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Mikhail E. Veisman
Serguei Y. Kalmykov

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CORRELATED COHERENT STATES OF THE TWO-DIMENSIONAL QUANTUM OSCILLATOR WITH A NONSTATIONARY MODE COUPLING

M. E. Veisman[1] and S. Yu. Kalmykov[1]

We investigate the squeezing and correlation phenomena of quantum fluctuations in correlated coherent states of a system of two oscillatory modes with a parametric coupling of the coordinates. In the regime of strong coupling between the modes, we discover an excitation and a strong correlation of the quantum fluctuations. The matrix of dispersions and covariances of the coordinates and momenta is found explicitly with the help of the method of the linear quantum integrals of motion.

1. The two-dimensional quantum oscillator with parametric mode coupling may serve as an example of two-mode squeezed light [1, 2], whose statistical properties are interesting in view of the use of parametric excitation in various quantum-optics systems [3] aimed at obtaining squeezed [4] and correlated [5] states of electromagnetic fields. In recent papers [6–8], analyses were given of the parametric [6, 7] and nonlinear [8] excitations of the Bose oscillators of the field and, also, of the statistical properties of the emerging nonclassical states of the field. Squeezing and correlations of quantum fluctuations in systems of nonstationary coupled oscillators were recently considered in [9, 10]. Nonstationary mode coupling can occur when coherent light propagates in a nonlinear medium whose index of refraction depends on the field amplitude. The results of the present work can be applied, therefore, to nonlinear problems of quantum optics [11].

The aim of this work is to investigate the squeezing and excitation of quantum fluctuations in the correlated coherent states of a two-dimensional quantum oscillator with a coupling of finite duration between the modes. The mode frequencies are assumed to be distinct real constants. The coordinates of the two-dimensional oscillator are assumed to be coupled during a limited period of time, the coupling frequency being a real constant [12]. The Hamiltonian of the oscillator modes with a nonstationary coupling of this type reads as

$$\widehat{H}(t) = \frac{1}{2}\widehat{\mathbf{Q}}\mathbf{B}\widehat{\mathbf{Q}}, \tag{1}$$

where $\widehat{\mathbf{Q}} = (\hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2)$ is a 4-column of momentum ($\hat{p}_i$) and coordinate ($\hat{q}_i$) operators (which correspond to the quadrature components of two-mode light in quantum optics), and $\mathbf{B}(t)$ is the 4×4 matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & \mathbf{B}_2 \end{pmatrix}, \qquad \mathbf{B}_2 = \begin{pmatrix} \omega^2 & \lambda(t) \\ \lambda(t) & \omega^{-2} \end{pmatrix},$$

with $\mathbf{I}_2$ being the unit 2×2 matrix. The Hamiltonian (1) is expressed through the dimensionless variables in which the energy is normalized to $\hbar\sqrt{\omega_1\omega_2}$ and $\hbar$ is Planck's constant. Further, $\omega_{1,2}$ is the frequency of the coupled modes and $\omega^2 = \omega_1/\omega_2$. The coupling $\lambda(t)$ is taken to be a piece-wise linear function $\lambda(0 < t < T) = \omega_0^2$, $\lambda(t < 0,\ t > T) = 0$, T is the duration of the mode coupling (where time is measured in the units of $(\omega_1\omega_2)^{-1/2}$), while ω_0 is the dimensionless coupling constant. In the general case, we have $\omega^2 \neq 1$. All of the frequencies are real and the oscillator masses are set equal to one. When $\omega^2 \neq 1$, it is

[1]Moscow Physicotechnical Institute, Russia.

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impossible to introduce normal coordinates by performing a canonical change of variables that preserves the commutation relations $[\widehat{Q}_a, \widehat{Q}_b] = -i\Sigma_{ab}$ $(a, b = 1, 2, 3, 4)$ satisfied by the dynamic variables, where

$$\mathbf{\Sigma} = \begin{pmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{pmatrix}.$$

Therefore, the problem is solved in the original variables.

2. The quantum-mechanical model under consideration can be exactly solved using the method of linear integrals of motion [13, 14]. It is known that the existence of the unitary operator implementing the evolution of the system enables one to construct $2N$ linear integrals of motion $\widehat{\mathbf{p}}_0(t)$ and $\widehat{\mathbf{q}}_0(t)$ (where, in our case, the number of the degrees of freedom is $N = 2$) by means of canonically transforming the dynamic variables as

$$\hat{I}(t) = \begin{pmatrix} \widehat{\mathbf{p}}_0(t) \\ \widehat{\mathbf{q}}_0(t) \end{pmatrix} = \mathbf{\Lambda}(t)\widehat{\mathbf{Q}}, \tag{2}$$

which does not affect the commutation relations $[\hat{I}_a, \hat{I}_b] = [\widehat{Q}_a, \widehat{Q}_b]$, whence $\mathbf{\Lambda}^{-1} = -\mathbf{\Sigma}\tilde{\mathbf{\Lambda}}\mathbf{\Sigma}$ (where the tilde denotes the transposed matrix). In physical terms, the integrals of motion (2) describe $2N$ initial values of the momenta and coordinates of the system. According to the Stone–von Neumann theorem [15], $\widehat{\mathbf{p}}_0(t)$ and $\widehat{\mathbf{q}}_0(t)$ make up a complete set of integrals of motion of a given quantum system [13], which means that any function of the $\widehat{\mathbf{p}}_0(t)$ and $\widehat{\mathbf{q}}_0(t)$ operators is necessarily an integral of motion. The necessary and sufficient condition for the invariance of $\widehat{\mathbf{p}}_0$ and $\widehat{\mathbf{q}}_0$ is the requirement that the $\mathbf{\Lambda}$-matrix evolve in accordance with the matrix equation

$$\dot{\mathbf{\Lambda}} = \mathbf{\Lambda}\mathbf{\Sigma}\mathbf{B} \tag{3}$$

with the initial condition $\mathbf{\Lambda}(0) = \mathbf{I}_4$, where $\mathbf{I}_4$ is the unit 4×4 matrix. System (3), in which the coupling coefficient is piece-wise constant, admits a solution

$$\mathbf{\Lambda}(t \leq T) = \mathbf{T}_+ \otimes \Xi_+ - \mathbf{T}_- \otimes \Xi_-, \tag{4}$$

where $\otimes$ denotes the tensor product of the auxiliary matrices

$$\mathbf{T}_\pm = \begin{pmatrix} \cos(\Omega_\pm t) & \Omega_\pm \sin(\Omega_\pm t) \\ -\Omega_\pm^{-1} \sin(\Omega_\pm t) & \cos(\Omega_\pm t) \end{pmatrix}, \qquad \Xi_\pm = \frac{1}{a_+ - a_-} \begin{pmatrix} a_\pm & 1 \\ 1 & -a_\mp \end{pmatrix},$$

which are expressed through the parameters $\omega_\pm^2 = \omega^2 \pm \omega^{-2}$, $\Omega_\pm^2 = \frac{1}{2}(\omega_+^2 \pm (\omega_-^4 + 4\omega_0^4)^{\frac{1}{2}})$, and $a_\pm = (\Omega_\pm^2 - \omega^{-2})/\omega_0^2$, where $a_+ a_- = -1$. The matrices $\Xi_\pm$ satisfy the relations $\Xi_\pm \mathbf{B}_2 = \Omega_\pm^2 \Xi_\pm$ and, also, have the properties $\Xi_\pm^2 = \pm\Xi_\pm$ and $\Xi_+\Xi_- = \mathbf{0}$. The quantities $\Omega_\pm$ coincide with the eigenvalues of the Hamilton equations for the classical two-dimensional coupled oscillator with Hamiltonian (1). When the coupling is strong ($\omega_0^2 > 1$), the potential energy of the classical two-dimensional coupled oscillator is not positive-definite ($\det \mathbf{B}_2 < 0$) and, thus, the oscillator is unstable. Then, eigenvalue Ω_- is purely imaginary, which leads to unstable behavior of the elements of the Λ-matrix implementing the canonical transformation (2). In the opposite case of weak coupling, the classical system performs a finite motion, the relevant eigenvalues are real, and the elements of the Λ-matrix are bounded.

3. The matrix of dispersions and covariances of the coordinates and momenta,

$$\sigma_Q = \frac{1}{2}\langle \widehat{Q}_\alpha \widehat{Q}_\beta + \widehat{Q}_\beta \widehat{Q}_\alpha \rangle - \langle \widehat{Q}_\alpha \rangle \langle \widehat{Q}_\beta \rangle = \begin{pmatrix} \sigma_{pp} & \sigma_{qp}^T \\ \sigma_{qp} & \sigma_{qq} \end{pmatrix}$$

is a positive-definite 4×4-matrix that is real and symmetric. Elements of this matrix satisfy certain constraints (the generalized indeterminacy relations [14]).

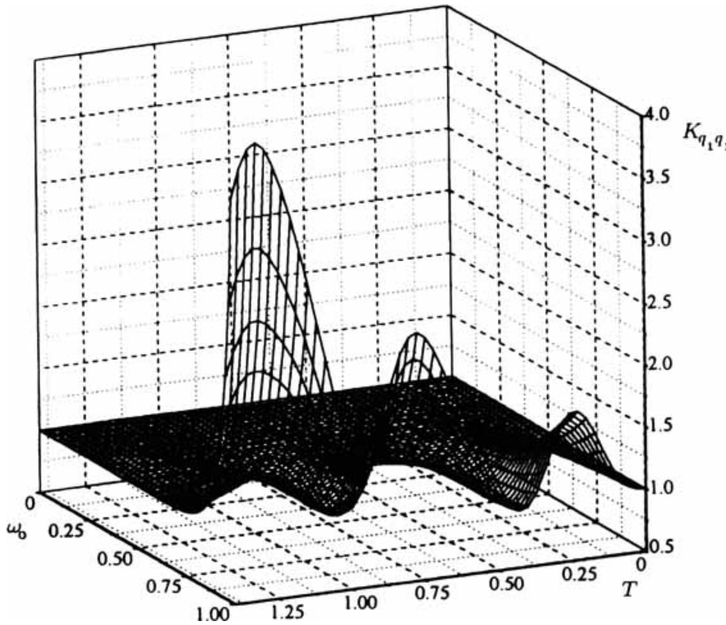

Fig. 1. Normalized dispersion of the coordinate $K_{q_1q_1}$.

It is assumed that the quantum state of the model under consideration with $t < 0$ is the coherent state [16] $|\alpha_1\rangle|\alpha_2\rangle$ (or, as a particular case, the vacuum state $|0\rangle|0\rangle$) of decoupled modes with the dispersion matrix given by

$$\sigma_Q(t \leq 0) = \begin{pmatrix} \mathbf{E}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_q \end{pmatrix},$$

where the diagonal blocks read

$$\mathbf{E}_p = \frac{1}{2}\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \mathbf{E}_q = \frac{1}{2}\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}.$$

In order to evaluate $\sigma_Q(t)$, one should use the dispersion matrix of the integrals of motion $\sigma_I = \frac{1}{2}\langle \hat{I}_a\hat{I}_b + \hat{I}_b\hat{I}_a\rangle - \langle \hat{I}_a\rangle\langle \hat{I}_b\rangle$. The average values of the integrals are independent of time, hence, $\sigma_I = \sigma_I(t = 0) = \sigma_Q(t = 0)$. Since $\widehat{Q}_a\widehat{Q}_b = (\mathbf{\Lambda}^{-1})_{ai}\hat{I}_i\hat{I}_j(\Lambda^{-1})_{jb}$, we have

$$\sigma_Q(0 \leq t \leq T) = \mathbf{\Lambda}^{-1}(t)\sigma_I\big(\mathbf{\Lambda}^T(t)\big)^{-1}. \tag{5}$$

This immediately implies the following expressions for the 2×2 blocks of the dispersion matrix:

$$\sigma_{pp}(t \leq T) = \sum_{\sigma=\pm}\left(\frac{\sin^2(\Omega_\sigma t)}{\Omega_\sigma^{-2}}\mathbf{A}_{q\sigma} + \cos^2(\Omega_\sigma t)\mathbf{A}_{p\sigma}\right) - \\ - \left(\frac{\sin(\Omega_+ t)\sin(\Omega_- t)}{\Omega_+^{-1}\Omega_-^{-1}} - \cos(\Omega_+ t)\cos(\Omega_- t)\right)(\mathbf{B}_q + \mathbf{B}_q^T), \tag{6}$$

$$\sigma_{qq}(t \leq T) = \sum_{\sigma=\pm}\left(\frac{\sin^2(\Omega_\sigma t)}{\Omega_\sigma^{2}}\mathbf{A}_{p\sigma} + \cos^2(\Omega_\sigma t)\mathbf{A}_{q\sigma}\right) - \\ - \left(\cos(\Omega_+ t)\cos(\Omega_- t) - \frac{\sin(\Omega_+ t)\sin(\Omega_- t)}{\Omega_+\Omega_-}\right)(\mathbf{B}_q + \mathbf{B}_q^T), \tag{7}$$

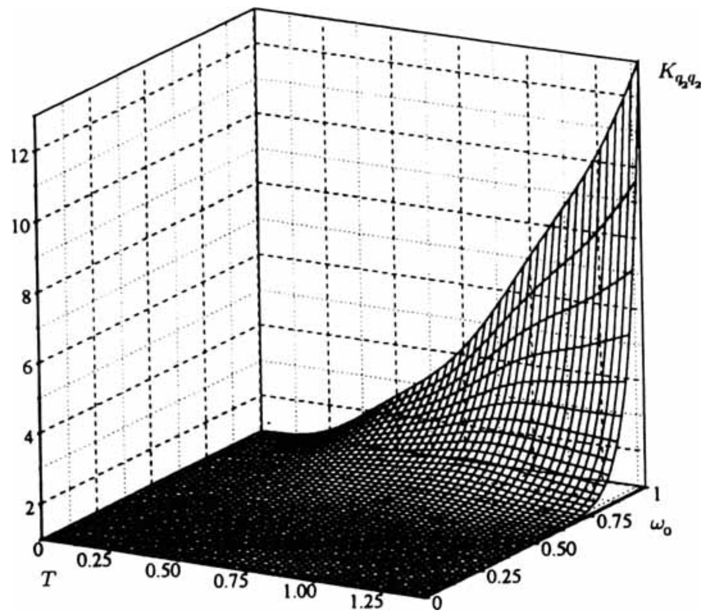

Fig. 2. Normalized dispersion of the coordinate $K_{q_2q_2}$.

$$\sigma_{qp}(t \le T) = \sum_{\sigma=\pm} \frac{1}{2}\sin(2\Omega_\sigma t)(\Omega_\sigma^{-1}\mathbf{A}_{p\sigma} + \Omega_\sigma \mathbf{A}_{q\sigma}) + \\ + \left(\frac{\sin(\Omega_- t)\cos(\Omega_+ t)}{\Omega_-^{-1}} - \frac{\sin(\Omega_+ t)\cos(\Omega_- t)}{\Omega_+}\right)\mathbf{B}_q + \\ + \left(-\frac{\sin(\Omega_- t)\cos(\Omega_+ t)}{\Omega_-} + \frac{\sin(\Omega_+ t)\cos(\Omega_- t)}{\Omega_+^{-1}}\right)\mathbf{B}_q^T, \tag{8}$$

where $\mathbf{A}_{q\pm} = \Xi_\pm \mathbf{E}_q \Xi_\pm$, $\mathbf{A}_{p\pm} = \Xi_\pm \mathbf{E}_p \Xi_\pm$, $\mathbf{B}_q = \Xi_+ \mathbf{E}_q \Xi_-$, and $\mathbf{B}_p = \Xi_+ \mathbf{E}_p \Xi_- = -\mathbf{B}_q$.

Under the evolution of the system governed by the Schrödinger equation with the nonstationary Hamiltonian (1), the two-dimensional oscillator state $|\alpha_1, \alpha_2, t\rangle$ remains a Gaussian packet described by the Wigner function [17],

$$W(\mathbf{Q}, t) = \left(\det \sigma_Q(t)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{Q}\sigma_Q^{-1}\mathbf{Q}\right),$$

where the two-dimensional, purely Gaussian (coherent) state $|\alpha_1, \alpha_2, t\rangle$ corresponds to the minimized indeterminacy relation [17] $\det \sigma_Q(t) = \det \sigma_Q(t=0) = \left(\frac{1}{4}\right)^2$. In these formulas, the four-dimensional vector $\mathbf{Q} = (p_1, p_2, q_1, q_2)$ consists of the Weyl symbols of the momentum (p_i) and the coordinate (q_i) operators. Thus, the time evolution of the state is completely determined by the evolution of the dispersion matrix with the block structure (6)–(8). The correlation coefficients $\Gamma_{q_i p_j} = \sigma_{q_i p_j} / \sqrt{\sigma_{q_i q_i} \sigma_{p_j p_j}}$ determine the statistical dependence of the coordinate and momentum operators of this state. Since the correlation coefficients that vanish in the initial coherent state of the decoupled modes become nonvanishing at all subsequent moments of time, we can say that the coherent state of the coupled modes is a "correlated" state.

In the absence of mode coupling, the time evolution of the dispersion matrix is governed by an equation similar to (5):

$$\sigma_Q(t > T) = \mathbf{\Lambda}_0^{-1}(t-T)\sigma_Q(t=T)\left(\mathbf{\Lambda}_0^T(t-T)\right)^{-1},$$

where $\mathbf{\Lambda}_0(t)$ is the $\mathbf{\Lambda}$-matrix (4) with $\omega_0 = 0$ (decoupled modes). The explicit form of $\sigma_Q(t > T)$ is not given here because it is very cumbersome.

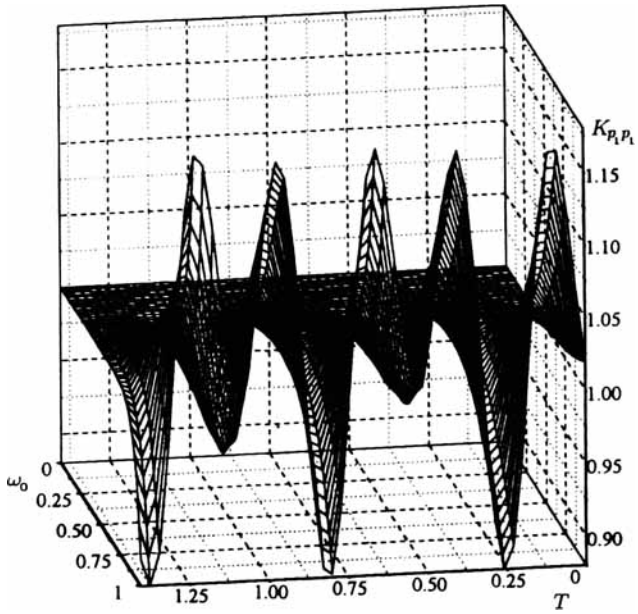

Fig. 3. Normalized momentum dispersion $K_{p_1p_1}$.

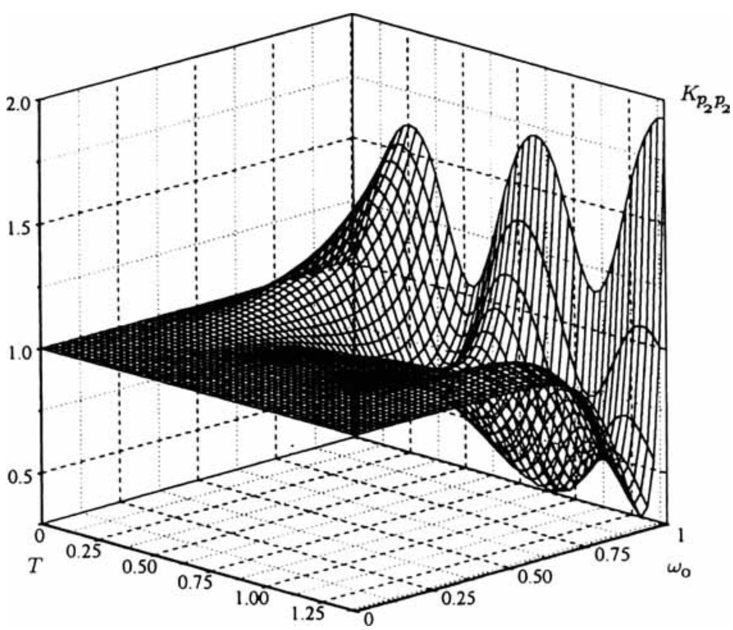

Fig. 4. Normalized momentum dispersion $K_{p_2p_2}$.

Figures 1–4 plot the coordinate and momentum dispersions ($\omega^2 = 5$) versus the duration T and the coupling constant ω_0, which is taken below the weak coupling limit ($\omega_0^2 < 1$). These dispersions are normalized to the respective dispersions in the initial coherent state, $K_{q_1q_1} = 2\omega\sigma_{q_1q_1}$, $K_{q_2q_2} = 2\omega^{-1}\sigma_{q_2q_2}$, $K_{p_1p_1} = 2\omega^{-1}\sigma_{p_1p_1}$, and $K_{p_2p_2} = 2\omega\sigma_{p_2p_2}$. A state of the system is called squeezed if the momentum dispersions $\sigma_{p_ip_i}$ and/or the coordinate dispersions $\sigma_{q_iq_i}$ are smaller than their corresponding values in the initial coherent state. In this sense, the correlated coherent state of the weakly coupled two-dimensional quantum oscillator turns out to be a squeezed state, since either $K_{p_ip_i}$ or $K_{q_iq_i}$ can become smaller than unity for certain values of the duration and the coupling constant (see Figs. 1, 3, and 4). On the other hand, dispersions of the quadrature components may increase above their initial values, provided $T\omega_0 > 1$ (as can be most clearly seen in Fig. 2). When a similar condition (coupling constant multiplied by the observation time is greater than one) holds, a dramatic increase in the dispersion of the number of photons has been observed for the parametric Bose oscillator [6, 7].

In the strong coupling regime, one can find the asymptotic form of the dispersion matrix blocks (6)–(8) for $\omega_0 \gg \omega$, $\omega_0 \gg \omega^{-1}$, once one takes into account that $\Omega_+ \approx \omega_0$, $\Omega_- \approx i\omega_0$, and neglects the terms growing slower than $e^{2\omega_0 t}$. In that case, we have

$$\sigma_{pp}(t \le T) \approx \frac{1}{32}\omega_0^2(\omega^{-1} + \omega)e^{2\omega_0 t}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{9}$$

and, moreover, $\sigma_{qq} = \omega_0^{-2}\sigma_{pp}$ and $\sigma_{pq} = -\omega_0^{-1}\sigma_{pp}$. Also, in the strong coupling regime, a strong excitation the of quantum fluctuations of the two-dimensional quantum oscillator occurs, which shows up in the exponential time growth (9) of their dispersions. Note that under these conditions, the oscillator state remains strongly correlated ($\Gamma_{p_i q_j} \neq 0$). In the regime of strong mode coupling, bound states clearly cannot be created in the two-dimensional potential of a classical coupled oscillator (since the potential energy is not positive-definite), which, in turn, means that the system performs an infinite motion in the phase space. In quantum language, this is expressed as the exponentially unstable behavior of the dispersions and covariances of the coordinates and momenta.

To summarize, we have shown that the quantum fluctuations in the correlated coherent state of a two-dimensional quantum oscillator can behave in essentially different ways, depending on the conditions of the mode coupling: squeezing occurs in the weak coupling regime and a strong correlation takes place in the strong mode coupling.

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