On the Dynamic Coloring of Strongly Regular Graphs

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Abstract

A proper vertex coloring of a graph G is called a dynamic coloring if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. It was conjectured that if G is a regular graph, then $\chi_2(G) - \chi(G) \leq 2$. In this paper we prove that, apart from the cycles $C_4$ and $C_5$ and the complete bipartite graphs $K_{n,n}$, every strongly regular graph G, satisfies $\chi_2(G) - \chi(G) \leq 1$.

1. Introduction

Let G be a graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The number of vertices of G is called the order of G. A proper vertex coloring of G is a function $c : V(G) \rightarrow L$, with this property: if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A vertex k-coloring is a proper vertex coloring with $|L| = k$. A proper vertex k-coloring of a graph G is called a dynamic coloring if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic k-coloring is called the dynamic chromatic number of G and is denoted by

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\( \chi_2(G) \). For every \( v \in V(G) \), \( N(v) \) denotes the neighbor set of \( v \). Let \( G \) be a graph with coloring \( c \). Then \( d(v) \) and \( c(N(v)) \) denote the degree of \( v \) and the set of all colors appearing on the neighbors of \( v \), respectively. In this paper we denote the cycle of order \( n \) and the complete bipartite graph with part sizes \( m \) and \( n \) by \( C_n \) and \( K_{m,n} \), respectively. In a vertex coloring of \( G \), we say that the dynamic property holds for vertex \( v \), if one of the following holds: (i) \( d(v) \leq 1 \), (ii) \( d(v) \geq 2 \) and there are at least two vertices with different colors incident with \( v \). A graph \( G \) of order \( n \) is called strongly \( k \)-regular if there are parameters \( k \), \( \lambda \) and \( \mu \) such that \( G \) is \( k \)-regular, every adjacent pair of vertices have \( \lambda \) common neighbors, and every nonadjacent pair of vertices have \( \mu \) common neighbors. Montgomery [5] conjectured that for every regular graph \( G \), \( \chi_2(G) - \chi(G) \leq 2 \). In this paper we show that if \( G \neq C_4, C_5 \) and \( K_{k,k} \), then for every strongly regular graph \( G \), \( \chi_2(G) - \chi(G) \leq 1 \).

**Conjecture 1.** [5] For every regular graph \( G \), \( \chi_2(G) - \chi(G) \leq 2 \).

**Remark 2.** If \( P \) is the Petersen graph, then clearly \( \chi(P) = 3 \). We want to show that \( \chi_2(P) = 4 \). By contradiction suppose that \( \chi_2(P) = 3 \). Consider the following figure:

![Figure 1](image)

Assume that \( c : V(P) \rightarrow \{1, 2, 3\} \) is a dynamic 3-coloring of \( P \). With no loss of generality, one may assume that \( c(v_1) = c(v_3) = 1 \), \( c(v_2) = c(v_4) = 2 \), and \( c(v_5) = 3 \). Since the dynamic property holds for vertices \( v_2 \) and \( v_3 \), we conclude that \( c(u_2) = c(u_3) = 3 \). Thus the dynamic property for \( u_5 \) does not hold, a contradiction. By Theorem 1 of [4], \( \chi_2(P) \leq 4 \). Hence \( \chi_2(P) = 4 \) and \( \chi_2(P) - \chi(P) = 1 \).

**Theorem 3.** Let \( G \) be a strongly \( k \)-regular graph with \( \mu = 1 \) and \( G \neq C_5, P \), where \( P \) is the Petersen graph. Then \( \chi_2(G) = \chi(G) \).

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Proof. If $\lambda > 0$, then every vertex is contained in a triangle and so we have $\chi_2(G) = \chi(G)$. Thus assume that $\lambda = 0$. If $k = 2$, then the assertion is trivial. By [2, p.855], the only strongly 3-regular graphs are $K_{3,3}$ and the Petersen graph. Therefore we can suppose that $k \geq 4$. To the contrary, assume that $\chi_2(G) \neq \chi(G)$. Let $c$ be a vertex $\chi(G)$-coloring of $G$ such that the number of vertices for which the dynamic property holds is maximum. Let $v \in V(G)$ and suppose that the dynamic property does not hold for $v$. Suppose that $N(v) = \{v_1, \ldots, v_k\}$. Without loss of generality, we can suppose that $c(v) = 1$ and $c(N(v)) = \{2\}$. Note that since $\lambda = 0$, $N(v_i) \setminus \{v\}$ is an independent set for each $i$, $i = 1, \ldots, k$ and since $\mu = 1$, for every $i, j \in \{1, \ldots, k\}, i \neq j$, $N(v_i) \cap N(v_j) = \{v\}$. Moreover, for every $j, j \neq i$, and $x \in N(v_i) \setminus \{v\}$, $|N(x) \cap N(v_j)| = 1$.

First, we claim that for every $w_i \in N(v_i) \setminus \{v\}$, $c(N(w_i)) = \{2, c_i\}$, where $c_i \in \{1, \ldots, \chi(G)\} \setminus \{2\}$. Clearly, $c(N(w_i)) \neq \{2\}$. Now, by contradiction assume that there are three distinct colors $\{2, x, y\} \subseteq c(N(w_i))$. One of the colors $x$ and $y$ is not 1. Now, change $c(v_i)$ to color $x$ and next change all colors $x$ in $N(v_i)$ to color 2 and call this coloring $c'$. Clearly, the dynamic property holds for $v$. We show that the dynamic property remains for those vertices which had the dynamic property before. Since $\lambda = 0$ and $\mu = 1$, using the equation $k(k - \lambda - 1) = \mu(n - k - 1)$, [7, p.465], we have $n = k^2 + 1$. This implies that $V(G) = N(v) \cup (\cup_{j=1}^k N(v_j))$. We note that for every $j, j \neq i$, $c'(v_j) = c(v_j)$ and $c'(N(v_j)) = c(N(v_j))$ and so $v_j$ has the dynamic property in $c$ if and only if $v_j$ has the dynamic property in $c'$. Now, assume that $j \neq i$ and $z \in N(v_j) \setminus \{v\}$. Since $k \geq 3$, there exists $q \neq j, i$, such that $N(z) \cap N(v_q) = \{a\}$. But $c'(a) \neq 2$ and so the dynamic property holds for $z$. Obviously, if the dynamic property holds for $v_i$ in coloring $c$, then it holds for $v_i$ in coloring $c'$. Now, we would like to show that the dynamic property holds for every $v \in N(v_i) \setminus \{v\}$. We have $\{x, y\} \subseteq c'(N(w_i))$ and so $w_i$ has the dynamic property. Let $z \in N(v_i) \setminus \{v, w_i\}$. Assume that $s \in N(w_i)$ and $c'(s) = x$. Suppose that $s \in N(v_r)$. Since $\mu = 1$, we have $sz \notin E(G)$ and so $N(s) \cap N(z) = \{p\}$. Since $ps \in E(G)$ and $c'(s) = x$, $c'(p) \neq x$ and the dynamic property holds for $z$. Thus the number of vertices in $c'$ for which the dynamic property holds is more than the number of vertices in $c$ for which the dynamic property holds, a contradiction. Hence, $c(N(w_i)) = \{2, c_i\}$.

Next, we want to prove that $|c(N(v_j) \setminus \{v\})| = k - 1$ for $j = 1, \ldots, k$. To the contrary and with no loss of generality assume that there is a color
Let $N(w_1) \cap N(v_2) = \{w_2\}$. Thus, as we did before, $c(N(w_2)) = \{2, b\}$. Since $\mu = 1$, $w_2 u_1 \not\in E(G)$. Thus, $w_2$ and $u_1$ should have a common neighbor, say $t$. But $c(t) = b$, a contradiction. Hence, $\chi(G) \geq k$. Now, Brook's Theorem \cite[p.197]{7} implies that $\chi(G) = k$. Since $c(v_j) = 2$ for every $j, 1 \leq j \leq k$, we conclude that $3 \in c(N(v_j) \setminus \{v\})$. For every vertex $w \in V(G)$ with $c(w) = 3$, change the color of $w$ to a color from the set $\{1, \ldots, \chi(G)\} \setminus (c(N(w)) \cup \{3\})$ to obtain a vertex $(\chi(G) - 1)$-coloring of $G$, a contradiction. Thus, every vertex has the dynamic property in $c$ and so $\chi_2(G) = \chi(G)$ and the proof is complete. \hfill $\square$

Now, we would like to prove that except for $C_4$, $C_5$, and $K_{k,k}$, for every strongly $k$-regular graph $G$ there is a vertex coloring by $\chi(G)$ colors such that the dynamic property does not hold for at most one vertex of $G$.

**Theorem 4.** Let $G \neq C_4, C_5, K_{k,k}$ be a strongly $k$-regular graph. Then we can color the vertices of $G$ by $\chi(G)$ colors such that the dynamic property does not hold for at most one vertex.

**Proof.** If $\chi(G) = 2$, then $G$ is bipartite. Thus $-k$ is an eigenvalue of $G$ \cite[p.53]{1}. If $G$ is a strongly regular graph which is not a complete graph, then it has three distinct eigenvalues, \cite[p.466]{7}. Since the eigenvalues of every bipartite graph are symmetric about the origin, we conclude that if $G \neq K_2$ is a strongly $k$-regular graph, then $\{-k, 0, k\}$ are eigenvalues of $G$ \cite[p.53]{1}. This yields that $G$ is a complete multipartite graph \cite[p.163]{3}. Hence $G$ is $K_{k,k}$, where $n = 2k$ and $n = |V(G)|$. Thus assume that $\chi(G) \geq 3$. If $\lambda > 0$, then every vertex of $G$ is contained in a triangle. So $\chi_2(G) = \chi(G)$. Thus assume that $\lambda = 0$. If $\mu = k$, then $0, -k = \frac{1}{2}(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})$ \cite[p.194]{3} are eigenvalues of $G$. So by \cite[p.399]{6}, $G$ is bipartite, a contradiction. Thus we can assume that $\mu \neq k$. Clearly, the assertion holds for $\mu = 0$. Assume that $\mu = 1$. If $G$ is the Petersen graph, then it is not hard to see that there is a vertex 3-coloring such that the dynamic property fails for exactly one vertex. Thus by Theorem 3 we can assume that $\mu \geq 2$. Now, consider a vertex $\chi(G)$-coloring such that the number of vertices of $G$ for which the dynamic property doesn’t hold is as small as possible. Let’s call this number $l$. It suffices to show that $l \leq 1$. To the contrary, suppose that $l \geq 2$. Consider that vertex coloring, say $c$, in which the dynamic property does not hold for exactly $l$ vertices. Assume that $v$ is one of these vertices. So, we can suppose that $c(v) = 1$.
and \( c(N(v)) = \{2\} \). Let \( H = G \setminus (\{v\} \cup (N(v))) \). None of the vertices of \( H \) can have color 2, because if \( w \in V(H) \) and \( c(w) = 2 \), then they should have \( \mu \) common neighbors, a contradiction. Since \( \mu \geq 2 \), every vertex of \( H \) should be adjacent to at least two vertices of \( N(v) \). Let \( x \in V(H) \). Since \( \mu \neq k \), \( N(x) \neq N(v) \). Thus the dynamic property holds for vertex \( x \). Now, assume that there exists \( y \in N(v) \) such that the dynamic property does not hold for \( y \). So, all neighbors of \( y \) should have color 1. Now, by changing \( c(y) \) to 3, the dynamic property holds for \( v \) in the new coloring. Moreover, since \( \mu \geq 2 \), every vertex \( z \in H \) is adjacent to a vertex with color 2 and also a vertex with a color different from 2 in \( H \). Thus, we obtain a coloring of \( G \) such that the number of vertices for which the dynamic property fails is less than \( l \), a contradiction. Hence \( l \leq 1 \) and the proof is complete. \( \square \)

We close the paper with the following corollary.

**Corollary 5.** If \( G \neq C_4, C_5, K_{k,k} \) is a strongly regular graph, then \( \chi_2(G) - \chi(G) \leq 1 \) and so Conjecture 1 is true for strongly regular graphs.

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**References**


