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The 1, 2-Conjecture for graphs with relatively small chromatic number

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Abstract

We apply the Combinatorial Nullstellensatz to prove the 1, 2-Conjecture and the 1, 2, 3-Conjecture for some classes of graphs. We prove that both conjectures hold for every r -regular graph such that $\chi(G) \leq p \leq \frac{r+2}{2}$ for some odd prime p . For a general graph G , the 1, 2-Conjecture holds if $\chi(G) \leq p \leq \frac{\delta(G)+4}{2}$ for some odd prime p . As a result, the 1, 2-Conjecture holds for any graph G such that $\chi(G) \leq \frac{\delta(G)+10}{4}$.

Keywords: 1, 2-Conjecture, 1, 2, 3-Conjecture, Combinatorial Nullstellensatz.

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1 Introduction

Variations on coloring problems in graph theory have involved different ways of generating a coloring of the vertices. When there are no restrictions on how a coloring is produced, the minimum number of colors that can be assigned to vertices of G such that adjacent vertices receive different colors is the *chromatic number* of G . We consider a restricted class of colorings produced using weights on the edges and vertices. Let $V(G)$, $E(G)$, $\delta(G)$, and $\chi(G)$ denote the vertex set, edge set, minimum degree, and the chromatic number of G , respectively. For $v \in V(G)$, let $\Gamma_G(v)$ be the set of edges of G that are incident to v .

A *total weighting* ω of a graph G is an assignment of integers to its vertices and its edges. Let $\phi_\omega(v) = \omega(v) + \sum_{e \in \Gamma_G(v)} \omega(e)$. A total weighting ω is *proper* if ϕ_ω is a proper coloring of G . When f assigns 0 to every vertex, we call it simply a *weighting* of G . If the values assigned by ω all lie in S , then ω is an *S-weighting* or a *total S-weighting*.

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Conjecture 1 (1, 2, 3-Conjecture; Karoński–Łuczak–Thomason [6]). *Every graph without isolated edges has a proper $\{1, 2, 3\}$ -weighting.*

Conjecture 2 (1, 2-Conjecture; Przybyło–Woźniak [7]). *Every graph has a proper $\{1, 2\}$ -total weighting.*

Many partial results are known toward these conjectures. Addario-Berry, Dalal, McDiarmid, Reed, and Thomason [1] showed that every graph without isolated edges has a proper $\{1, \dots, 30\}$ -weighting. Kalkowski, Karoński, and Pfender [5] improved the guarantee to a proper $\{1, \dots, 5\}$ -weighting. Przybyło and Woźniak [7] proved the 1, 2-Conjecture for complete graphs and for graphs with chromatic number at most 3. By showing that the “irregularity strength” is 3, Faudree, Jacobson, Kinch, and Lehel [4] proved the 1, 2, 3-Conjecture for graphs with minimum degree larger than $n - \sqrt{n/18}$.

A (k, k') -list assignment for a graph G is a map L that assigns each vertex a set of k numbers and each edge a set of k' numbers. A graph G is (k, k') -weight-choosable if for every (k, k') -list assignment, there is a proper total weighting such that the weight on each object is chosen from its list. It is k -weight-choosable if a proper weighting can be chosen from the lists whenever the edges are assigned lists of size k .

Conjecture 3 (Bartnicki–Grytczuk–Niwczyk [3]). *Every graph without isolated edges is 3-weight-choosable.*

Conjecture 4 (Wong–Zhu [10]). *Every graph is $(2, 2)$ -weight-choosable. Every graph without isolated edges is $(1, 3)$ -weight-choosable.*

Wong, Yang, and Zhu [9] proved that complete multipartite graphs of the form $K_{n,m,1,1,\dots,1}$ are $(2, 2)$ -weight-choosable and that complete bipartite graphs other than K_2 are $(1, 2)$ -weight-choosable.

Several authors have applied the Combinatorial Nullstellensatz [2] to these problems. Bartnicki, Grytczuk, and Niwczyk [3] used it to prove Conjecture 3 for complete graphs, complete bipartite graphs, and trees other than K_2 . Wong and Zhu [10] used it to prove Conjecture 4 for complete graphs (and in general all complements of linear forests). They also proved that every tree with an even number of edges is $(1, 2)$ -choosable. Przybyło and Woźniak [8] used it to prove that graphs with at most one cycle and wheels are $(2, 2)$ -weight-choosable.

Lemma 5 (Combinatorial Nullstellensatz). *Let f be a polynomial of degree t in m variables over a field \mathbb{F} . If there is a monomial $\prod x_i^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero (in \mathbb{F}), then f is nonzero at some point of $\prod S_i$, where each S_i is a set of $t_i + 1$ values in \mathbb{F} .*

Our results also use the Combinatorial Nullstellensatz. We prove the 1, 2-Conjecture for every graph G such that $\chi(G) \leq \frac{\delta(G)+4}{2}$ and there is an odd prime between them. Note that trees do not satisfy this condition. For regular graphs, a slightly tighter inequality gives also the 1, 2, 3-Conjecture.

2 Results

We apply the Combinatorial Nullstellensatz to orientations of special subgraphs having large minimum outdegree. The tool for finding such subgraphs is the following elementary lemma.

Lemma 6. *Every graph G has an orientation in which for each $v \in V(G)$, the outdegree of v is at least $\lfloor d(v)/2 \rfloor$.*

Proof. Add a new vertex u to G adjacent to all the vertices of odd degree. Each component of the new graph is Eulerian. Orient the edges of the new graph following an Eulerian circuit in each component. Finally drop u to obtain the desired orientation of G . \square

Theorem 7. *If G is an r -regular graph and p is an odd prime such that $\chi(G) \leq p \leq \frac{r+2}{2}$, then G has a proper $\{1, 2\}$ -weighting. Consequently the 1, 2-Conjecture and the 1, 2, 3-Conjecture hold for G .*

Proof. If G has a proper $\{1, 2\}$ -weighting, then giving weight 1 to each vertex yields a proper total $\{1, 2\}$ -weighting. Also, a proper $\{1, 2\}$ -weighting is a proper $\{1, 2, 3\}$ -weighting.

Let V_1, \dots, V_p be the color classes in a proper coloring of G with p colors. Since G is r -regular and $r/2 \geq p - 1$, the graph G has an orientation in which the outdegree of each vertex is at least $p - 1$, by Lemma 6. By deleting extra edges that exit from vertices with excess outdegree, we obtain a spanning subgraph G' having an orientation D in which the outdegree of each vertex is exactly $p - 1$.

Our goal is to prove that G' has a spanning subgraph H such that if $uv \in E(G)$, then $d_H(u) \not\equiv d_H(v) \pmod{p}$. Given such a subgraph H , we define ω by assigning weight 1 to edges of H and weight 2 to edges not in H . If $uv \in E(G)$, then $\phi_\omega(u) - \phi_\omega(v) = (2r - d_H(u)) - (2r - d_H(v)) = d_H(v) - d_H(u)$. Hence such a subgraph H yields a proper $\{1, 2\}$ -weighting.

Define a variable x_e for each edge e in G' , and define a polynomial f over \mathbb{Z}_p by

$$f(x) = \prod_{i=1}^p \prod_{v \in V_i} \prod_{a \in \mathbb{Z}_p - \{i\}} \left(-a + \sum_{e \in \Gamma_{G'}(v)} x_e \right)$$

We claim that the coefficient of $\prod_{e \in E(G')} x_e$ in f is not a multiple of p . With each occurrence of $\prod_{e \in E(G')} x_e$ in the expansion of f (always contributing $+1$ to the coefficient), we associate an orientation of G' . The variable x_e is chosen from a factor in f having the form $(-a + \sum_{e \in \Gamma_{G'}(w)} x_e)$ for some a , for exactly one of its endpoints, w ; let the tail of e in the orientation be w . Let D' be the resulting orientation; since there are $p - 1$ factors in f associated with each vertex, in D' the outdegree of each vertex is exactly $p - 1$. Each such orientation D' arises $(p - 1)!$ times, because the choices of variables from the factors associated with a vertex can be made in any order.

Let D'' be the spanning subdigraph within the special orientation D whose edges are those oriented oppositely in D and D' . Since each vertex has outdegree $p - 1$ in both D and D' , the digraph D'' is a circulation. Conversely, reversing the edges of a circulation does not change any outdegree, so the map sending D' to D'' is a bijection. Hence the number of unit contributions to the coefficient of $\prod_{e \in E(G)} x_e$ is the number of circulations of D times $(p - 1)!^n$. Note that $(p - 1)!^n \not\equiv 0 \pmod{p}$.

Let A be a basis for the binary space of circulations of D . The number of circulations is $2^{|A|}$, which is nonzero in \mathbb{Z}_p when p is an odd prime. Hence the coefficient in f of the monomial $\prod_{e \in E(G)} x_e$ is nonzero modulo p . By the Combinatorial Nullstellensatz, there exists $x^* \in \{0, 1\}^{(p-1)n}$ such that $f(x^*) \neq 0$. Let H be the spanning subgraph of G' with edge set $\{e \in E(G') : x_e^* = 1\}$. In H , each vertex of V_i has degree congruent to $i \pmod{p}$, since otherwise $f(x^*) = 0$. We have accomplished our goal. As remarked earlier, assigning weight 1 to edges of H and weight 2 to the remaining edges yields a proper 1, 2-weighting. \square

Theorem 8. *If G is a graph and p is an odd prime such that $\chi(G) \leq p \leq \frac{\delta(G)+4+2k}{2}$, where k is a nonnegative integer, then G has a proper total weighting using vertex weights in $\{1, \dots, k+2\}$ and edge weights in $\{1, 2\}$.*

Proof. We may assume that G has no isolated vertex and $k = \lceil p - 2 - \frac{\delta(G)}{2} \rceil$. Let V_1, \dots, V_p be the color classes in a proper coloring of G with p colors. Since $\delta(G) \geq 2(p - 2 - k)$, Lemma 6 implies that G contains a spanning subgraph G' having an orientation D in which the outdegree of each vertex is $p - 2 - k$.

For $v \in V(G)$ and $i \in \mathbb{Z}_p$, let $S(i, v) = \{i - d_G(v) + j : 0 \leq j \leq p - 3 - k\}$. For each edge $e \in E(G')$ define a variable x_e , and consider the following polynomial over \mathbb{Z}_p :

$$f(x) = \prod_{i=1}^p \prod_{v \in V_i} \prod_{a \in S(i, v)} \left(-a + \sum_{e \in \Gamma_{G'}(v)} x_e \right)$$

Arguing as in the proof of Theorem 7, the coefficient of $\prod_{e \in E(G')} x_e$ in f is $(p - 2 - k)!2^{|A|}$, where A is a basis for the space of circulations of D . This number is nonzero in \mathbb{Z}_p . Hence by the Combinatorial Nullstellensatz, there exists $x^* \in \{0, 1\}^{(p-2-k)n}$ such that $f(x^*) \neq 0$.

Let H be the spanning subgraph of G' with edge set $\{e \in E(G') : x_e^* = 1\}$. In H , each vertex of V_i has degree in $\mathbb{Z}_p - S(i, v)$, which consists of $k + 2$ consecutive congruent classes, since otherwise $f(x^*) = 0$.

For each $e \in E(H)$, let $\omega(e) = 2$. If $e \in E(G) - E(H)$, define $\omega(e) = 1$.

For $v \in V_i$, we have $d_H(v) \equiv i - d_G(v) - s \pmod{p}$ for some s with $1 \leq s \leq k + 2$. let $\omega(v) = s$. Now

$$\omega(v) + \sum_{e \in \Gamma(v)} \omega(e) \equiv s + 2 \cdot (i - d_G(v) - s) + 1 \cdot (2d_G(v) - i + s) \equiv i \pmod{p}$$

Hence ω is a proper total weighting of G . \square

Putting $k = 0$ in Theorem 8 gives us another family of graphs for which the 1,2-Conjecture holds.

Corollary 9. *If G is a graph and p is an odd prime such that $\chi(G) \leq p \leq \frac{\delta(G)+4}{2}$, then the 1,2-Conjecture holds for G .*

Corollary 10. *If $\chi(G) \leq \frac{\delta(G)+10}{4}$, then the 1,2-Conjecture holds for G .*

Proof. By the Prime Number Theorem, for every integer n greater than 1, there is a prime number strictly between n and $2n$. Apply this with $n = \chi(G) - 1$ to obtain a prime p with $\chi(G) \leq p \leq 2\chi(G) - 3$. Since $\delta(G) \geq 4\chi(G) - 10$, we have $\delta(G) \geq 4(\frac{p+3}{2}) - 10 = 2(p - 2)$. Now Corollary 9 gives the desired conclusion. \square

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