On the Dynamic Coloring of Cartesian Product Graphs

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Abstract

Let $G$ and $H$ be two graphs. A proper vertex coloring of $G$ is called a dynamic coloring, if for every vertex $v$ with degree at least 2, the neighbors of $v$ receive at least two different colors. The smallest integer $k$ such that $G$ has a dynamic coloring with $k$ colors denoted by $\chi_2(G)$. We denote the cartesian product of $G$ and $H$ by $G \square H$. In this paper, we prove that if $G$ and $H$ are two graphs and $\delta(G) \geq 2$, then $\chi_2(G \square H) \leq \max(\chi_2(G), \chi(H))$. We show that for every two natural numbers $m$ and $n$, $m, n \geq 2$, $\chi_2(P_m \square P_n) = 4$. Also, among other results it is shown that if $3|m|n$, then $\chi_2(C_m \square C_n) = 3$ and otherwise $\chi_2(C_m \square C_n) = 4$.

1. Introduction

Let $G$ be a graph. We denote the edge set and the vertex set of $G$, by

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$E(G)$ and $V(G)$, respectively. The number of vertices of $G$ is called the order of $G$. A proper vertex coloring of $G$ is a function $c : V(G) \rightarrow L$, with this property: if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A vertex $k$-coloring is a proper vertex coloring with $|L| = k$. The smallest integer $k$ such that $G$ has a vertex $k$-coloring is called the chromatic number of $G$ and denoted by $\chi(G)$. A proper vertex $k$-coloring of a graph $G$ is called dynamic if for every vertex $v$ with degree at least 2, the neighbors of $v$ receive at least two different colors. The smallest integer $k$ such that $G$ has a dynamic $k$-coloring is called the dynamic chromatic number of $G$ and denoted by $\chi_2(G)$. Recently, the dynamic coloring of graphs has been studied by several authors, see [1], [2], [3]. For any $v \in V(G)$, $N_G(v)$ denotes the neighbor set of $v$ in $G$. Let $c$ be a proper vertex coloring of $G$. For any $v \in V(G)$, we mean $c(N_G(v))$ the set of all colors appearing in the neighbors of $v$ in $G$. In this article, $P_n$ and $C_n$ denote the path and cycle of order $n$, respectively. In the proof of our results we need the following lemma.

Lemma 1. [4, p.5] Let $n \geq 3$ be a natural number. Then we have,

(i) $\chi_2(P_n) = 3$

(ii) $\chi_2(C_n) = \begin{cases} 3 & |n - 3| \\ 4 & 3 \n, \ n \neq 5 \\ 5 & n = 5 \end{cases}$

Let $G$ and $H$ be two graphs. We recall that the cartesian product of $G$ and $H$, $G \square H$, is a graph with the vertex set $V(G) \times V(H)$ such that two vertices $(u, v)$ and $(u', v')$ are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. Clearly, $\Delta(G \square H) = \Delta(G) + \Delta(H)$. For any $(u, v) \in V(G \square H)$, $N_{G \square H}((u, v))$ denotes the neighbor set of $(u, v)$ in $G \square H$.

In the next theorem, we provide an upper bound for the dynamic chromatic number of cartesian product of two graphs.

Theorem 1. Let $G$ and $H$ be two graphs. If $\delta(G) \geq 2$, then $\chi_2(G \square H) \leq$
\[ \max(\chi_2(G), \chi(H)). \]

**Proof.** Suppose that there are dynamic coloring \( c_1 : V(G) \rightarrow \{1, \ldots, \chi_2(G)\} \) and the vertex coloring \( c_2 : V(H) \rightarrow \{1, \ldots, \chi(H)\} \). Assume that \( k = \max(\chi_2(G), \chi(H)) \). For every \( u \in V(G) \) and \( v \in V(H) \), define a vertex coloring \( c : V(G \square H) \rightarrow \{1, \ldots, k\}, c((u, v)) \equiv c_1(u) + c_2(v) \pmod{k}. \) Now, we claim that \( c \) is a dynamic coloring of \( G \square H \). Clearly, \( c \) is a proper coloring. Moreover, for every vertex \( u \in V(G) \), \( |c_1(N_G(u))| \geq 2 \). Thus for every vertex \((u, v) \in V(G \square H)\), \( |c(N_{G \square H}((u, v)))| \geq 2 \) and the proof is complete. \( \Box \)

**Theorem 2.** For every two natural numbers \( m \) and \( n \), \( m, n \geq 2 \), we have 
\[ \chi_2(P_m \square P_n) = 4. \]

**Proof.** Let \( V(P_m) = \{u_1, \ldots, u_m\} \), \( V(P_n) = \{v_1, \ldots, v_n\} \) and \( G = P_m \square P_n \). First note that since \( \Delta(G) \geq 2 \), \( \chi_2(G) \geq 3 \). We claim that \( \chi_2(G) \geq 4 \). To the contrary, assume that \( \chi_2(G) = 3 \). Consider a dynamic 3-coloring \( c \) of \( G \). With no loss of generality we can assume that \( c((u_1, v_1)) = 1 \) and \( c((u_2, v_1)) = 2 \). Also, since \( N_G((u_1, v_1)) = \{(u_1, v_2), (u_2, v_1)\} \) and \( c \) is a dynamic coloring of \( G \), \( c((u_1, v_2)) = 3 \). Now, \( \{2, 3\} \subseteq c(N_G((u_2, v_2))) \) and so \( c((u_2, v_2)) = 1 \). Also, since \( N_G((u_2, v_1)) = \{(u_1, v_1), (u_2, v_2), (u_3, v_1)\} \) and the dynamic property holds for \((u_2, v_1)\), \( c((u_3, v_1)) = 3 \). Now, \( \{1, 3\} \subseteq c(N_G((u_3, v_2))) \) and so \( c((u_3, v_2)) = 2 \). By repeating this procedure, we conclude that the colors of the vertices \((u_1, v_1), \ldots, (u_m, v_1)\) are 1, 2, 3, 1, 2, 3, \ldots, and the colors of the vertices \((u_1, v_2), \ldots, (u_m, v_2)\) are 3, 1, 2, 3, 1, 2, \ldots, respectively. Since \( N_G((u_m, v_1)) = \{(u_{m-1}, v_1), (u_m, v_2)\} \) and also \( c((u_{m-1}, v_1)) = c((u_m, v_2)) \) we have \( |c(N_G((u_m, v_1)))| = 1 \), a contradiction. So \( \chi_2(G) \geq 4 \).

Now, we claim that the function \( c : V(G) \rightarrow \{1, 2, 3, 4\}, c((u_i, v_j)) \equiv i + 2j \pmod{4} \) is a dynamic 4-coloring of \( G \). Since a pair of adjacent vertices is as \((u_i, v_j)\) and \((u_{i+1}, v_j)\) or \((u_i, v_j)\) and \((u_i, v_{j+1})\) for some \( i, j \), \( c \) is a proper coloring of \( G \). In order to see that \( c \) is a dynamic coloring, it suffices to show that in the vertices of each subgraph isomorphic to \( C_4 \) of \( G \), four different colors are appeared. Clearly, the vertices of each subgraph
isomorphic to $C_4$ of $G$, are $(u_i, v_j), (u_i, v_{j+1}), (u_{i+1}, v_{j+1})$ and $(u_{i+1}, v_j)$, for some $i, j$. We have $c((u_i, v_j)) \equiv i + 2j$, $c((u_i, v_{j+1})) \equiv i + 2j + 2$, $c((u_{i+1}, v_j)) \equiv i + 2j + 1$ and $c((u_{i+1}, v_{j+1})) \equiv i + 2j + 3 \mod 4$. Obviously, these four colors are different and so $c$ is a dynamic $4$-coloring of $G$ and the claim is proved. Thus for every two natural numbers $m$ and $n$, $m, n \geq 2$, $\chi_2(P_m \square P_n) = 4$.

In the following theorem, we obtain the dynamic chromatic number of the cartesian product of $C_m$ and $P_n$.

**Theorem 3.** For every two natural numbers $m$ and $n$ ($m \geq 3$),

$$\chi_2(C_m \square P_n) = \begin{cases} 
\chi_2(C_m) & n = 1 \\
3 & 3 \mid m \\
4 & \text{otherwise}
\end{cases}$$

**Proof.** Let $V(C_m) = \{u_1, \ldots, u_m\}$, $V(P_n) = \{v_1, \ldots, v_n\}$ and $G = C_m \square P_n$. If $n = 1$, then $G \simeq C_m$ and the assertion is trivial. So we can assume that $n \neq 1$. Since $\Delta(G) \geq 2$, $\chi_2(G) \geq 3$. If $3 \mid m$, then by Lemma 1 and Theorem 1, we conclude that in this case, $\chi_2(G) = 3$. Now, suppose that $3 \nmid m$ and $m \neq 5$. By Theorem 1, $\chi_2(G) \leq 4$. We claim that in this case, $\chi_2(G) = 4$. To the contrary, assume that $\chi_2(G) = 3$. Consider a dynamic $3$-coloring $c$ of $G$. Since $3 \nmid m$, by Lemma 1, $\chi_2(C_m) \geq 4$. Thus, there exists a vertex in the first copy of $C_m$ in $G$, say $(u_1, v_1)$, for which the dynamic property does not hold. With no loss of generality assume that $c((u_1, v_1)) = 1$ and $c((u_2, v_1)) = c((u_m, v_1)) = 2$. Since the dynamic property holds for $(u_1, v_1)$ in $G$, $c((u_1, v_2)) = 3$. Also, since $\{(u_2, v_1), (u_1, v_2)\} \subseteq N_G((u_2, v_2))$ and $\{(u_m, v_1), (u_1, v_2)\} \subseteq N_G((u_m, v_2))$, $c((u_2, v_2)) = c((u_m, v_2)) = 1$. Moreover, since $c$ is a dynamic coloring of $G$, $c((u_1, v_3)) = 2$. By repeating this procedure, we conclude that $|c(N_G((u_1, v_n)))| = 1$, a contradiction. So, in this case $\chi_2(G) = 4$. Now, suppose that $m = 5$. Since $n \neq 1$, then for every odd number $j$, $1 \leq j \leq n$, define $c((u_1, v_j)) = 1$, $c((u_2, v_j)) =
2, \( c((u_3, v_j)) = 3 \), \( c((u_4, v_j)) = 4 \), \( c((u_5, v_j)) = 2 \) and for every even number \( j \), \( 1 \leq j \leq n \), define \( c((u_1, v_j)) = 3 \), \( c((u_2, v_j)) = 1 \), \( c((u_3, v_j)) = 2 \), \( c((u_4, v_j)) = 1 \), \( c((u_5, v_j)) = 4 \). Clearly, this provides a dynamic 4-coloring of \( C_5 \square P_n \) and so \( \chi_2(C_5 \square P_n) \leq 4 \). By a similar argument, as we did before, we have \( \chi_2(C_5 \square P_n) \geq 4 \). Hence, \( \chi_2(C_5 \square P_n) = 4 \) and the proof is complete.

**Theorem 4.** Let \( G \) be a graph and \( m \geq 3 \) be a natural number. Then the following hold:

(i) If \( 3 \nmid m \), then \( \chi_2(C_m \square G) = \max \{3, \chi(G)\} \).

(ii) If \( 3 \nmid m \) and \( \chi_2(G) = 3 \), then \( \chi_2(C_m \square G) = \begin{cases} 3 & \delta(G) \geq 2 \\ 4 & \delta(G) = 1 \end{cases} \)

(iii) If \( 3 \nmid m \) and \( \chi_2(G) > 3 \), then \( \chi_2(C_m \square G) \geq 4 \). Moreover, if \( G \) is a bipartite graph with no isolated vertex, then \( \chi_2(C_m \square G) = 4 \).

**Proof.** Let \( V(C_m) = \{u_1, \ldots, u_m\} \), \( V(G) = \{v_1, \ldots, v_n\} \) and \( H = C_m \square G \). For every \( i, \ 1 \leq i \leq m \), call the \( i \)-th copy of \( G \) in \( H \), by \( G_i \).

(i) Note that by Theorem 1, \( \chi_2(H) \leq \max(3, \chi(G)) \). Moreover, since \( \Delta(H) \geq 2 \) and \( G \) is a subgraph of \( H \), \( \chi_2(H) \geq \max(3, \chi(G)) \). So \( \chi_2(H) = \max(3, \chi(G)) \).

(ii) If \( \delta(G) \geq 2 \), then using Theorem 1, \( \chi_2(H) = 3 \). Now, assume that \( \delta(G) = 1 \). First we prove that \( \chi_2(H) \leq 4 \). If \( m \neq 5 \), then by Theorem 1, \( \chi_2(H) \leq 4 \). Now, suppose that \( m = 5 \). We can assume that \( G \) is a connected graph. Let \( c_1 : V(G) \rightarrow \{1, 2, 3\} \) be a dynamic 3-coloring of \( G \). For every vertex \( (u_i, v_j) \), \( 1 \leq i \leq 5 \) and \( 1 \leq j \leq n \), define the vertex 3-coloring \( c \) of \( H \) as follows:

\[ c((u_i, v_j)) = c_1(v_j) + i \ (\text{mod } 3) \]

Since \( c_1 \) is a dynamic coloring of \( G \), for every vertex \( (u, v) \) in \( H \) with \( d_G(v) \geq 2 \), the dynamic property holds for this vertex in \( H \). Also, clearly for every \( 2 \leq i \leq 4 \) and \( 1 \leq j \leq n \), \( |c(N_H((u_i, v_j)))| \geq 2 \). Now, for every \( j, \ 1 \leq j \leq n \), if \( d_G(v_j) = 1 \), then we change the colors of vertices \( (u_2, v_j) \) and \( (u_4, v_j) \) to 4. Since \( G \) has
no two adjacent vertices of degree one, the new coloring is still a proper coloring, moreover the dynamic property holds for every vertex of $H$ and so $\chi_2(H) \leq 4$. Now, it suffices to prove that $\chi_2(H) \geq 4$. To the contrary, suppose that $c$ is a dynamic 3-coloring of $H$ with colors $\{1, 2, 3\}$. With no loss of generality let $v_1 \in V(G)$ be a vertex of $G$ such that $N_G(v_1) = \{v_2\}$, $c((u_1, v_1)) = 1$ and $c((u_1, v_2)) = 2$. Since the dynamic property holds for $(u_1, v_1)$ in $H$, with no loss of generality we may assume that $c((u_2, v_1)) = 3$. Now, $\{2, 3\} \subseteq c(N_H((u_2, v_2)))$ and so $c((u_2, v_2)) = 1$. Similarly, since the dynamic property holds for $(u_2, v_1)$ in $H$, $c((u_3, v_1)) = 2$. Now, $\{1, 2\} \subseteq c(N_H((u_3, v_2)))$ and so $c((u_3, v_2)) = 3$. By repeating this procedure, we conclude that $c((u_4, v_1)) = 1, c((u_5, v_1)) = 3, c((u_6, v_1)) = 2, \ldots$. Now, if $c((u_m, v_1)) = 3$, then $c(N_H((u_m, v_1))) = \{1\}$, a contradiction. Thus, $c((u_m, v_1)) = 2$. This implies that $3 \mid m$, a contradiction. Thus, $\chi_2(H) = 4$.

(iii) To the contrary, suppose that $c$ is a dynamic 3-coloring of $H$ with colors $\{1, 2, 3\}$. Note that $\chi_2(G) > 3$ and so there exists a vertex, say $(u_1, v_1)$, such that $c((u_1, v_1)) = 1$ and for every $v_i \in N_G(v_1), c((u_1, v_i)) = 2$. Since the dynamic property holds for $(u_1, v_1)$ in $H$, with no loss of generality we may assume that $c((u_2, v_1)) = 3$. Hence for every $v_i \in N_G(v_1)$, $c((u_2, v_i)) = 1$. Thus $c((u_3, v_1)) = 2$. By repeating this procedure, we conclude that $c((u_4, v_1)) = 1, c((u_5, v_1)) = 3, c((u_6, v_1)) = 2, \ldots$ Now, if $c((u_m, v_1)) = 3$, then $c(N_H((u_m, v_1))) = \{1\}$, a contradiction. Thus, $c((u_m, v_1)) = 2$. This implies that $3 \mid m$, a contradiction. Thus, $\chi_2(H) \geq 4$.

Now, assume that $G = (X, Y)$ is a bipartite graph such that $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$. If $m = 5$, then consider two vertex 4-colorings $c$ and $c'$ of $C_5$, $c(u_1) = 1, c(u_2) = 2, c(u_3) = 3, c(u_4) = 4, c(u_5) = 2$ and $c'(u_1) = 3, c'(u_2) = 4, c'(u_3) = 1, c'(u_4) = 2, c'(u_5) = 1$. Now, define the dynamic 4-coloring $c''$ of $H$ as follows:

For $1 \leq i \leq 5$ and $1 \leq j \leq s$, let $c''((u_i, x_j)) = c(u_i)$ and for $1 \leq i \leq 5$ and $1 \leq k \leq t$, let $c''((u_i, y_k)) = c'(u_i)$. This shows that in this case $\chi_3(H) \leq 4$ and so $\chi_2(H) = 4$. Now, suppose that $m \neq 5$. Since $3 \nmid m$, then $\chi_2(C_m) = 4$. Consider a dynamic 4-coloring $c'$ of $C_m$. Then for every vertex $(u_i, x_j), 1 \leq i \leq m$ and $1 \leq j \leq s$, define $c((u_i, x_j)) = c'(u_i)$ and also for
every vertex \((u_i, y_k), 1 \leq i \leq m\) and \(1 \leq k \leq t\), define \(c((u_i, y_k)) = c'(u_i) + 1 \pmod{4}\). Clearly, \(c\) is a dynamic 4-coloring of \(H\). Thus, we conclude that \(\chi_2(H) \leq 4\). So, \(\chi_2(H) = 4\). \(\square\)

**Theorem 5.** Let \(m, n \geq 3\) be two natural numbers. Then

\[
\chi_2(C_m \square C_n) = \begin{cases} 
3 & \text{if } 3 \mid mn \\
4 & \text{if } 3 \nmid mn
\end{cases}
\]

**Proof.** Let \(V(C_m) = \{u_1, \ldots, u_m\}\), \(V(C_n) = \{v_1, \ldots, v_n\}\) and \(G = C_m \square C_n\). Since \(\Delta(G) \geq 2\), \(\chi_2(G) \geq 3\). First suppose that \(3 \mid mn\). By Theorem 1, \(\chi_2(G) = 3\). Now, suppose that \(3 \nmid mn\). By Lemma 1 and Theorem 4, Part (iii), \(\chi_2(G) \geq 4\). If one of the \(m\) and \(n\) is not 5, then by Theorem 1, \(\chi_2(G) \leq 4\) and we are done. So, suppose that \(m = n = 5\). Now, we define the dynamic 4-coloring \(c\) of \(C_5 \square C_5\) as follows:

Consider the following \(5 \times 5\) matrix \(A\), \(A = [a_{ij}]\) and define \(c((u_i, v_j)) = a_{ij}\), for every \(i\) and \(j\), \(1 \leq i, j \leq 5\).

\[
A = \begin{pmatrix}
1 & 2 & 1 & 2 & 3 \\
2 & 3 & 2 & 3 & 1 \\
3 & 1 & 3 & 1 & 2 \\
2 & 4 & 2 & 4 & 1 \\
4 & 1 & 4 & 1 & 2
\end{pmatrix}
\]

So \(\chi_2(C_5 \square C_5) \leq 4\). Thus, \(\chi_2(C_5 \square C_5) = 4\) and the proof is complete. \(\square\)

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References


