a cycle in G containing a subset of $\{u_1, \ldots, u_r\} \cup \{v_1, \ldots, v_r\}$. Hence we find a cycle C in G all whose vertices have degree 3. This is a contradiction with Lemma 10.

Now suppose $r = 3$. If v_1, v_2, and v_3 are distinct vertices, then similar to the above argument we obtain a contradiction by finding a cycle in G all whose vertices have degree 3. Hence suppose $v_1 = v_2$. Therefore v_1 is adjacent to u_1, u_2, and u_3 in G. Recall that B is a pendant block of C^*. Therefore at least two vertices of B have degree 2 in C^*. As a result, at least two vertices in $\{u_1, u_2, u_3\}$ have four 4^+-vertices on their second neighborhood. In fact, those two vertices belong to A_3^*, because each of them has a neighbor (v_1) all of whose neighbors are 3^-neighbors and has two other neighbors whose neighbors are 4^+-vertices. This is a contradiction because as we argued above C^* contains no vertex of A_3^*.

Case 2: At least one vertex in $V(B)$ is part of a 3-cycle in G.

Let wv_1v_2 be a triangle in G such that $\{w, v_1, v_2\} \cap V(B) \neq \emptyset$. By Lemma 5, we may suppose that $d_G(w) \geq 4$ and $d_G(v_1) = d_G(v_2) = 3$. Recall that vertices of B are 3-vertices in G. Hence either v_1 and v_2 both belong to $V(B)$ or only one of them belongs to $V(B)$. Let $N_G(v_1) - \{w, v_2\} = \{v_1'\}$ and $N_G(v_2) - \{w, v_1\} = \{v_2'\}$. We consider two subcases.

Subcase 1. $v_1 \in V(B)$ and $v_2 \in V(B)$. By Lemmas 7 and 8 we may suppose that $d_G(v_2') \geq 4$. By the construction of G'', there exists a neighbor v_3 of w such that $N^*(w) = \{v_1, v_2, v_3\}$. Lemmas 4 and 6 imply that v_1', v_2', and v_3 are distinct vertices.

Since $d_G(v_2') \geq 4$ by the construction of G', the vertex v_2' has two neighbors v_4 and v_5 in G such that $N^*(v_2') = \{v_2, v_4, v_5\}$. Note that since G has no 4-cycle containing a vertex in C^*, the vertices v_4 and v_5 are distinct from v_1 and v_3.

The vertex v_2 is adjacent to v_4 and v_5 in C^*. If v_2 is not a cut-vertex of B or if v_4 and v_5 belong to B, then B contains at least 5 vertices ($\{v_1, \ldots, v_5\}$). Hence B cannot be a cycle, because v_2 is adjacent to v_1, v_3, v_4, v_5 in B. Therefore B is a complete graph. Hence vertices v_4 and v_5 must be adjacent to v_1 in B. Equivalently, v_4 and v_5 must have common neighbors with v_1 in G. If $v_4w \in E(G)$ or $v_5w \in E(G)$, then v_2 belongs to a 4-cycle in G, which is not accepted. Hence we must have $v_4v_1' \in E(G)$ and $v_5v_1' \in E(G)$. This is a contradiction, because $v_2'v_4v_1'v_5v_2'$ forms a 4-cycle in G.

Hence v_2 must be a cut-vertex in C^*. If v_4 is a vertex of B, knowing that v_4 is not a cut-vertex of B, then we conclude that v_5 belongs to B. But we argued above that the case $v_4 \in V(B)$ and $v_5 \in V(B)$ cannot happen. Hence none of the vertices v_4 and v_5 belongs to B.

We use a similar argument as above to show that $d_G(v_1') = 3$. If $d_G(v_1') \geq 4$, then let $N^*(v_1') = \{v_1, v_6, v_7\}$. Since v_1 is not a cut-vertex of C^*, the vertices v_6 and v_7 belong to B. Hence B contains at least five vertices ($\{v_1, v_2, v_3, v_6, v_7\}$). Hence B cannot be a cycle, because v_1 is adjacent to v_2, v_3, v_6, v_7 in B. Therefore B is a complete graph. Hence vertices v_6 and v_7 must be adjacent to v_2 in B. Equivalently, v_6 and v_7 must have common neighbors with v_2 in G. If $v_6w \in E(G)$ or $v_7w \in E(G)$, then v_1 belongs to a 4-cycle in G, which is not accepted. Hence we must have $v_6v_2' \in E(G)$ and $v_7v_2' \in E(G)$. This is a contradiction, because $v_1'v_6v_2'v_7v_1'$ forms a 4-cycle in G. Hence we have $d_G(v_1') \leq 3$, and so by Lemma 1, we have $d_G(v_1') = 3$.

Since C^* has no vertex in A_3^*, the vertex v_1' does not have two 4^+-neighbors in G, otherwise $v_1 \in A_3^*$. Hence v_1' must have at least one other 3-neighbor v_6 beside v_1. The vertex v_6 is adjacent to v_1 in B, and as a result it must also be adjacent to v_2 in B. Therefore v_6 must have a common neighbor with v_2 in G that belongs to $N^*(v_2)$. That common neighbor is not w, because otherwise we find a 4-cycle containing v_1 in G. Hence v_6 must belong to $N^*(v_2')$. In other words $v_6 = v_4$ or $v_6 = v_5$. But this is a contradiction, because v_6 is a vertex of B while v_4 and v_5 are not vertices of B.

Subcase 2. $v_1 \in V(B)$ but $v_2 \notin V(B)$. By the construction of G', there exist neighbors v_3 and v_4 of w such that $N^*(w) = \{v_1, v_3, v_4\}$. If $v_3v_4 \in E(G)$, then we can repeat Subcase 1 for the triangle wv_3v_4. Hence suppose $v_3v_4 \notin E(G)$. Therefore by the choice of $N^*(w)$, we have $v_2 \in A_3^*$, $v_3 \notin A_3^*$, and

$v_4 \notin A_3^*$, since otherwise $\{v_1, v_2, v_3\}$ or $\{v_1, v_2, v_4\}$ would give us a better option for $N^*(w)$, according to the choice of $N^*(w)$.

Since $v_2 \in A_3^*$, the vertex v_2' has degree 3 in G and has two 4^+-neighbors in G. By the same reason $d_G(v_1') \geq 4$. Let $N^*(v_1') = \{v_1, v_5, v_6\}$. Note that we know $v_1 \in N^*(v_1')$, since otherwise the vertex v_1 has a list of size larger than its degree in G''. We have $\{v_5, v_6\} \cap \{v_2, v_3, v_4\} = \emptyset$, since otherwise G contains a 4-cycle containing v_1, which is not accepted. Therefore according to the adjacencies we have determined so far in G, the vertex v_1 has neighbors $\{v_2', v_3, \ldots, v_6\}$ in C^*. Therefore $d_{C^*}(v_1) = 5$.

Let v_7 and v_8 be the 4^+-neighbors of v_2'. Since vertex v_2' has two 4^+-neighbors and since v_2' belongs to C^* (because it is adjacent to v_1 in C^*), we must have $v_2' \in N^*(v_7)$ and $v_2' \in N^*(v_8)$, since otherwise the list of v_2' in G'' has size larger than its degree in G'', which is not accepted. Therefore $d_{C^*}(v_2') = 5$.

Let $N_G(v_3) = \{w, v_3', v_3''\}$ and $N_G(v_4) = \{w, v_4', v_4''\}$. If the neighbors of v_3 in C^* are only v_1 and v_4, then v_3 has to be a vertex in A_3^*, which is not accepted. If v_3 has at most one more neighbor besides v_1 and v_4 in C^*, then we must have $d_G(v_3') = d_G(v_3'') = 3$, one vertex in $\{v_3', v_3''\}$ has exactly one 3-neighbor x, and one vertex in $\{v_3', v_3''\}$ has two 4^+-neighbors. When $x \neq w$ we get a contradiction with Lemma 2 and when $x = w$ we get a contradiction with Lemmas 7 and 8. Therefore $d_{C^*}(v_3) \geq 4$. By a similar argument, we have $d_{C^*}(v_4) \geq 4$, $d_{C^*}(v_5) \geq 4$, and $d_{C^*}(v_6) \geq 4$.

By the above arguments, the vertices $v_1, v_2', v_3, v_4, v_5, v_5$ belong to C^* and all of them have degree at least 4 in C^*. We know moreover that $N_{C^*}(v_1) = \{v_2', v_3, v_4, v_5, v_6\}$ and the vertex v_1 is a vertex of the block B. Hence B has 5 or 6 vertices. Since v_1, v_3, v_4 and v_1, v_5, v_6 form triangles in C^*, we conclude that either $V(B) = \{v_1, v_2', v_3, v_4, v_5, v_6\}$ or $V(B) = \{v_1, v_3, v_4, v_5, v_6\}$. In the both cases B cannot be an odd cycle, so it is a complete graph.

Hence v_3 and v_5 have a common neighbor z in G. Also v_3 and v_6 have a common neighbor z' in G. We have $z \neq z'$ and $\{z, z'\} \cap \{w, v_1, \ldots, v_6, v_1', v_2'\}$, since otherwise a 4-cycle containing a vertex of B exists in G or Subcase 1 can be applied. Similarly there are disjoint vertices y and y' in G such that y is a common neighbor of v_4 and v_5 in G, y' is a common neighbor of v_4 and v_6 in G, and $\{y, y'\} \cap \{w, v_1, \ldots, v_6, v_1', v_2'\}$. We also have $\{z, z'\} \cap \{y, y'\} = \emptyset$, since otherwise v_3 or v_5 is inside a 4-cycle in G.

Now the vertices w, v_5, v_6 and v_1, v_3, v_4 are the branch vertices of a $K_{3,3}$-minor in G, which implies G is not planar, a contradiction.

Case 3: B is a complete graph.

By Case 1 we may suppose B is a complete graph with four, five, six, or seven vertices, as each vertex in G'' has degree at most 6. Since B is a pendant block, in G'' all but at most one vertex of B has all its neighbors in $V(B)$. Let v be one of the vertices of B all whose neighbors in G'' are in $V(B)$, i.e. v is not a cut-vertex of C^*. Let u_1, u_2, u_3 be the neighbors of v in G. By Case 2, $\{u_1, u_2, u_3\}$ forms an independent set in G.

We consider three subcases.

Subcase 1. Two of the neighbors of v in B, say w_1 and w_2, are neighbors of u_1 in G, and two of the neighbors of v in B, say w_3 and w_4, are neighbors of u_2 in G.

By Case 2, we may suppose that $\{w_1, w_2, w_3, w_4\} \cap \{u_1, u_2, u_3\} = \emptyset$. Since G is planar, we may suppose that the vertices $w_1, \ldots, w_4$ appear in the counterclockwise direction in the drawing of G. Note that $w_1, \ldots, w_4$ have degree 3 in G. Since B is a complete graph, the four vertices $w_1, \ldots, w_4$ are pairwise adjacent in B, and hence each pair of them must have a common neighbor in G.

Let y_1 be the common neighbor of w_1 and w_3 in G. We have $y_1 \neq w_4$, since otherwise $w_3w_4 \in E(G)$ and Case 2 can be applied on the triangle $u_2w_3w_4$. Similarly $y_1 \neq w_2$. Hence all the vertices $v, u_1, u_2, w_1, w_2, w_3, w_4, y_1$ are distinct. Now consider the cycle $C' : vu_1w_1y_1w_3u_2v$. Since the vertices $w_1, \ldots, w_4$ are in counterclockwise direction, the cycle C' separates the vertex w_2 from the vertex w_4 in G. In order to have a common neighbor for w_2 and w_4 in G, both of w_2 and w_4 have to be adjacent to a vertex x in the cycle C'. We have $x \neq v$, because the only neighbors of v in G are u_1, u_2, u_3. We have $x \neq u_1$, $x \neq u_2$, and $x \neq y_1$, since otherwise G contains a 4-cycle containing w_2 or w_4, which is not accepted. We have $x \neq w_1$ and $x \neq w_3$, because otherwise Case 2 can be applied. Therefore this subcase does not happen.

Subcase 2. Two of the neighbors of v in B, say w_1 and w_2, are neighbors of u_1 in G, and one of the neighbors of v in B, say w_3, is a neighbor of u_2 in G.

Since G is planar, we may suppose that the vertices w_1, w_2, w_3 appear in the counterclockwise direction in G. Note that when $d_B(v) = 6$ or $d_B(v) = 5$, Subcase 1 can be applied to get a contradiction. Hence we may suppose that $d_B(v) \leq 4$. By Subcase 1, we may also suppose that u_2 has a neighbor of degree at least 4. As a result, $d_G(u_2) = 3$. By a similar argument we have $d_G(u_3) = 3$. Let z be the 4^+-neighbor of u_2.

If $d_G(u_1) \geq 4$, then u_1, v, u_2, z, u_3, w_3 form a configuration as of Lemma 3, which is a contradiction. Therefore we have $d_G(u_1) = 3$. The vertices w_1 and w_3 must have a common neighbor y_1 in G. By Case 2, the vertex y_1 is different from vertices w_2 and z. Therefore the vertices $v, u_1, u_2, w_1, w_2, w_3, z, y_1$ are all distinct vertices in G. If $d_G(y_1) \leq 3$, then $y_1 w_1 u_1 v u_2 w_3 y_1$ forms a cycle of all 3^--vertices, which contradicts Lemma 10. Hence $d_G(y_1) \geq 4$.

By the construction of G'', the vertex y_1 has a neighbor w_4 in G such that w_4 is adjacent to w_1 and w_3 in B, i.e. $N^*(y_1) = \{w_1, w_3, w_4\}$. Note that $w_4 \neq w_2$, since otherwise a 4-cycle containing w_2 exists in G. On the other hand since B is a complete graph, w_4 must be in the second neighborhood of v. Therefore w_4 must be adjacent to u_3.

If w_3 has only one 4^+-neighbor in G (the vertex y_1), then y_1, w_3, u_2, z form a configuration as the one in Lemma 2, which is a contradiction. Similarly, if w_4 has only one 4^+-neighbor in G (the vertex y_1), then the vertices y_1, u_4, u_3, and the 4^+-neighbor of u_3 form a configuration as the one in Lemma 2, which is a not accepted. Therefore both of w_3 and w_4 have two 4^+-neighbors in G. As a result, each of them has degree 5 in C^*. We can repeat Subcase 1 for a vertex in $\{w_3, w_4\}$ that is not a cut-vertex of C^*.

Subcase 3. Exactly one neighbor of v in B, say w_1 is a neighbor of u_1 in G, exactly one neighbor of v in B, say w_2 is a neighbor of u_2 in G, and exactly one neighbor of v in B, say w_3 is a neighbor of u_3 in G.

Therefore, by Subcases 1 and 2, we may suppose that each of u_1, u_2, and u_3 has a 4^+-neighbor in G. Suppose z_1 is the 4^+-neighbor of u_1 in G, z_2 is the 4^+-neighbor of u_2 in G, and z_3 is the 4^+-neighbor of u_3 in G. Hence $d_G(u_1) = d_G(u_2) = d_G(u_3) = 3$. Note that in this case B is a complete graph with vertices w_1, w_2, w_3, and v. Hence w_1 and w_2 must have a common neighbor, say y_1, in G.

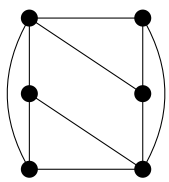
If $d_G(y_1) = 3$, then $v u_1 w_1 y_1 w_2 u_2 v$ is a cycle in G all whose vertices have degree 3, a contradiction with Lemma 10. Hence we must have $d_G(y_1) \geq 4$. Since $|N^*(y_1)| = 3$, all vertices in $N^*(y_1)$ have degree at most 3, and since B has only four vertices, the vertex y_1 must be adjacent to w_3 in G. Recall that at most one vertex in $\{w_1, w_2, w_3\}$ is a cut-vertex of C^*. With no loss of generality suppose w_1 is not a cut-vertex of C^*. Now subcase 2 can be applied on w_1 to get a contradiction.

□

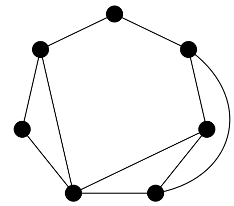
5 Future Work

At the moment, we know of no planar graph with 3-weak-dynamic number 6. However, there are planar graphs with 3-weak-dynamic number 5, as we can see in Figure 8. Therefore the best general upper bound for 3-weak-dynamic number of planar graphs is either 5 or 6.

Question 1. *Are there planar graphs that have 3-weak-dynamic number 6?*



(a) $wd_3(G) = 5$



(b) $wd_3(G) = 5$

Figure 8: Graphs with 3-weak-dynamic number 5.

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