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Spring March, 2018

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Splish-splash: Center of mass, stability, and a fun pool toy

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Center of mass is a common topic in physics courses. It appears in relation to studies of stable and unstable equilibrium, momentum, and rotation. Science products suppliers frequently include gadgets that demonstrate the concepts of center of mass and stability, such as the classic balancing bird. Additionally, *The Physics Teacher* has featured articles studying the center of mass of a rotating baton,¹ locating the center of mass of a hanging Slinky toy,² and describing a wide range of interesting systems.³

Project inspiration

While attending a swim practice for my kids, I was inspired by a pool toy similar to the one pictured in Fig. 1 to create a dynamic, challenging center of mass problem. Mounted about 3.0 m above the pool is a large cone with an axle passing horizontally through the cone, allowing the cone to rotate freely while water pours into the cone from above. Videos depicting the dynamics of the cone can be found online.⁴

Fig. 1. Example of a water cone (also called bucket dump). The water can faintly be seen pouring into the top of the cone. Used with permission from My Splash Pad (<https://mysplashpad.net/>).

Initially, the cone is empty with the pointed end (vertex) directed towards the pool. Water pours into the cone from above, and when the center of mass of the water-cone system rises above the support (the axle), it enters a state of unstable equilibrium. The turbulence of the water pouring in quickly causes the cone to tip and empty its contents on the joyous swimmers frolicking below.

The problem

Students are presented this problem after a brief review of center of mass principles, which includes an example of finding the center of mass of a solid cone.

$$\begin{aligned} X_{\text{CM}} &= \frac{1}{M} \int x \, dm, Y_{\text{CM}} = \frac{1}{M} \int y \, dm, \\ Z_{\text{CM}} &= \frac{1}{M} \int z \, dm. \end{aligned} \quad (1)$$

The cone is oriented with the vertex at the origin and the symmetry axis of the cone along the $+z$ -axis, and the center of mass is shown to be $X_{\text{CM}} = 0$, $Y_{\text{CM}} = 0$, and $Z_{\text{CM}} = 3H/4$,⁵ where the X_{CM} and Y_{CM} values are clear from

Fig. 2. Schematic drawing of hollow cone.

symmetry considerations and H is the overall height of the cone. Next, students are shown a video of this pool toy and asked to determine the center of mass of the hollow cone with no water in it. To provide a common starting point for students, I simplify the cone geometry to have wall thickness T , height H , and radius at the top R , and specify the material to be aluminum. The schematic shown in Fig. 2 highlights that the thickness is measured perpendicular through the wall of the cone, shows that the axle passes through the cone at a height of $0.6H$, and shows that for our model of the cone, we opt to have a pointed tip (rather than the rounded cone as seen in Fig. 1). The angle θ and the position $z = b$ are given by

$$\theta = \tan^{-1} \left[\frac{R}{H} \right] \text{ and } b = \left[\frac{T}{\sin \theta} \right].$$

Note that a solid cone of height H could be constructed using the hollow cone and exactly filling it using a smaller solid cone of height $H-b$ with the same θ . Thus the center of mass of the hollow cone can be determined from

$$\begin{aligned} Z_{\text{CM}}^{\text{large solid cone}} &= \frac{3H}{4} = \\ \frac{m_{\text{hollow cone}} Z_{\text{CM}}^{\text{hollow cone}} + m_{\text{small solid cone}} Z_{\text{CM}}^{\text{small solid cone}}}{m_{\text{total}}}. \end{aligned} \quad (2)$$

The volume of a solid cone and the density of aluminum,

$$V = \pi R^2 H / 3 \text{ and } \rho_{\text{aluminum}} = 2700 \frac{\text{kg}}{\text{m}^3},$$

allows the masses to be computed, and we know that

$$Z_{\text{CM}}^{\text{small solid cone}} = b + \frac{3(H-b)}{4}.$$

Fighting through a bit of algebra yields the result

$$\begin{aligned} Z_{\text{CM}}^{\text{hollow cone}} &= \\ \frac{3R^2 H^2 - \left(R - \frac{T}{\cos \theta} \right)^2 \left(H - \frac{T}{\sin \theta} \right) \left(3H + \frac{T}{\sin \theta} \right)}{4 \left[R^2 H - \left(R - \frac{T}{\cos \theta} \right)^2 \left(H - \frac{T}{\sin \theta} \right) \right]}. \end{aligned} \quad (3)$$

In order to make sense of this result, it helps to provide dimensions for our cone. I had selected $H = 0.70$ m, $R = 0.20$ m, and $T = 0.03$ m, which gives $Z_{CM}^{\text{hollow cone}} = 0.6912H = 0.4838$ m. Note that this result compares sensibly to a cone with infinitesimally thin walls, which has center of mass at $2H/3$. Obtaining this result leads students to realize that the center of mass of the hollow cone is already higher than the axle. We counter this problem by introducing a point mass located at the vertex of the cone. At this point students are unleashed to tackle the remainder of the project independently. Students determine what $M_{\text{point mass}}$ must be in order to shift the center of mass of the hollow cone plus point mass system to $Z_{CM} = 0.55H$, which can be determined by setting $Z_{CM}^{\text{point mass}} = 0$ in the expression.

$$Z_{CM}^{\text{hollow cone+point mass}} = 0.55H = \frac{M_{\text{hollow cone}} Z_{CM}^{\text{hollow cone}} + M_{\text{point mass}} Z_{CM}^{\text{point mass}}}{M_{\text{hollow cone}} + M_{\text{point mass}}} \quad (4)$$

The fundamental question that remains is to determine the height of the water in the cone when it flips. An equation similar to Eq. (4) is used to accomplish this, but it now includes a term for the water, which is essentially a small “solid” cone. At a particular fill level, $Z = Z_{\text{water}}$, the system’s center of mass rises above the axle and the cone tips. The vertex of the water cone lies at

$$Z = \frac{T}{\sin \theta},$$

and so its properties include: height $H_{\text{water}} = Z_{\text{water}} - \frac{T}{\sin \theta}$, volume

$$V_{\text{water}} = \frac{\pi}{3} (\text{radius})^2 (\text{height}) = \frac{\pi}{3} \left(\frac{RZ_{\text{water}}}{H} - \frac{T}{\cos \theta} \right)^2 (H_{\text{water}}),$$

and center of mass

$$Z_{CM}^{\text{water}} = \frac{3}{4} (H_{\text{water}}) + \frac{T}{\sin \theta}.$$

Students will often choose to insert numerical values to make the algebraic solution simpler. Alternately, students may be encouraged to employ a computational program such as Mathematica, Maple, or Excel to aid in finding the solution. Extensions to this problem can include, for example, asking students what is the minimum density fluid that could be used that would still flip the cone, or to determine the water flow rate based on the amount of time (from the video) it takes the empty cone to fill and dump.

The experiment

I constructed a model version of this water cone toy using an Imhoff cone. Holes were drilled at $0.6H$ to match the dimensions of our problem. The overall dimensions and material of the hollow cone differ from the problem tackled above; however, the process developed can be applied to this model system. The first thing I show the students is that, as predicted, the cone flips vertex up due to the center of mass location. To create the “point mass,” I glued several washers to

Fig. 3. Model version of pool toy water cone. In the photo on the left, the center of mass lies between the axle and the top of the cone, causing the vertex to point upward. In the photo on the right, a “point mass” has been added to the vertex to shift the center of mass to lie between the axle and vertex.

the bottom of the cap that screws off the bottom of the Imhoff cone—when this cap is replaced, the cone flips as desired with the vertex downward. The model cone can be seen in Fig. 3.

Students record measurements of the model system and make a prediction about its fill level when they expect it to flip and pour out the water. They test those predictions by taking the model outside and performing tests pouring water from a pitcher.

Conclusions and future directions

This project has grown larger than I had initially anticipated when I was sitting at a swim practice pondering this pool toy. Students have reported that they find it surprising that what appears to be a rather simple system can involve such a challenging solution. The incorporation of the experimental components of the project using the model system have enhanced the project, as students must apply their earlier solutions to a real system to make and test predictions.

References

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2. T. Toepker, “Center of mass of a suspended Slinky: An experiment,” *Phys. Teach.* **42**, 16 (Jan. 2004).
3. E. McFarland, “Center of mass revisited,” *Phys. Teach.* **21**, 42 (Jan. 1983).
4. Links to examples of the water cone pool toy (a.k.a. “bucket dump”) in action: <https://youtu.be/Is6yzP31H7M>, <https://vimeo.com/173851451>, and <https://youtu.be/jEYU8yStEFw>.
5. J. Taylor, *Classical Mechanics* (University Science Books, 2005), pp. 87–89.