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## EXTREMAL THEOREMS FOR DEGREE SEQUENCE PACKING AND THE TWO-COLOR DISCRETE TOMOGRAPHY PROBLEM\*

JENNIFER DIEMUNSCH<sup>†</sup>, MICHAEL FERRARA<sup>‡</sup>, SOGOL JAHANBEKAM<sup>‡</sup>,  
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**Abstract.** A sequence  $\pi = (d_1, \dots, d_n)$  is graphic if there is a simple graph  $G$  with vertex set  $\{v_1, \dots, v_n\}$  such that the degree of  $v_i$  is  $d_i$ . We say that graphic sequences  $\pi_1 = (d_1^{(1)}, \dots, d_n^{(1)})$  and  $\pi_2 = (d_1^{(2)}, \dots, d_n^{(2)})$  pack if there exist edge-disjoint  $n$ -vertex graphs  $G_1$  and  $G_2$  such that for  $j \in \{1, 2\}$ ,  $d_{G_j}(v_i) = d_i^{(j)}$  for all  $i \in \{1, \dots, n\}$ . Here, we prove several extremal degree sequence packing theorems that parallel central results and open problems from the graph packing literature. Specifically, the main result of this paper implies degree sequence packing analogues to the Bollobás–Eldridge–Catlin graph packing conjecture and the classical graph packing theorem of Sauer and Spencer. In discrete tomography, a branch of discrete imaging science, the goal is to reconstruct discrete objects using data acquired from low-dimensional projections. Specifically, in the  $k$ -color discrete tomography problem the goal is to color the entries of an  $m \times n$  matrix using  $k$  colors so that each row and column receives a prescribed number of entries of each color. This problem is equivalent to packing the degree sequences of  $k$  bipartite graphs with parts of sizes  $m$  and  $n$ . Here we also prove several Sauer–Spencer-type theorems with applications to the two-color discrete tomography problem.

**Key words.** degree sequence, discrete tomography, packing

**AMS subject classifications.** 05C35, 05C70

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**1. Introduction.** A sequence of nonnegative integers  $\pi = (d_1, d_2, \dots, d_n)$  is *graphic* if there is a (simple) graph  $G$  of order  $n$  having degree sequence  $\pi$ . In this case,  $G$  is said to *realize* or be a *realization* of  $\pi$ , and we write  $\pi = \pi(G)$ . If a sequence  $\pi$  consists of the terms  $d_1, \dots, d_t$  having multiplicities  $\mu_1, \dots, \mu_t$ , then we may write  $\pi = (d_1^{\mu_1}, \dots, d_t^{\mu_t})$ .

There are a number of necessary and sufficient conditions for a sequence to be graphic, including the seminal Havel–Hakimi algorithm [21, 23] and the Erdős–Gallai criteria [16]. However, a given graphic sequence may have a large family of nonisomorphic realizations, and as such considerable attention has been given to the study of when a graphic sequence has a realization with a given property. Such problems can be divided into two broad classes, described as “forcible” problems and “potential” problems in [30]. Given a graph property  $\mathcal{P}$ , we say that a graphic sequence  $\pi$  is *forcibly  $\mathcal{P}$ -graphic* if every realization of  $\pi$  has property  $\mathcal{P}$ , and that  $\pi$  is *potentially  $\mathcal{P}$ -graphic* if at least one realization of  $\pi$  has property  $\mathcal{P}$ .

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Results on forcible degree sequences are often stated as traditional problems in structural or extremal graph theory, where a necessary and/or sufficient condition is given in terms of the degrees of the vertices (or equivalently the number of edges) of a given graph. For instance, minimum degree thresholds for the existence of certain graph structures, such as the threshold for hamiltonicity in Dirac's theorem [12], can be thought of as forcible theorems. Two older but exceptionally thorough surveys on forcible and potential problems are due to Hakimi and Schmeichel [22] and Rao [31], and a more recent survey on forcible "Chvátal-type" theorems (in the spirit of [9]) is due to Bauer et al. [3].

A number of degree sequence analogues to classical problems in extremal graph theory appear throughout the literature, including potentially graphic sequence variants of Hadwiger's conjecture [15, 32], graph Ramsey numbers [6], and the Turán problem (cf. [17]). In this paper, we consider an extension of the classical graph packing literature to degree sequences. In particular, we prove a potentially  $\mathcal{P}$ -graphic analogue to a widely studied graph packing conjecture of Bollobás and Eldridge [4] and, independently, Catlin [8], which implies a graphic sequence version of the Sauer–Spencer graph packing theorem [33]. We conclude by using similar techniques to prove a pair of related results that have applications to discrete imaging science.

**1.1. Graph packing.** Two  $n$ -vertex graphs  $G_1$  and  $G_2$  *pack* if  $G_1$  is a subgraph of  $\overline{G_2}$ , or alternatively if  $G_1$  and  $G_2$  can be expressed as edge-disjoint subgraphs of  $K_n$ , the complete graph on  $n$  vertices. Graph packing has received a great deal of attention in the literature ([26], [35], and [36] are detailed and useful surveys).

In 1978, Sauer and Spencer [33] proved the following classical theorem.

**THEOREM 1.1.** *Let  $G_1$  and  $G_2$  be graphs of order  $n$  with maximum degree  $\Delta_1$  and  $\Delta_2$ , respectively. If*

$$\Delta_1 \Delta_2 < \frac{n}{2},$$

*then  $G_1$  and  $G_2$  pack.*

Likely the most notable open conjecture in graph packing is due to Bollobás and Eldridge [4] and, independently, Catlin [8].

**CONJECTURE 1.** *Let  $G_1$  and  $G_2$  be  $n$ -vertex graphs with maximum degrees  $\Delta(G_i) = \Delta_i$  for  $i = 1, 2$ . If*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1,$$

*then  $G_1$  and  $G_2$  pack.*

If true, Conjecture 1 implies Theorem 1.1. The Bollobás–Eldridge–Catlin conjecture has been settled in several cases, including when  $\Delta_1 \leq 2$  by Aigner and Brandt [1] and Alon and Fischer [2]. The case when  $\Delta_1 = 3$  was shown by Csaba, Shokoufandeh, and Szemerédi [11] for large  $n$  utilizing the regularity lemma. For  $\Delta_1, \Delta_2 \geq 300$ , Kaul, Kostochka and Yu [25] showed that  $(\Delta_1 + 1)(\Delta_2 + 1) \leq 0.6n + 1$  implies that the two graphs pack, which improves the Sauer–Spencer theorem, and is a partial solution to Conjecture 1. Other partial results were obtained by Corrádi and Hajnal [10] and Hajnal and Szemerédi [20].

**1.2. Packing graphic sequences.** The notion of packing graphic sequences was investigated in [7], where the following key definition appears. If  $\pi_1$  and  $\pi_2$  are (not necessarily monotone) graphic sequences, with  $\pi_1 = (d_1^{(1)}, \dots, d_n^{(1)})$  and  $\pi_2 = (d_1^{(2)}, \dots, d_n^{(2)})$ , then  $\pi_1$  and  $\pi_2$  *pack* if there exist edge-disjoint graphs  $G_1$  and  $G_2$ ,

both with vertex set  $\{v_1, \dots, v_n\}$ , such that

$$d_{G_1}(v_i) = d_i^{(1)} \quad \text{and} \quad d_{G_2}(v_i) = d_i^{(2)}.$$

It is critical to note here that the order of the terms in  $\pi_1$  and  $\pi_2$  is fixed, so that the statement “ $\pi_1$  and  $\pi_2$  pack” is not equivalent to “ $\pi_1$  and  $\pi_2$  have realizations that pack.” This framework allows for some interesting distinctions between packing graphs and packing graphic sequences. On the other hand, by fixing the ordering of  $\pi_1$  and  $\pi_2$ , the study of degree sequence packing provides insight into how a pair of graphs with these degree sequences might feasibly pack, if in fact they do.

Given a sequence  $\pi$ , let  $\Delta(\pi)$  and  $\delta(\pi)$  denote the maximum and minimum terms in  $\pi$ , respectively. Further, given two sequences  $\pi_1$  and  $\pi_2$  of the same length, let  $\pi_1 + \pi_2$  denote the “vector sum” of  $\pi_1$  and  $\pi_2$ . One of the main results from [7] is the following.

**THEOREM 1.2.** *Let  $\pi_1$  and  $\pi_2$  be  $n$ -term graphic sequences with  $\Delta = \Delta(\pi_1 + \pi_2)$  and  $\delta = \delta(\pi_1 + \pi_2)$ . If*

$$\Delta \leq \sqrt{2\delta n} - (\delta - 1),$$

*then  $\pi_1$  and  $\pi_2$  pack, except that strict inequality is required when  $\delta = 1$ . This result is sharp for all  $n$  and  $\delta$ .*

As was noted in [7], this theorem can be viewed as an “additive” analogue to the Sauer–Spencer theorem, since  $\Delta_1 + \Delta_2 < \sqrt{2n}$  implies that  $\Delta_1\Delta_2 < \frac{n}{2}$ . We modify and strengthen the techniques introduced in the proof of Theorem 1.2 to obtain our main results here.

**1.3. Statement of main results.** Throughout the statement and proof of the following results, given graphic sequences  $\pi_1$  and  $\pi_2$  we let  $\Delta_i = \Delta(\pi_i)$  and  $\delta_i = \delta(\pi_i)$  for  $i \in \{1, 2\}$ . Our main result is as follows.

**THEOREM 1.3.** *Let  $\pi_1$  and  $\pi_2$  be graphic sequences with  $\Delta_2 \geq \Delta_1$  and  $\delta_1 \geq 1$ . If*

$$\begin{cases} (\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1 & \text{when } \Delta_2 + 2 \geq \Delta_1 + \delta_1, \\ \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4} \leq \delta_1 n + 1 & \text{when } \Delta_2 + 2 < \Delta_1 + \delta_1, \end{cases}$$

*then  $\pi_1$  and  $\pi_2$  pack.*

Theorem 1.3 holds regardless of the orderings of  $\pi_1$  and  $\pi_2$ , although these orderings are fixed. Given this, we cannot assume that  $\delta(\pi_1) = \delta(\pi_2) = 0$ , as it would be possible to order  $\pi_1$  and  $\pi_2$  so that the zero terms correspond, which would impact the relative strength of the hypothesis. It seems feasible that the conditions that  $\Delta_1 \leq \Delta_2$  and  $\delta(\pi_1) \geq 1$  could be replaced by the weaker hypothesis that  $\delta(\pi_1 + \pi_2) \geq 1$ , although we are unable to obtain such a result at this time.

Theorem 1.3 implies the following analogue to the Bollobás–Eldridge–Catlin conjecture.

**COROLLARY 1.4.** *Let  $\pi_1$  and  $\pi_2$  be graphic sequences with  $\Delta_2 \geq \Delta_1$  and  $\delta_1 \geq 1$ . If*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1,$$

*then  $\pi_1$  and  $\pi_2$  pack. This result is best possible.*

Much as the Bollobás–Eldridge–Catlin conjecture implies the Sauer–Spencer theorem, we also obtain the following.

**COROLLARY 1.5.** *Let  $\pi_1$  and  $\pi_2$  be graphic sequences with  $\Delta_2 \geq \Delta_1$  and  $\delta_1 \geq 1$ . If  $\Delta_1\Delta_2 < \frac{n}{2}$ , then  $\pi_1$  and  $\pi_2$  pack. This result is best possible.*

**1.4. Sharpness.** In [24], Kaul and Kostochka characterized the sharpness examples for Theorem 1.1. Specifically, graphs  $G_1$  and  $G_2$  satisfying  $\Delta_1\Delta_2 = \frac{n}{2}$  pack, unless  $n$  is even,  $G_1$  is a matching of size  $\frac{n}{2}$ , and either  $\frac{n}{2}$  is odd and  $G_2 = K_{\frac{n}{2}, \frac{n}{2}}$  or  $G_2$  is any graph that contains  $K_{\frac{n}{2}+1}$  as a component.

In a similar manner, to see that Corollaries 1.4 and 1.5 are sharp, let  $n$  be even and consider  $\pi_1 = (1^n)$  and  $\pi_2 = (\frac{n}{2}^{\frac{n+2}{2}}, 0^{\frac{n-2}{2}})$ . These sequences are uniquely realized as a perfect matching and  $K_{\frac{n}{2}+1} \cup (\frac{n}{2} - 1)K_1$ , which do not pack, regardless of the orderings of  $\pi_1$  and  $\pi_2$ . The proof of the following theorem is inherent in the proofs of Corollaries 1.4 and 1.5, so we omit the proof in the interest of concision.

**THEOREM 1.6.** *Theorem 1.3 is strictly stronger than Corollary 1.4 unless  $\delta_1 = 1$ . Further, Corollary 1.4 is strictly stronger than Corollary 1.5 unless  $\Delta_1 = \delta_1 = 1$ .*

A  $k$ -factor of a graph  $G$  is a spanning  $k$ -regular subgraph of  $G$ . Kundu’s  $k$ -factor theorem [28], proved independently by Lovász for  $k = 1$  [29], states that a graphic sequence  $\pi = (d_1, \dots, d_n)$  has a realization containing a  $k$ -factor if and only if  $\pi' = (d_1 - k, \dots, d_n - k)$  is also graphic. Together with Theorem 1.6, this allows us to partially characterize the sharpness of Corollary 1.4 and completely characterize the sharpness of Corollary 1.5. The latter characterization is analogous to the characterization for graph packing from [24].

**THEOREM 1.7.** *Let  $\pi_1$  and  $\pi_2$  be graphic sequences with  $\Delta_2 \geq \Delta_1$  and  $\delta_1 \geq 1$ .*

- (a) *If  $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 2$  and  $\delta_1 \neq 1$ , then  $\pi_1$  and  $\pi_2$  pack.*
- (b) *If  $\Delta_1\Delta_2 = \frac{n}{2}$ , then  $\pi_1$  and  $\pi_2$  pack unless  $\Delta_1 = 1$  and  $\pi_1 + \pi_2$  is not graphic.*

**2. Proofs of Theorem 1.3 and Corollaries 1.4 and 1.5.** Let  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  be graphs. We say a vertex pair  $(x, y)$  is a *bad pair* for  $(G_1, G_2)$  or a  $(G_1, G_2)$ -*bad pair* if  $xy \in E_1 \cap E_2$ . Let  $b(G_1, G_2)$  denote the number of  $(G_1, G_2)$ -bad pairs. We begin by proving Theorem 1.3.

*Proof of Theorem 1.3.* Let  $\pi_1$  and  $\pi_2$  be graphic sequences that do not pack. Choose  $G_1 = G(\pi_1)$  and  $G_2 = G(\pi_2)$  to have the fewest bad pairs among all realizations of  $\pi_1$  and  $\pi_2$  and let  $G = G_1 \cup G_2$ . For a given  $(G_1, G_2)$ -bad pair  $(x, y)$  we define  $I(x, y) = V - (N_G(x) \cup N_G(y))$ . Among all choices of  $G_1$  and  $G_2$  that minimize  $b(G_1, G_2)$ , choose  $G_1, G_2$  and a bad pair  $(x, y)$  such that the size of  $I = I(x, y)$  is maximum. For  $i \in \{1, 2\}$ , let  $Q_i(y)$  be  $N_{G_i}(y) - N_G[x]$  and define  $Q_i(x)$  similarly. If either  $Q_1(x)$  or  $Q_1(y)$  is nonempty, assume without loss of generality that  $|Q_1(x)| \leq |Q_1(y)|$ . Otherwise, if both  $Q_1(x)$  and  $Q_1(y)$  are empty, then assume without loss of generality that  $|Q_2(x)| \leq |Q_2(y)|$ .

Throughout the proof we will make use of the following sets. First, let  $\bar{Y} = V(G) - N_G[y]$ . Define  $A$  to be a subset of  $N_{G_1}(\bar{Y})$  such that every vertex of  $A$  has at least two neighbors in  $G_1$  in  $\bar{Y}$ . Finally, let  $B = N_{G_1}(\bar{Y}) - A$  and  $R = A \cup \{v \in N_G[y] : A \subseteq N_G(v)\}$ .

We prove Theorem 1.3 by counting the number of edges in  $G_1$  between  $R$  and  $V(G) - R$  to reach a contradiction. In order to gain the desired count, we first show particular edge structures in  $I, \bar{Y}$ , and  $N_{G_1}(\bar{Y})$ . We then show that  $A$  is not empty and further that  $R$  is a vertex cover of  $G_1$ .

We proceed by proving a sequence of claims, the first of which follows immediately from the straightforward fact that  $4xy \leq (x + y)^2$  for all real  $x$  and  $y$ .

**CLAIM 1.**  $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{1}{4}(\Delta_2 + 1 + \Delta_1 + \delta_1)^2$ .

**CLAIM 2.** *If  $u$  and  $v$  are vertices in  $G$  such that  $xu$  and  $yv$  are not in  $E(G)$ , then  $uv$  is not in  $E(G)$ .*

*Proof.* Assume otherwise, and without loss of generality let  $uv$  be an edge of  $G_1$ . We may then exchange the edges  $xy$  and  $uv$  with the nonedges  $xu$  and  $yv$  in  $G_1$

to create another realization of  $\pi_1$ . Since  $xu$  and  $yv$  are not in  $G$ , this reduces the number of bad pairs, a contradiction.  $\square$

Claim 2 immediately implies that  $I$  is an independent set in  $G$ .

CLAIM 3.  $\overline{Y} \neq \emptyset$ .

*Proof.* Toward contradiction, suppose that  $N_G[y] = V(G)$ . Thus  $|N_G[y]| = n$ , and therefore  $\Delta_1 + \Delta_2 \geq n$ . By assumption,  $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$ , which implies that

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1(\Delta_1 + \Delta_2) + 1.$$

Expanding and rearranging, this yields

$$0 \leq \Delta_1(\delta_1 - 1 - \Delta_2) - \delta_1 + 1.$$

However,  $\delta_1 - 1 - \Delta_2 < 0$  and  $-\delta_1 + 1 \leq 0$ , a contradiction. Consequently,  $N_G[y] \neq V(G)$ .  $\square$

CLAIM 4.  $\overline{Y}$  is independent in  $G_1$ .

*Proof.* Otherwise, suppose there are vertices  $u$  and  $v$  in  $\overline{Y}$  that form an edge in  $G_1$ . By Claim 2, both  $u$  and  $v$  must be adjacent to  $x$ . If there is some vertex  $z \in Q_1(y)$ , then removing the edges  $uv$ ,  $xy$ , and  $yz$  from  $G_1$  and adding the nonedges  $yu$ ,  $yv$ , and  $xz$  to  $G_1$  would create another realization of  $G'_1$  of  $\pi_1$  such that  $b(G'_1, G_2) < b(G_1, G_2)$ . If  $Q_1(y)$  is empty, then since  $|Q_1(x)| \leq |Q_1(y)|$ , we have that  $Q_1(x)$  is also empty, and therefore since we have assumed  $|Q_2(x)| \leq |Q_2(y)|$  and  $u \in Q_2(x)$ , there is some  $z$  in  $Q_2(y)$ . We then exchange the edges  $yz$  and  $xu$  in  $G_2$  and the edges  $uv$  and  $xy$  in  $G_1$  for the nonedges  $yu$  and  $xz$  in  $G_2$  and the nonedges  $xu$  and  $yv$  in  $G_1$  to again create realizations of  $\pi_1$  and  $\pi_2$  with fewer than  $b(G_1, G_2)$  bad pairs. Thus,  $\overline{Y}$  is independent in  $G_1$ .  $\square$

CLAIM 5.  $N_{G_1}(\overline{Y}) \cup \{x, y\}$  is a clique in  $G$ .

*Proof.* Let  $u \in \overline{Y}$  and  $w \in N_{G_1}(u)$ . By Claim 4,  $w \notin \overline{Y}$  and therefore  $w \in N_G[y]$ . If  $w \neq x$ , then since  $uy \notin E(G)$ , by Claim 2,  $wx \in E(G)$ . Thus,  $N_{G_1}(u) \subseteq N_G[x] \cap N_G(y)$ .

Consequently, suppose  $w, w' \in N_{G_1}(\overline{Y})$  are such that  $ww' \notin E(G)$ . Let  $u \in N_{G_1}(w) \cap \overline{Y}$  and  $u' \in N_{G_1}(w') \cap \overline{Y}$  ( $u$  and  $u'$  need not be distinct). Note that without loss of generality  $x \neq w$  since  $xw' \in E(G)$ . If  $u \in I$ , then replacing the edges  $uw$ ,  $u'w'$ , and  $xy$  in  $G_1$  with the nonedges  $xu$ ,  $yu'$ , and  $ww'$  contradicts the minimality of  $b(G_1, G_2)$ . Thus  $u \notin I$ , and likewise  $u' \notin I$ .

Next, assume there is some  $z \in Q_1(y)$ . By Claim 2,  $uz \notin E(G)$ . Remove the edges  $wu$ ,  $w'u'$ , and  $yz$  from  $G_1$  and add the edges  $ww'$ ,  $yu'$ , and  $zu$  to create a realization  $G'_1$  of  $\pi_1$  with  $b(G'_1, G_2) = b(G_1, G_2)$ . However, neither  $x$  nor  $y$  is adjacent to vertices in  $\{z\} \cup I(x, y)$ , which contradicts the maximality of  $I$ .

It remains to consider the case where  $Q_1(y) = \emptyset$ . Similar to the proof of Claim 4, since  $Q_1(y)$  is empty,  $Q_1(x)$  is empty, and therefore  $u, u' \in Q_2(x)$  and there must be a vertex  $z$  in  $Q_2(y)$ . Also note that since  $u, u' \in Q_2(x)$  the edges  $xu$  and  $xu'$  are in  $G_2$ . Exchanging the edges  $wu$ ,  $w'u'$ , and  $xy$  in  $G_1$  with  $ux$  and the nonedges  $u'y$  and  $ww'$  creates another realization  $G'_1$  of  $\pi_1$  such that  $(u, x)$  is a  $(G'_1, G_2)$ -bad pair and  $b(G'_1, G_2) = b(G_1, G_2)$ . However, by Claim 2  $u$  is not adjacent to vertices in  $\{z\} \cup I(x, y)$ , and  $x$  is not adjacent to vertices in  $\{z\} \cup I(x, y)$ . Therefore  $I(u, x) > I(x, y)$ . Hence,  $N_{G_1}(\overline{Y}) \cup \{x, y\}$  is a clique in  $G$ .  $\square$

CLAIM 6.  $A \neq \emptyset$ .

*Proof.* For the sake of contradiction, suppose  $A$  is empty, and therefore  $N_{G_1}(\overline{Y}) = B$ . Since  $\overline{Y}$  is independent in  $G_1$  we have that  $\delta_1|\overline{Y}| \leq |B|$ . Thus,

$$n = |\overline{Y}| + |N_G[y]| \leq \frac{|B|}{\delta_1} + \Delta_1 + \Delta_2.$$

We proceed by showing that  $|B| \leq \Delta_1 + \Delta_2 - 2$ , which establishes the desired contradiction. By the definition of  $\overline{Y}$ ,  $y$  is not adjacent to vertices in  $\overline{Y}$ , and therefore  $y \notin B$ . If  $x \notin B$ , then  $|B| \leq |N_G(y) - \{x\}| \leq \Delta_1 + \Delta_2 - 2$ . If  $x \in B$ , then since  $x$  has a neighbor in  $\overline{Y}$ ,  $Q_1(x) \neq \emptyset$ . By assumption  $|Q_1(x)| \leq |Q_1(y)|$ ; thus there is some vertex  $z$  in  $N_G[y]$  not adjacent to  $x$ . Now we have that  $|B| \leq |N_G[y] - \{y, z\}| \leq \Delta_1 + \Delta_2 - 2$ . Inserting this upper bound of  $|B|$  into the above inequality we have that

$$\delta_1 n + 1 \leq (\delta_1 + 1)(\Delta_1 + \Delta_2) - 1.$$

By Claim 1,

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4},$$

so that the hypothesis of the theorem yields

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq (\delta_1 + 1)(\Delta_1 + \Delta_2) - 1,$$

which implies

$$\Delta_2(\Delta_1 - 1) < \delta_1(\Delta_1 - 1),$$

so that either  $0 < 0$ , if  $\Delta_1 = 1$ , or  $\delta_1 > \Delta_2$ , a contradiction.  $\square$

By Claim 6,  $A \neq \emptyset$ , and by Claim 5,  $N_{G_1}(\overline{Y}) \cup \{x, y\}$  is a clique in  $G$  and therefore  $N_{G_1}(\overline{Y}) \cup \{x, y\} \subseteq R$ .

CLAIM 7. *Every edge of  $G_1$  is incident with  $R$ .*

*Proof.* Toward contradiction let  $zz'$  be an edge of  $G_1$  not incident with  $R$ . By Claim 4 we know that  $z$  and  $z'$  must be in  $N_G[y] - R$ , so there exist vertices  $w$  and  $w'$  (not necessarily distinct) in  $A$  which are not adjacent to  $z$  and  $z'$  (respectively). Also, we have distinct vertices  $u$  and  $u'$  in  $\overline{Y}$  such that  $wu$  and  $w'u'$  are edges in  $G_1$ .

We can remove the edges  $zz'$ ,  $uw$ , and  $u'w'$  from  $G_1$  and add the nonedges  $wz$ ,  $w'z'$ , and  $uu'$  to form a realization  $G'_1$  of  $\pi_1$ . It is possible that, via this edge-exchange,  $(u, u')$  is a bad pair of  $(G'_1, G_2)$ , implying that  $b(G'_1, G_2) = b(G_1, G_2) + 1$ . However, the sets  $Q_1(x)$ ,  $Q_2(x)$ ,  $Q_1(y)$ , and  $Q_2(y)$  are not affected by these exchanges. Now,  $\overline{Y}$  is no longer independent in  $G_1$  and  $(x, y)$  is still a bad pair. As in the proof of Claim 4 we now exchange edges to obtain a realization  $G''_1$  of  $\pi_1$  such that  $(x, y)$  and  $(u, u')$  are no longer bad pairs and no other bad pairs are created. Thus,  $b(G''_1, G_2) < b(G_1, G_2)$ , a contradiction.

Therefore  $R$  is a vertex cover of  $G_1$ , as desired.  $\square$

We conclude the proof by finding lower and upper bounds on the number of edges in  $G_1$  between  $R$  and  $V - R$ , which we denote by  $e_1(R, V - R)$ . The necessary lower bound follows easily from the assertion that  $V - R$  is independent in  $G_1$ ,

$$\delta_1(n - |R|) \leq e_1(R, V - R).$$

While  $\Delta_1|R|$  is a straightforward upper bound for  $e_1(R, V - R)$ , we require a stronger bound to obtain the desired result.

Suppose  $|R| \leq \Delta_2 + 1$ . Since  $\{x, y\} \subseteq R$ , in  $G_1$  both  $x$  and  $y$  have at most  $\Delta_1 - 1$  neighbors in  $V - R$ . The remaining vertices of  $R$  each have at most  $\Delta_1$  neighbors in  $V - R$ . Thus,  $e_1(R, V - R)$  is bounded above by

$$(|R| - 2)\Delta_1 + 2(\Delta_1 - 1) = |R|\Delta_1 - 2.$$

Combining the upper and lower bounds on  $e_1(R, V - R)$  yields

$$\delta_1 n + 1 < |R|(\Delta_1 + \delta_1).$$

By our assumption on  $|R|$  we have the following contradiction:

$$\delta_1 n + 1 < (\Delta_2 + 1)(\Delta_1 + \delta_1).$$

Now assume that  $|R| = \Delta_2 + 1 + t$ , where  $t$  is a positive integer. Notice that  $|N_G[y]| \leq \Delta_1 + \Delta_2$  implies that

$$|N_G[y] - R| \leq \Delta_1 + \Delta_2 - (\Delta_2 + 1 + t) = \Delta_1 - t - 1.$$

As  $y$  has no neighbors in  $\overline{Y}$ ,  $y$  has at most  $\Delta_1 - t - 1$  neighbors of  $G_1$  in  $V - R$ . If there is another vertex  $w \in R - N_G(\overline{Y})$ , then  $w$  also has no neighbors in  $\overline{Y}$  and thus has at most  $\Delta_1 - t - 1$  neighbors of  $G_1$  in  $V - R$ . If  $R - N_G(\overline{Y}) = \{y\}$ , then  $R$  is a clique. In this case,  $x$  has at most  $\Delta_1 + \Delta_2 - |R|$  neighbors of  $G_1$  in  $V - R$ . As  $\Delta_1 + \Delta_2 - |R| = \Delta_1 - t - 1$ , we have that there are at least two vertices in  $R$  with at most  $\Delta_1 - t - 1$  neighbors of  $G_1$  in  $V - R$ .

Each of the remaining vertices of  $R$  have at most  $\Delta_1 - t$  neighbors of  $G_1$  in  $V - R$ . In particular, if  $v \in B$ , then  $v$  has one neighbor of  $G_1$  to  $\overline{Y}$  and at most  $\Delta_1 - t - 1$  neighbors of  $G_1$  to  $N_G[y] - R$ . If  $v \in A$ , then  $v$  is adjacent to every vertex of  $A$  and therefore has at most  $\Delta_1 + \Delta_2 - |R| + 1$  neighbors of  $G_1$  to  $V - R$ , which is  $\Delta_1 - t$ .

Therefore, we have that

$$e_1(R, V - R) \leq 2(\Delta_1 - t - 1) + (|R| - 2)(\Delta_1 - t) = |R|(\Delta_1 - t) - 2.$$

Combining this with the lower bound of  $e_1(R, V - R)$ , we have

$$\delta_1 n + 1 < |R|(\Delta_1 + \delta_1 - t).$$

Since  $\Delta_2 + 1 + t = |R|$ , we expand the right side to obtain

$$\delta_1 n + 1 < (\Delta_2 + 1)(\Delta_1 + \delta_1) - t(\Delta_2 + 1 - (\Delta_1 + \delta_1)) - t^2.$$

If  $\Delta_2 + 2 \geq \Delta_1 + \delta_1$ , then we contradict our claim that  $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$ . Otherwise,  $\Delta_2 + 2 < \Delta_1 + \delta_1$ . In this case, the right side is maximized when  $t = \frac{1}{2}(-\Delta_2 - 1 + \Delta_1 + \delta_1)$ , which yields

$$\delta_1 n + 1 < \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4}.$$

This contradiction completes the proof.  $\square$

We next prove Corollary 1.4.

*Proof.* Assume that  $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ . Then

$$\begin{aligned} (\Delta_2 + 1)(\Delta_1 + \delta_1) &= (\Delta_2 + 1)(\Delta_1 + 1 + \delta_1 - 1) \\ &= (\Delta_2 + 1)(\Delta_1 + 1) + (\Delta_2 + 1)(\delta_1 - 1) \\ &\leq \delta_1 n + 1, \end{aligned}$$

where the last inequality follows from the hypothesis and the fact that  $\delta_1 \geq 1$ . Thus the result follows when  $\Delta_2 + 2 \geq \Delta_1 + \delta_1$ .

Suppose then that  $\Delta_2 + 2 < \Delta_1 + \delta_1$ , which implies  $\delta_1 \geq 2$  and also that

$$\frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4} < \frac{[2(\Delta_1 + \delta_1)]^2}{4} = (\Delta_1 + \delta_1)^2.$$

Note that  $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$  implies that  $\Delta_1 \leq \sqrt{n}$ , while  $\delta_1 \geq 2$  implies that  $\sqrt{\delta_1 n} - \delta_1 > \sqrt{n}$ . Hence  $(\Delta_1 + \delta_1)^2 \leq \delta_1 n$ , and the result follows.  $\square$

Finally, we give the straightforward proof that Corollary 1.4 implies Corollary 1.5.

*Proof.* For  $\Delta_1 > 1$ , since  $\Delta_1 \geq \Delta_2$ , we have that  $\Delta_1 + \Delta_2 \leq 2\Delta_2$  and  $2\Delta_2 \leq \Delta_1\Delta_2$ , thus

$$\Delta_1\Delta_2 + \Delta_1 + \Delta_2 \leq 2\Delta_1\Delta_2.$$

By assumption,  $2\Delta_1\Delta_2 < n$ , and therefore  $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ . If, instead,  $\Delta_1 = 1$ , then  $2\Delta_2 < n$  or  $2\Delta_2 + 1 \leq n$ , and since  $(\Delta_1 + 1)(\Delta_2 + 1) - 1 = 2\Delta_2 + 1$ , we have that  $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$  as desired.  $\square$

**3. Discrete tomography.** *Tomography* is the process of imaging through sectioning, for example, constructing a three-dimensional image from a series of two-dimensional cross sections or projections. Of interest here is *discrete tomography*, which uses low-dimensional projections to reconstruct discrete objects, such as the atomic structure of crystalline lattices and other polyatomic structures.

**3.1. The  $k$ -color tomography problem.** Numerous papers (cf. [13, 14, 18, 19]) study the  *$k$ -color tomography problem*, in which the goal is to color the entries of an  $m \times n$  matrix using  $k$  colors so that each row and column receives a prescribed number of entries of each color. The colors represent different types of atoms appearing in a crystal, and the number of times an atom appears in a given row or column is generally obtained using high-resolution transmission electron microscopes [27, 34]. This is precisely the problem of packing the degree sequences of  $k$  bipartite graphs with partite sets of size  $m$  and  $n$ .

**3.2. Sauer–Spencer-type theorems.** A sequence  $\pi = (a_1, \dots, a_r; b_1, \dots, b_s)$  is *bigraphic* if there is a bipartite graph  $G$  such that  $\pi = \pi(G)$  with partite sets  $X$  and  $Y$ , and the degrees of the vertices in  $X$  and  $Y$  are  $a_1, \dots, a_r$  and  $b_1, \dots, b_s$ , respectively. Two bigraphic sequences,  $\pi_1 = (a_1^{(1)}, \dots, a_r^{(1)}; b_1^{(1)}, \dots, b_s^{(1)})$  and  $\pi_2 = (a_1^{(2)}, \dots, a_r^{(2)}; b_1^{(2)}, \dots, b_s^{(2)})$ , pack if there exist edge-disjoint bipartite graphs  $G_1$  and  $G_2$ , both with partite sets  $X = \{x_1, \dots, x_r\}$  and  $Y = \{y_1, \dots, y_s\}$ , such that for  $j \in \{1, 2\}$ ,

$$d_{G_j}(x_i) = a_i^{(j)}$$

for  $1 \leq i \leq r$  and

$$d_{G_j}(y_i) = b_i^{(j)}$$

for  $1 \leq i \leq s$ .

The following is a tomographic analogue to Corollary 1.5.

**THEOREM 3.1.** *Let  $\pi_1$  and  $\pi_2$  be bigraphic sequences with parts of sizes  $r$  and  $s$ , and  $\Delta_i = \Delta(\pi_i)$  and  $\delta_i = \delta(\pi_i)$  for  $i \in \{1, 2\}$ , such that  $\Delta_1 \leq \Delta_2$  and  $\delta_1 \geq 1$ . If*

$$\Delta_1 \Delta_2 \leq \frac{(r+s)}{4},$$

*then  $\pi_1$  and  $\pi_2$  pack.*

The other main result of this section, which takes  $\delta_1$  into account, improves on Theorem 3.1 when  $\delta_1 \geq 3$ .

**THEOREM 3.2.** *Let  $\pi_1$  and  $\pi_2$  be bigraphic sequences with parts of sizes  $r$  and  $s$ , and  $\Delta_i = \Delta(\pi_i)$  and  $\delta_i = \delta(\pi_i)$  for  $i \in \{1, 2\}$ , such that  $\Delta_1 \leq \Delta_2$  and  $\delta_1 \geq 1$ . If*

$$\Delta_1 \Delta_2 \leq \delta_1 \frac{(r+s)}{8},$$

*then  $\pi_1$  and  $\pi_2$  pack.*

As before, we say a vertex pair  $(x, y)$  is a *bad pair* for  $(G_1, G_2)$  or a  $(G_1, G_2)$ -*bad pair* if  $xy \in E(G_1) \cap E(G_2)$ .

Let  $\pi_1$  and  $\pi_2$  be bigraphic sequences that do not pack, choose  $G_1 = G(\pi_1)$  and  $G_2 = G(\pi_2)$  to have the fewest bad pairs among all realizations of  $\pi_1$  and  $\pi_2$ , and let  $G = G_1 \cup G_2$ . Fix a  $(G_1, G_2)$ -bad pair  $(x, y)$  and let  $X$  and  $Y$  be the partite sets of  $G$ , where  $x \in X$  and  $y \in Y$ . Let  $I_X = X - N_G(y)$  and  $I_Y = Y - N_G(x)$ . We now have the following lemmas, the first of which is analogous to Claim 2.

**LEMMA 3.3.** *The set  $I_X \cup I_Y$  is independent.*

*Proof.* Suppose otherwise, so in particular let  $z \in I_X$  and  $z' \in I_Y$  such that  $zz' \in E(G)$ . Exchanging the edges  $xy$  and  $zz'$  with the nonedges  $zy$  and  $z'x$  decreases the number of  $(G_1, G_2)$ -bad pairs, contradicting the choice of  $G_1$  and  $G_2$ .  $\square$

**LEMMA 3.4.** *The subgraph of  $G$  induced by  $N_{G_1}(I_Y) \cup N_{G_1}(I_X) \cup \{x, y\}$  is a complete bipartite graph.*

*Proof.* First, note that by Lemma 3.3 and the definition of  $I_Y$ ,  $x$  is adjacent to every vertex in  $N_{G_1}(I_X)$ , and likewise,  $y$  is adjacent to every vertex in  $N_{G_1}(I_Y)$ . Suppose then that there is some  $w \in N_{G_1}(I_Y)$  and  $w' \in N_{G_1}(I_X)$  such that  $ww'$  is not an edge in  $G$ . Now we have that there is some  $z' \in I_Y$  and  $z \in I_X$  such that  $wz'$  and  $w'z$  are edges in  $G_1$ . Exchanging the edges  $w'z, wz'$ , and  $xy$  (all in  $G_1$ ) with the nonedges  $ww', xz'$ , and  $yz$  decreases the number of bad pairs in  $G$ , a contradiction.  $\square$

We are now ready to prove Theorems 3.1 and 3.2.

*Proof of Theorem 3.1.* Observe first that each vertex in  $N_{G_1}(I_X)$  (respectively,  $N_{G_1}(I_Y)$ ) can have at most  $\Delta_1$  neighbors in  $I_X$  (respectively,  $I_Y$ ) so that

$$|I_X| + |I_Y| \leq \Delta_1(|N_{G_1}(I_X)| + |N_{G_1}(I_Y)|).$$

We further have that

$$n - (|N_G(x)| + |N_G(y)|) \leq |I_X| + |I_Y|$$

and

$$|N_{G_1}(I_Y)| + |N_{G_1}(I_X)| \leq |N_G(x)| + |N_G(y)| - 2 \leq 2(\Delta_1 + \Delta_2) - 4.$$

Taken together, these yield that

$$n - (2(\Delta_1 + \Delta_2) - 2) \leq \Delta_1(2(\Delta_1 + \Delta_2) - 4),$$

so

$$\frac{n}{2} \leq (\Delta_1 + 1)(\Delta_1 + \Delta_2) - 2\Delta_1 - 1.$$

As  $\Delta_1\Delta_2 \leq \frac{n}{4}$ , it follows that

$$2\Delta_1\Delta_2 \leq \Delta_1^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - 2\Delta_1 - 1,$$

so that

$$\Delta_1\Delta_2 - \Delta_2 \leq \Delta_1^2 - \Delta_1 - 1.$$

However, then

$$\Delta_2 \leq \Delta_1 - \frac{1}{\Delta_1 - 1},$$

a contradiction, since  $\Delta_1 \leq \Delta_2$ .  $\square$

*Proof of Theorem 3.2.* By Lemma 3.3,  $I_X$  and  $I_Y$  are independent, so every vertex in  $I_X \cup I_Y$  must have at least  $\delta_1$  neighbors in  $N_{G_1}(I_Y) \cup N_{G_1}(I_X)$ . Also, as in Theorem 3.1, each vertex in  $N_{G_1}(I_Y) \cup N_{G_1}(I_X)$  has at most  $\Delta_1$  neighbors in  $I_X \cup I_Y$ . Therefore,

$$\delta_1(|I_X| + |I_Y|) \leq \Delta_1(|N_{G_1}(I_Y)| + |N_{G_1}(I_X)|)$$

so that

$$|I_X| + |I_Y| \leq \frac{\Delta_1}{\delta_1} (|N_{G_1}(I_Y)| + |N_{G_1}(I_X)|).$$

Again, we have that

$$|N_{G_1}(I_Y)| + |N_{G_1}(I_X)| \leq |N_G(x)| + |N_G(y)| - 2 \leq 2(\Delta_1 + \Delta_2 - 2).$$

Let  $r + s = n$ , so that

$$|I_X| + |I_Y| = n - (|N_G(x)| + |N_G(y)|).$$

Combining the above equations yields

$$n - 2(\Delta_1 + \Delta_2) + 2 \leq 2\frac{\Delta_1}{\delta_1}(\Delta_1 + \Delta_2 - 2).$$

By isolating  $\Delta_2$ ,

$$\frac{\delta_1 n}{2(\Delta_1 + \delta_1)} - \Delta_1 + \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1} \leq \Delta_2.$$

Notice that  $\Delta_1 + \delta_1 \leq 2\Delta_1$ , so we have

$$\frac{\delta_1 n}{4\Delta_1} \leq \Delta_2 + \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1}.$$

By assumption,  $\Delta_2 \leq \frac{\delta_1 n}{8\Delta_1}$ , so

$$2\Delta_2 \leq 2\left(\frac{\delta_1 n}{8\Delta_1}\right) \leq \Delta_2 + \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1},$$

which implies

$$\Delta_2 \leq \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1}.$$

Since  $\frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1} > 0$ , and  $\Delta_2 \geq \Delta_1$ , we arrive at a contradiction, completing the proof.  $\square$

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## REFERENCES

- [1] M. AIGNER AND S. BRANDT, *Embedding arbitrary graphs of maximum degree two*, J. London Math. Soc., 28 (1993), pp. 39–51.
- [2] N. ALON AND E. FISCHER, *2-factors in dense graphs*, Discrete Math., 152 (1996), pp. 13–23.
- [3] D. BAUER, H. J. BROERSMA, J. VAN DEN HEUVEL, N. KAHL, A. NEVO, E. SCHMEICHEL, D. R. WOODALL, AND M. YATAURO, *A survey of best monotone degree conditions for graph properties*, Graphs. Combin., 31 (2015), pp. 1–22.
- [4] B. BOLLOBÁS AND S. E. ELDRIDGE, *Packings of graphs and applications to computational complexity*, J. Combin. Theory Ser. B, 25 (1978), pp. 105–124.
- [5] B. BOLLOBÁS, A. KOSTOCHKA, AND K. NAKPRASIT, *Packing  $d$ -degenerate graphs*, J. Combin. Theory Ser. B, 98 (2008), pp. 85–94.
- [6] A. BUSCH, M. FERRARA, S. HARTKE, AND M. JACOBSON, *Ramsey-type numbers for degree sequences*, Graphs. Combin., 30 (2014), pp. 847–859.
- [7] A. BUSCH, M. FERRARA, M. JACOBSON, H. KAUL, S. HARTKE, AND D. WEST, *Packing of graph  $n$ -tuples*, J. Graph Theory, 70 (2012), pp. 29–39.
- [8] P. CATLIN, *Embedding Subgraphs and Coloring Graphs Under Extremal Degree Conditions*, Ph.D. thesis, Ohio State University, Columbus, 1976.
- [9] V. CHVÁTAL, *On Hamilton's ideals*, J. Combin. Theory Ser. B, 12 (1972), pp. 163–168.
- [10] H. CORRÁDI AND A. HAJNAL, *On the Maximal number of independent circuits in a graph*, Acta Math. Hungar., 14 (1963), pp. 423–439.
- [11] B. CSABA, A. SHOKOUFANDEH, AND E. SZEMERÉDI, *Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three*, Combinatorica, 23 (2003), pp. 35–72.
- [12] G. A. DIRAC, *Some theorems on abstract graphs*, Proc. London Math. Soc., 2 (1952), pp. 69–81.
- [13] P. DULIO AND C. PERI, *Discrete tomography and plane partitions*, Adv. Appl. Math., 50 (2013), pp. 390–408.
- [14] C. DÜRR, F. GUÍÑEZ, AND C. MATAMALA, *Reconstructing 3-colored grids from horizontal and vertical projections is NP-hard: A solution to the 2-atom problem in discrete tomography*, SIAM J. Discrete Math., 26 (2012), pp. 330–352.
- [15] Z. DVORÁK AND B. MOHAR, *Chromatic number and complete graph substructures for degree sequences*, Combinatorica, 33 (2013), pp. 513–529.
- [16] P. ERDOS AND T. GALLAI, *Graphs with prescribed degrees*, Mat. Lapor, 11 (1960), pp. 264–274 (in Hungarian).
- [17] M. FERRARA, T. LESAULNIER, C. MOFFATT, AND P. WENGER, *On the sum necessary to ensure a degree sequence is potentially  $H$ -graphic*, Combinatorica, to appear.
- [18] P. GRITZMANN, B. LANGFELD, AND M. WIEGELMANN, *Uniqueness in discrete tomography: Three remarks and a corollary*, SIAM J. Discrete Math., 25 (2011), pp. 1589–1599.
- [19] F. GUÍÑEZ, C. MATAMALA, AND S. THOMASSÉ, *Realizing disjoint degree sequences of span at most two: A tractable discrete tomography problem*, Discrete Appl. Math., 159 (2011), pp. 23–30.
- [20] A. HAJNAL AND E. SZEMERÉDI, *Proof of a conjecture of Erdős*, in Combinatorial Theory and Its Applications, II, P. Erdős, and V. T. Sós, eds., Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, 1970.
- [21] S. L. HAKIMI, *On realizability of a set of integers as degrees of the vertices of a linear graph*, I, J. SIAM, 10 (1962), pp. 496–506.
- [22] S. HAKIMI AND E. SCHMEICHEL, *Graphs and their degree sequences: A survey*, in Theory and Applications of Graphs, Lecture Notes in Math. 642, Springer, Berlin, 1978, pp. 225–235.
- [23] V. HAVEL, *A remark on the existence of finite graphs*, Časopis Pěst. Mat., 80 (1955), pp. 477–480 (in Czech).
- [24] H. KAUL AND A. KOSTOCHKA, *Extremal graphs for a graph packing theorem of Sauer and Spencer*, Combin. Probab. Comput., 16 (2007), pp. 409–416.
- [25] H. KAUL, A. KOSTOCHKA, AND G. YU, *On a graph packing conjecture by Bollobás, Eldridge and Catlin*, Combinatorica, 28 (2008), pp. 469–485.
- [26] H. KIERSTEAD, A. KOSTOCHKA, AND G. YU, *Extremal graph packing problems: Ore-type versus Dirac-type*, London Math. Soc. Lecture Note Ser. 365, Cambridge University Press, Cambridge, UK, 2009.

- [27] C. KISIELOWSKI, P. SCHWANDER, F. BAUMANN, M. SEIBT, Y. KIM, AND A. OURMAZD, *An approach to quantitate high resolution transmission electron microscopy of crystalline materials*, Ultramicroscopy, 58 (1995), pp. 131–155.
- [28] S. KUNDU, *The  $k$ -factor conjecture is true*, Discrete Math., 6 (1973), pp. 367–376.
- [29] L. LOVÁSZ, *Valencies of graphs with 1-factors*, Period. Math. Hungar., 5 (1974), pp. 149–151.
- [30] A. R. RAO, *The clique number of a graph with a given degree sequence*, in Proceedings of the Symposium on Graph Theory, ISI Lecture Notes 4, Macmillan of India, New Delhi, 1979, pp. 251–267.
- [31] S. B. RAO, *A survey of the theory of potentially  $P$ -graphic and forcibly  $P$ -graphic degree sequences*, in Combinatorics and Graph Theory, Lecture Notes in Math. 885, Springer, Berlin, 1981, pp. 417–440.
- [32] N. ROBERTSON AND Z. SONG, *Hadwiger number and chromatic number for near regular degree sequences*, J. Graph Theory, 64 (2010), pp. 175–183.
- [33] N. SAUER AND J. SPENCER, *Edge disjoint placement of graphs*, J. Combin. Theory Ser. B, 25 (1978), pp. 295–302.
- [34] P. SCHWANDER, C. KISIELOWSKI, M. SEIGT, F. BAUMANN, Y. KIM, AND A. OURMAZD, *Mapping projected potential interfacial roughness and composition in general crystalline solids by quantitative transmission electron microscopy*, Phys. Rev. Lett., 71 (1993), pp. 4150–4153.
- [35] M. WOZNAK, *Packing of graphs*, Dissertationes Math., 362 (1997).
- [36] H. YAP, *Packing of graphs: A survey*, Discrete Math., 72 (1988), pp. 395–404.