Spring May, 1998

Quantum-statistical properties of two coupled modes of electromagnetic field

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(Received 12 August 1996; revised manuscript received 19 August 1997)

Squeezing the quantum fluctuations of the two-mode light due to the nonstationary coupling between the quadrature components $\hat{q}_1$ and $\hat{q}_2$ is examined. The δ- and step-function mode couplings are considered. The conditions of weak and strong step-function coupling are distinguished, the latter being the condition of the instability for the classical counterpart of the quantum system under study. Under the conditions of weak coupling the quadrature squeezing is established in both a two-mode electromagnetic noise in thermal equilibrium (the thermal state) and a two-mode correlated coherent state (CCS). Squeezing in the thermal state is suppressed at a high temperature. The photon distribution function (PDF) in the thermal state reveals oscillatory behavior for both high and low temperature, the oscillations decrease while the temperature increases. The PDF in the CCS can be either oscillatory or smooth, whereas the photon statistics is essentially non-Poissonian. The nonclassical intermode photon-number correlations in the CCS are briefly studied.
[S1050-2947(98)03005-4]

PACS number(s): 42.50.Dv, 42.50.Lc, 42.50.Ar

I. INTRODUCTION

Quantum fluctuations of parametric systems were studied in recent years by a number of investigators [1–6]. It was established that squeezing [7] the quantum fluctuations of the electromagnetic field may be achieved in a resonator with moving boundaries or with a refractive index varying with time. The squeezed states of a radiation field were created experimentally using various quantum optical systems [7]. A great deal of attention was paid to nonclassical properties of distributions of the photon numbers (see Ref. [8], and references therein). An oscillatory behavior of the distribution function of photons in the squeezed coherent state of a single-mode system was discovered in Ref. [9], and it was suggested that such behavior of the photon distribution may be regarded as a sign of the nonclassical nature of the state involved. Sub-Poissonian statistics and antibunching of the photons were also studied actively [10]. Just recently, the schemes for measuring the quantum state of a *two-mode* optical field have been proposed [11]. The statistical properties of the two-mode light were discussed [12–17] since the concept of the two-mode squeezing was proposed in Refs. [18,19]. An opportunity to create the two-mode squeezed state of light by means of the nonstationary coupling between the modes with δ-excited frequencies was pointed out in Ref. [17]. Physically, coupling between the modes of light may occur, when the coherent light propagates in the nonlinear medium with a refractive index depending on the amplitude of the field [20]. The mode coupling can also appear in physics of superconductivity in the case of a superconducting circuit made of two integrating Josephson junctions [17].

In the present article we examine the quadrature squeezing and specific features of photon statistics in certain quantum states of the pair of coupled modes of radiation with different constant frequencies. It is presumed that quadrature components $\hat{q}_1$ and $\hat{q}_2$ are coupled over the course of a restricted time interval. The observable quantities corresponding to different modes become statistically dependent since the coupling emerges; hence, the quantum states of the coupled modes are correlated [21,22]. We consider below two coupled modes of the electromagnetic noise in thermal equilibrium (the thermal state) and a two-mode correlated coherent state (CCS).

The basic theoretical approach of this work relies on quantum integrals of the motion [23,24] of the system under consideration. In Sec. II we introduce those integrals in terms of the canonical transformation of the quadrature operators. The reciprocal matrix of the canonical transformation describes the semiclassical behavior of the quantum-mechanical averages of the quadrature components of the two-mode light. The symplectic matrix of the transformation is found explicitly for both δ- and step-function types of mode coupling. The conditions of weak and strong step-function coupling are defined so that the strongly coupled classical two-dimensional oscillator (i.e., the counterpart of the quantum system under study) is unstable.

In Sec. III variances of the quadrature components are calculated. Under the conditions of weak coupling, those are bounded and the phenomenon of quadrature squeezing occurs in both the thermal state and the CCS. In the thermal state, squeezing disappears for the temperature exceeding $\hbar(\omega_1\omega_2)^{1/2}$ ($\hbar$ is the Planck constant, $\omega_{1,2}$ are the mode frequencies). The quadrature variances increase with time exponentially under the conditions of strong coupling, so does the average number of photons in either mode. In this case, the larger mean number of photons corresponds to the mode with a lower frequency.

In Sec. IV the two-mode photon distribution function (PDF) in the thermal state is examined for both low and high temperature. The oscillatory behavior of the distribution function is emphasized, and it is found that the oscillations decrease while the temperature increases. The asymmetry of the PDF with respect to the photon numbers is associated

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with the difference of the mode frequencies.

In Sec. V we demonstrate the numerical examples of the PDF in the CCS. The distribution function of photons may be either oscillating or smooth depending on parameters of the state involved. The photon statistics, though, is always non-Poissonian due to the nonclassical intermode correlations of the numbers of photons.

Section VI contains a summary of the results.

II. INTEGRALS OF MOTION

Two light modes with nonstationary mode coupling are described by the Hamiltonian of the form

$$\hat{H}(t)=\frac{1}{2}\,\hat{\mathbf{Q}}^T\mathbf{B}\hat{\mathbf{Q}},$$

$$\mathbf{B}=\begin{pmatrix}\mathbf{I}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2\end{pmatrix},\quad \mathbf{B}_2=\begin{pmatrix}\omega^2 & \lambda(t)\\ \lambda(t) & \omega^{-2}\end{pmatrix}. \tag{1}$$

Hereafter $\mathbf{I}_N$ denotes N-dimensional identity matrix, and the superscript T means transposition. Time is measured in units of $(\omega_1\omega_2)^{-1/2}$ so that the unit of energy is $\hbar(\omega_1\omega_2)^{1/2}$, and $\omega^2=\omega_1/\omega_2$. The column vector $\hat{\mathbf{Q}}=(\hat{\mathbf{p}},\hat{\mathbf{q}})^T$ consists of two pairs of the quadrature components $\hat{p}_j=i\omega^{\delta_j/2}(\hat{a}_j^\dagger-\hat{a}_j)/\sqrt{2}$ and $\hat{q}_j=\omega^{-\delta_j/2}(\hat{a}_j^\dagger+\hat{a}_j)/\sqrt{2}$ of the photon creation $\hat{a}_j^\dagger$ and annihilation $\hat{a}_j$ operators ($j=1,2$, $\delta_1=1$, $\delta_2=-1$). The sign $\dagger$ means the Hermitian conjugation, and i denotes the imaginary unit. When $\omega\neq 1$, an introduction of normal variables by a nonstationary canonical replacement seems possible. However, obtaining the replacement is not more simple than solving the problem in the original variables.

Statistical properties of the interacting modes are examined below using quantum integrals of the motion (or invariant operators) of the system under consideration. An average value of a quantum integral of the motion $\langle\hat{\mathcal{I}}\rangle=\mathrm{Tr}\,\hat{\rho}\hat{\mathcal{I}}$ (hereafter $\hat{\rho}$ denotes a density operator of a system under study) is independent of time. Therefore the invariant operator is a solution to the operator equation

$$\partial\hat{\mathcal{I}}/\partial t=i[\hat{\mathcal{I}},\hat{H}]. \tag{2}$$

The invariant operators, which are quadratic with respect to position and momentum operators, were found for a one-dimensional quantum oscillator with a variable frequency in Ref. [25]. The integrals of the motion, which are linear in the position and momentum operators, were obtained for a one-dimensional quantum oscillator in Refs. [26,27]. The integrals of the motion, which are both linear and quadratic in the position and momentum operators, were obtained for a nonstationary multidimensional quantum oscillator in Ref. [28]. According to Refs. [23,24], where the general theory of multidimensional quantum systems with time-dependent quadratic Hamiltonians is developed, we introduce the linear invariant operators $\hat{\mathcal{I}}(t)=(\hat{\mathbf{p}}_0,\hat{\mathbf{q}}_0)^T$ via a homogeneous canonical transform of the quadrature components:

$$\hat{\mathcal{I}}(t)=\mathbf{\Lambda}(t)\hat{\mathbf{Q}}. \tag{3}$$

The symplectic matrix $\mathbf{\Lambda}(t)$ obeys the condition $\mathbf{\Lambda}^{-1}=\mathbf{\Sigma}_2\mathbf{\Lambda}^T\mathbf{\Sigma}_2$ to preserve the commutation relations $[\hat{\mathcal{I}}_a,\hat{\mathcal{I}}_b]=[\hat{Q}_a,\hat{Q}_b]=\Sigma_{2_{ab}}$, $a,b=1,\ldots,4$, with the matrix $\mathbf{\Sigma}_2$ being the block Pauli matrix $\mathbf{\Sigma}_2=\sigma_2\otimes\mathbf{I}_2$ (the symbol $\otimes$ denotes the tensor product of the matrices). According to the Stone–von Neumann theorem, the quantum integrals of the motion $\hat{\mathbf{p}}_0$, $\hat{\mathbf{q}}_0$ form a complete set [24,26]. The quantum averages $\langle\hat{\mathbf{p}}\rangle$ and $\langle\hat{\mathbf{q}}\rangle$ develop with time according to the equation

$$\langle\hat{\mathbf{Q}}\rangle=\mathbf{\Lambda}^{-1}\langle\hat{\mathbf{Q}}\rangle_{t=0} \tag{4}$$

[see Eq. (3), and note that $\langle\hat{\mathcal{I}}(t)\rangle=\langle\hat{\mathbf{Q}}\rangle_{t=0}$]. As is known, the quantum average values follow classical paths. Hence, the reciprocal matrix $\mathbf{\Lambda}^{-1}$ describes the evolution of the classical two-dimensional oscillator in a phase space $(\mathbf{p},\mathbf{q})$ (here $\mathbf{p}$ and $\mathbf{q}$ are the classical canonical momenta and coordinates). Equation (2) with the initial condition $\hat{\mathcal{I}}(0)=\hat{\mathbf{Q}}$ yields the linear matrix equation

$$\partial\mathbf{\Lambda}/\partial t=i\mathbf{\Lambda}\mathbf{\Sigma}_2\mathbf{B},\quad \mathbf{\Lambda}(0)=\mathbf{I}_4, \tag{5}$$

for the matrix $\mathbf{\Lambda}(t)$ of the canonical transform (3).

We specify the coupling function so that $\int_{-\infty}^{+\infty}\lambda(t)dt=\lambda_0<+\infty$ and assume, first, that $\lambda(t)>0$ during a time interval that is much less than a nominal unit of time $(\omega_1\omega_2)^{-1/2}$. As a limiting case, the δ-function coupling $\lambda(t)=\lambda_0\delta(t)$ [$\delta(t)$ is the Dirac delta function] is introduced. Then, the solution to Eq. (5) reads

$$\mathbf{\Lambda}(t\geqslant+0)=\begin{pmatrix}\cos(\omega t) & -\lambda_0\omega\sin\left(\frac{t}{\omega}\right) & \omega\sin(\omega t) & \lambda_0\cos\left(\frac{t}{\omega}\right)\\ -\frac{\lambda_0}{\omega}\sin(\omega t) & \cos\left(\frac{t}{\omega}\right) & \lambda_0\cos(\omega t) & \frac{1}{\omega}\sin\left(\frac{t}{\omega}\right)\\ -\frac{1}{\omega}\sin(\omega t) & 0 & \cos(\omega t) & 0\\ 0 & -\omega\sin\left(\frac{t}{\omega}\right) & 0 & \cos\left(\frac{t}{\omega}\right)\end{pmatrix}. \tag{6}$$

For the noninfinitesimal duration of the mode coupling, Eq. (5) can be solved numerically for a certain $\lambda(t)$. We consider Gaussian, triangular, and trapezoidal coupling functions that may exceed unity. Numerical solutions to Eq. (5) for such $\lambda(t)$ have a tendency to grow while $\lambda(t)>1$ (strong coupling). Note that the strongly coupled oscillator has a nonpositive definite potential energy $\frac{1}{2}\mathbf{q}^T\mathbf{B}_2(t)\mathbf{q}$. To obtain an appropriate analytic description of the instability, we introduce the piecewise constant coupling function $\lambda(0<t\leqslant t_0)=\omega_0^2$, $\lambda(t<0,t>t_0)=0$ (step-function coupling). The solution to Eq. (5) reads

$$\mathbf{\Lambda}(t\leqslant t_0)=\mathbf{T}_+\otimes\mathbf{\Xi}_+-\mathbf{T}_-\otimes\mathbf{\Xi}_-\,, \tag{7}$$

where the auxiliary matrices

$$\mathbf{\Xi}_\pm=\frac{1}{a_+-a_-}\begin{pmatrix} a_\pm & 1\\ 1 & -a_\mp\end{pmatrix},$$

$$\mathbf{T}_\pm=\begin{pmatrix}\cos(\Omega_\pm t) & \Omega_\pm\,\sin(\Omega_\pm t)\\ -\Omega_\pm^{-1}\,\sin(\Omega_\pm t) & \cos(\Omega_\pm t)\end{pmatrix}$$

are expressed in terms of the parameters $\omega_\pm^2=\omega^2\pm\omega^{-2}$, $\Omega_\pm^2=\frac{1}{2}[\omega_+^2\pm(\omega_-^4+4\omega_0^4)^{1/2}]$, $a_\pm=(\Omega_\pm^2-\omega^{-2})/\omega_0^2$, $a_+a_-=-1$. The matrices $\mathbf{\Xi}_\pm$ satisfy the relations $\mathbf{\Xi}_\pm\mathbf{B}_2=\Omega_\pm^2\mathbf{\Xi}_\pm$ and have the properties $\mathbf{\Xi}_\pm^2=\pm\mathbf{\Xi}_\pm$, and $\mathbf{\Xi}_+\mathbf{\Xi}_-=\mathbf{0}$. The complex numbers $\pm i\Omega_-$ and $\pm i\Omega_+$ are the characteristic values of the two-dimensional classical oscillator. As the Hamiltonian (1) is independent of time inside the interval $0<t\leqslant t_0$, the positive definite potential energy of the system is the necessary and sufficient condition for the stability of the motion. Under the conditions of strong coupling, $\omega_0>1$, the potential energy has a saddle point at $\mathbf{q}=\mathbf{0}$ rather than a global minimum, and Ω_- is the imaginary number. Therefore the classical system executes the unstable motion till $t=t_0$. Under the conditions of weak coupling, $\omega_0\leqslant 1$, the values $\Omega_\pm$ are both real, the elements of the matrix $\mathbf{\Lambda}$ are bounded, and the classical system reveals stable behavior.

The quantum integrals of the motion, which are linear in the ladder operators $\hat{a}_j$, $\hat{a}_j^\dagger$, are constructed via a homogeneous canonical transformation

$$\begin{pmatrix}\hat{\mathbf{b}}\\ \hat{\mathbf{b}}^\dagger\end{pmatrix}=\mathbf{\Omega}(t)\begin{pmatrix}\hat{\mathbf{a}}\\ \hat{\mathbf{a}}^\dagger\end{pmatrix},\quad \mathbf{\Omega}=\begin{pmatrix}\zeta & \eta\\ \eta^* & \zeta^*\end{pmatrix}. \tag{8}$$

The 4×4 symplectic matrix $\mathbf{\Omega}$ consists of 2×2 complex blocks ζ, η. These blocks are used in Sec. V. The relation $\mathbf{\Omega}=\mathbf{K}^{-1}\mathbf{\Lambda}\mathbf{K}$ connects the symplectic matrices $\mathbf{\Omega}$ and $\mathbf{\Lambda}$, where

$$\mathbf{K}=\frac{1}{\sqrt{2}}\begin{pmatrix}-i\mathbf{E}_\omega & i\mathbf{E}_\omega\\ \mathbf{E}_\omega^{-1} & \mathbf{E}_\omega^{-1}\end{pmatrix},\quad \mathbf{E}_\omega=\begin{pmatrix}\omega^{1/2} & 0\\ 0 & \omega^{-1/2}\end{pmatrix}.$$

The blocks ζ, η, which correspond to the δ-function coupling, are

$$\zeta=\begin{pmatrix} e^{i\omega t} & i\frac{\lambda_0}{2}e^{it/\omega}\\ i\frac{\lambda_0}{2}e^{i\omega t} & e^{it/\omega}\end{pmatrix},\quad \eta=i\frac{\lambda_0}{2}\begin{pmatrix}0 & e^{-it/\omega}\\ e^{-i\omega t} & 0\end{pmatrix}.$$

The expressions for the blocks ζ, η obtained for the step-function coupling are rather cumbersome and are not introduced here.

III. VARIANCES OF QUADRATURE COMPONENTS

In the present section we calculate the variances of the quadrature components $M_{ab}=\frac{1}{2}\langle\hat{Q}_a\hat{Q}_b+\hat{Q}_b\hat{Q}_a\rangle-\langle\hat{Q}_a\rangle\langle\hat{Q}_b\rangle$, $a,b=1,\ldots,4$, which form the 4×4 real symmetric dispersion matrix

$$\mathbf{M}=\begin{pmatrix}\mathbf{M_{pp}} & \mathbf{M_{qp}^T}\\ \mathbf{M_{qp}} & \mathbf{M_{qq}}\end{pmatrix}. \tag{9}$$

The variances obey certain restrictions, which are none other than generalized uncertainty relations [22,23]. Making the canonical replacement (3) in Eq. (9) yields the expression

$$\mathbf{M}(t)=\mathbf{\Lambda}^{-1}\mathbf{M}(t=0)(\mathbf{\Lambda}^{-1})^T, \tag{10}$$

which describes the evolution of the initial dispersion matrix $\mathbf{M}(t=0)$. We specify an initial state with zero quadrature correlations that possesses a diagonal dispersion matrix:

$$\mathbf{M}(t=0)=\mathrm{diag}(\mathbf{E_p},\mathbf{E_q}),$$
$$\mathbf{E_p}=\mathrm{diag}(\Delta_{p_1},\Delta_{p_2}),\quad \mathbf{E_q}=\mathrm{diag}(\Delta_{q_1},\Delta_{q_2}). \tag{11}$$

The quantities Δ_{p_j} and Δ_{q_j} are the initial dispersions (or mean square deviations) of the quadrature components. The operators of the quadrature components cease to be statistically independent for $t>0$, and nonzero covariances appear in the dispersion matrix $\mathbf{M}(t>0)$. We introduce the correlation coefficients $\Gamma_{q_kp_j}=M_{q_kp_j}/(M_{q_kq_k}M_{p_jp_j})^{1/2}$, $k,j=1,2$, which describe the statistical correlation between the quadrature components $\hat{p}_j$ and $\hat{q}_k$. Also, quadrature squeezing may appear provided that at least one among the dispersions $M_{p_jp_j}$, $M_{q_jq_j}$ turns out to be smaller than a corresponding dispersion $\Delta_{p_j}^{\mathrm{coh}}=\omega^{\delta_j}/2$, $\Delta_{q_j}^{\mathrm{coh}}=1/(2\omega^{\delta_j})$ in a coherent state $|\alpha_j\rangle$. (The coherent state $|\alpha_j\rangle$ is the eigenvector of the photon annihilation operator $\hat{a}_j$: $\hat{a}_j|\alpha_j\rangle=\alpha_j|\alpha_j\rangle$, the eigenvalues α_j covering the entire complex plane.) We define the squeezing coefficients as $K_{p_j}^2=M_{p_jp_j}/\Delta_{p_j}^{\mathrm{coh}}$ and $K_{q_j}^2=M_{q_jq_j}/\Delta_{q_j}^{\mathrm{coh}}$, their classical values being restricted from below by unity.

Under the conditions of the δ-function mode coupling, the matrix $\mathbf{\Lambda}(t)$ of the canonical transformation is given by Eq. (6), and the dispersion matrix $\mathbf{M}(t\geqslant+0)$ consists of elements

$$M_{p_jp_j}=(\Delta_{p_j}+\Delta_{q_s}\lambda_0^2)\cos^2(\omega^{\delta_j}t)+\Delta_{q_j}\omega^{2\delta_j}\,\sin^2(\omega^{\delta_j}t), \tag{12a}$$

$$M_{p_1p_2}=\lambda_0\sum_{j=1}^{2}\omega^{\delta_j}\Delta_{q_j}\sin(\omega^{\delta_j}t)\cos(\omega^{-\delta_j}t), \quad (12b)$$

$$M_{q_jq_j}=(\Delta_{p_j}+\Delta_{q_s}\lambda_0^2)\omega^{-2\delta_j}\sin^2(\omega^{\delta_j}t)+\Delta_{q_j}\cos^2(\omega^{\delta_j}t), \quad (12c)$$

$$M_{q_1q_2}=-\lambda_0\sum_{j=1}^{2}\omega^{\delta_j}\Delta_{q_j}\sin(\omega^{-\delta_j}t)\cos(\omega^{\delta_j}t), \quad (12d)$$

$$M_{q_jp_j}=\frac{1}{2}\sin(2\omega^{\delta_j}t)[\omega^{-\delta_j}(\Delta_{p_j}+\lambda_0^2\Delta_{q_s})-\omega^{\delta_j}\Delta_{q_j}], \quad (12e)$$

$$M_{q_kp_j}=-\lambda_0[\Delta_{q_k}\cos(\omega^{\delta_k}t)\cos(\omega^{\delta_j}t) -\Delta_{q_j}\omega^{2\delta_j}\sin(\omega^{\delta_k}t)\sin(\omega^{\delta_j}t)], \quad k\neq j, \quad (12f)$$

where $s(j)=2/j$. If both $K_{p_j}^2(t=0)$ and $K_{q_j}^2(t=0)$ are not less than unity, squeezing is not the case [see Eqs. (12a), (12c)].

Under the conditions of step-function mode coupling, the symplectic matrix $\mathbf{\Lambda}(t)$ is determined by Eq. (7), and Eq. (10) yields the dispersion matrix $\mathbf{M}(0\leqslant t\leqslant t_0)$, which consists of the 2×2 blocks

$$\mathbf{M_{pp}}=\sum_{\sigma=\pm}\left(\frac{\sin^2(\Omega_\sigma t)}{\Omega_\sigma^{-2}}\mathbf{A_{q}}_\sigma+\cos^2(\Omega_\sigma t)\mathbf{A_{p}}_\sigma\right) -\left(\frac{\sin(\Omega_+t)\sin(\Omega_-t)}{\Omega_+^{-1}\Omega_-^{-1}}+\xi\cos(\Omega_+t)\cos(\Omega_-t)\right) \times(\mathbf{B_q}+\mathbf{B_q}^T), \quad (13a)$$

$$\mathbf{M_{qq}}=\sum_{\sigma=\pm}\left(\frac{\sin^2(\Omega_\sigma t)}{\Omega_\sigma^{2}}\mathbf{A_{p}}_\sigma+\cos^2(\Omega_\sigma t)\mathbf{A_{q}}_\sigma\right) -\left(\cos(\Omega_+t)\cos(\Omega_-t)+\xi\frac{\sin(\Omega_+t)\sin(\Omega_-t)}{\Omega_+\Omega_-}\right) \times(\mathbf{B_q}+\mathbf{B_q}^T), \quad (13b)$$

$$\mathbf{M_{qp}}=\sum_{\sigma=\pm}\frac{1}{2}\sin(2\Omega_\sigma t)\left(\frac{\mathbf{A_{p}}_\sigma}{\Omega_\sigma}+\frac{\mathbf{A_{q}}_\sigma}{\Omega_\sigma^{-1}}\right) +\left(\frac{\sin(\Omega_-t)\cos(\Omega_+t)}{\Omega_-^{-1}}+\xi\frac{\sin(\Omega_+t)\cos(\Omega_-t)}{\Omega_+}\right)\mathbf{B_q} +\left(\xi\frac{\sin(\Omega_-t)\cos(\Omega_+t)}{\Omega_-}+\frac{\sin(\Omega_+t)\cos(\Omega_-t)}{\Omega_+^{-1}}\right)\mathbf{B_q}^T, \quad (13c)$$

with $\xi=(\Delta_{p_1}-\Delta_{p_2})/(\Delta_{q_1}-\Delta_{q_2})$; the auxiliary 2×2 matrices are $\mathbf{A_{q(p)}}_\pm=\mathbf{\Xi}_\pm\mathbf{E_{q(p)}}\mathbf{\Xi}_\pm$, $\mathbf{B_{q(p)}}=\mathbf{\Xi}_+\mathbf{E_{q(p)}}\mathbf{\Xi}_-=-\mathbf{B_{p(q)}}$. For $\omega_0>1$ and $|\Omega_-|^{-1}\ll t<t_0$, the variances (13) reveal asymptotic behavior

$$\mathbf{M_{pp}}\sim\frac{1}{4}\exp(2|\Omega_-|t)(|\Omega_-|^2\mathbf{A_{q-}}+\mathbf{A_{p-}}),$$

$$\mathbf{M_{qq}}\sim\frac{1}{4}\exp(2|\Omega_-|t)(|\Omega_-|^{-2}\mathbf{A_{p-}}+\mathbf{A_{q-}}),$$

$$\mathbf{M_{qp}}\sim\frac{1}{4}\exp(2|\Omega_-|t)(|\Omega_-|^{-1}\mathbf{A_{p-}}-|\Omega_-|\mathbf{A_{q-}}).$$

Squeezing the quantum fluctuations is impossible under such conditions. The 2×2 matrix of the correlation coefficients $\Gamma_{q_kp_j}(t\leqslant t_0)$ is asymptotically equal to

$$\mathbf{\Gamma}\sim\mathcal{C}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},$$

$$\mathcal{C}=\frac{\Delta_{p_1}-|\Omega_-|^2\Delta_{q_1}+a_+^2(\Delta_{p_2}-|\Omega_-|^2\Delta_{q_2})}{\Delta_{p_1}+|\Omega_-|^2\Delta_{q_1}+a_+^2(\Delta_{p_2}+|\Omega_-|^2\Delta_{q_2})}.$$

When the condition of weak coupling is fulfilled, the eigenvalue Ω_- is a real number, and the values of the quadrature dispersions are bounded. Then, the two-mode quadrature squeezing can occur. We treat below the thermal state of the coupled modes, which reproduces the temporal evolution of the original state with a density operator

$$\hat\rho(t=0)=\hat\rho_1\hat\rho_2, \quad \hat\rho_j=2\sinh(\beta\omega^{\delta_j}/2)\exp[-\beta\omega^{\delta_j}(\hat a_j^\dagger\hat a_j+1/2)], \quad (14)$$

β^{-1} means the temperature in units $\hbar(\omega_1\omega_2)^{1/2}$. The initial quadrature dispersions are

$$\Delta_{p_j}=(\omega^{\delta_j}/2)\coth(\beta\omega^{\delta_j}/2), \quad \Delta_{q_j}=\Delta_{p_j}/\omega^{2\delta_j}.$$

The covariances of the quadrature components in the state (14) are equal to zero. We set $\omega=2$ and examine the dependence of squeezing on temperature β^{-1} and coupling frequency ω_0 and duration t_0. The contour plot in Fig. 1(a) shows $K_{p_2}^2$ for $\beta=1.2$ versus ω_0 and t_0. Squeezing ($K_{p_2}^2<1$) appears near the threshold of strong coupling, i.e., in the neighborhood of $\omega_0\approx0.95$, for $t_0\approx5.7$ and $t_0\approx8$. To find out the temperature effect on squeezing, we calculate $K_{p_2}^2$ for $\omega_0=0.92$ as a function of β and t_0 [its contour plot is seen in Fig. 1(b)]. It is clear that the quadrature fluctuations grow while the temperature increases, and the squeezing disappears for $\beta<1$. Thus the thermal noise damps squeezing when the temperature exceeds $\hbar(\omega_1\omega_2)^{1/2}$. We set $\omega_0=0.92$ and calculate $K_{p_2}^2$ in the time domain $0\leqslant t_0\leqslant3\pi$ up to very low (almost zero) temperature (viz., $\beta^{-1}=10^{-3}$). For vanishing temperature, the thermal state turns into a correlated vacuum state (CVS) $|\mathbf{0},t\rangle$, which is a common eigenvector of the formal operators $\hat b_1^\dagger\hat b_1$, $\hat b_2^\dagger\hat b_2$ of the number of photons: $\hat b_j^\dagger\hat b_j|\mathbf{0},t\rangle=0$. The dispersion matrix of the CVS is the same as in the correlated coherent state $|\boldsymbol{\alpha},t\rangle$, which is a common eigenvector of the invariant operators $\hat{\mathbf{b}}(t)$: $\hat b_j|\boldsymbol{\alpha},t\rangle=\alpha_j|\boldsymbol{\alpha},t\rangle$, $|\boldsymbol{\alpha},0\rangle=|\alpha_1\rangle|\alpha_2\rangle$. The squeezing coefficient $K_{p_2}^2$ in the thermal state decreases monotonically while

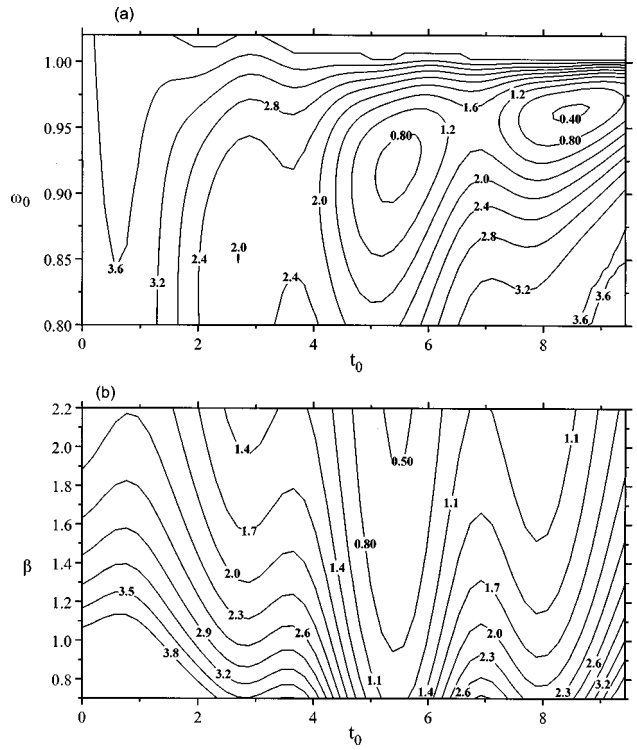

FIG. 1. Contour plot of $K^2_{p_2}(\omega_0,t_0,\beta)$ for $\omega=2$: (a) $\beta=1.2$, (b) $\omega_0=0.92$. Squeezing ($K^2_{p_2}<1$) is seen for $t_0\approx5.7$ and $t_0\approx8$ (a) near $\omega_0=0.95$, (b) for $\beta>1$.

the temperature tends to zero, and, therefore, the squeezing in the CCS is the case. However, in the given range of parameters, the minimum value of $K^2_{p_2}$ is not less than 0.4.

Photon statistics in both the thermal state and the CCS is the subject of the foregoing sections. The statistical moments of the photon distribution function can be expressed in terms of the quadrature means and variances. Note that a non-negative integer n_j is an eigenvalue of the operator $\hat{a}_j^\dagger\hat{a}_j=\frac{1}{2}(\omega^{\delta_j}\hat{q}_j^2+\omega^{-\delta_j}\hat{p}_j^2-1)$. Hence, for the first-order averages, we get

$$\langle n_j\rangle=\frac{1}{2}\left(\omega^{\delta_j}(M_{q_jq_j}+\langle\hat{q}_j\rangle^2)+\frac{1}{\omega^{\delta_j}}(M_{p_jp_j}+\langle\hat{p}_j\rangle^2)-1\right). \tag{15}$$

The mean numbers of photons are independent of the quadrature correlation coefficients. (But the width of the photon distribution does depend on the correlation between the quadratures [30].) For $\omega_0>1$ and $|\Omega_-|^{-1}\ll t_0$, the average values (15) are exponentially large:

$$\begin{aligned}\langle n_1(t_0)\rangle &\sim\frac{\exp(2|\Omega_-|t_0)}{8\omega(a_+-a_-)^2}\left(1+\frac{\omega^2}{|\Omega_-|^2}\right)[|\Omega_-|^2(a_-^2\Delta_{q_1}+\Delta_{q_2})\\&\quad+a_-^2\Delta_{p_1}+\Delta_{p_2}+[|\Omega_-|(a_-\langle\hat{q}_1\rangle_{t=0}+\langle\hat{q}_2\rangle_{t=0})\\&\quad+a_-\langle\hat{p}_1\rangle_{t=0}+\langle\hat{p}_2\rangle_{t=0}]^2],\end{aligned}$$

and

$$\langle n_1(t_0)\rangle\sim\langle n_2(t_0)\rangle\frac{a_-^2(\omega^2+|\Omega_-|^2)}{1+\omega^2|\Omega_-|^2}. \tag{16}$$

When ω_0^2 is very large in comparison with $\omega^{\pm2}$, the values a_-^2 and $|\Omega_-|^2$ are close to unity and ω_0^2, respectively. Then $\langle n_1(t_0)\rangle/\langle n_2(t_0)\rangle\sim1/\omega^2$, and the larger mean photon number corresponds to the lower-frequency mode. In order to examine that asymmetry of the two-mode photon distribution we set $\omega=2$ for all the calculations of Sec. IV and Sec. V.

IV. PHOTON STATISTICS OF COUPLED MODES IN THERMAL EQUILIBRIUM

The two-mode photon distribution function is the probability to have n_1 photons in the first mode and n_2 photons in the second mode. The desired probability is a diagonal element $F(\mathbf{n},t)=\langle\mathbf{n}|\hat{\rho}|\mathbf{n}\rangle$ of the density operator of the system under consideration; $|\mathbf{n}\rangle=|n_1\rangle|n_2\rangle$, where $|n_j\rangle$ is an eigenvector of the operator $\hat{a}_j^\dagger\hat{a}_j$: $\hat{a}_j^\dagger\hat{a}_j|n_j\rangle=n_j|n_j\rangle$. According to Ref. [30], the PDF in the thermal state is expressed in terms of a ''diagonal'' Hermite polynomial [29] of four variables:

$$F(\mathbf{n},t)=\left[\det\left(\mathbf{M}+\frac{1}{2}\mathbf{I}_4\right)\right]^{-1/2}\frac{H_{\mathbf{nn}}^{\{\mathbf{R}\}}(\mathbf{0})}{\mathbf{n}!}, \tag{17}$$

where $\mathbf{n}!=n_1!n_2!$, and $\mathbf{n}$ denotes a pair of non-negative integers. The matrix $\mathbf{R}=\mathbf{U}^\dagger(\mathbf{I}_4-2\mathbf{M})(\mathbf{I}_4+2\mathbf{M})^{-1}\mathbf{U}^*$ is defined in terms of a unitary matrix

$$\mathbf{U}=\begin{pmatrix}-i\mathbf{I}_2 & i\mathbf{I}_2\\ \mathbf{I}_2 & \mathbf{I}_2\end{pmatrix}$$

satisfying the relations $\mathbf{U}^\dagger\mathbf{U}=\mathbf{U}^T\mathbf{U}^*=\mathbf{I}_4$, $\mathbf{U}^\dagger\mathbf{U}^*=\mathbf{U}^T\mathbf{U}=\mathbf{\Sigma}_1$; $\mathbf{\Sigma}_1=\sigma_1\otimes\mathbf{I}_2$ is the four-dimensional block Pauli matrix.

The examples of the distribution (17) presented below are obtained numerically using the known recurrence relation for Hermite polynomials of N variables ($N=4$ in our case)

$$\begin{aligned}H_{n_1,\ldots,n_j+1,\ldots,n_N}^{\{\mathbf{R}\}}(\mathbf{y})&=\left(\sum_{k=1}^N R_{jk}y_k\right)H_{n_1,\ldots,n_j,\ldots,n_N}^{\{\mathbf{R}\}}(\mathbf{y})\\&\quad-\sum_{k=1}^N R_{jk}n_kH_{n_1,\ldots,n_k-1,\ldots,n_N}^{\{\mathbf{R}\}}(\mathbf{y}),\end{aligned}$$

$$H_{\mathbf{0}}^{\{\mathbf{R}\}}=1,\quad H_{0,\ldots,1_j,\ldots,0}^{\{\mathbf{R}\}}(\mathbf{y})=\sum_{k=1}^N R_{jk}y_k.$$

The photon distribution functions displayed in Fig. 2 are calculated in the regimes of both strong step-function coupling, $\omega_0=1.3$, $t_0=3\pi/4$ [Figs. 2(a) and 2(b)], and δ-function coupling, $\lambda_0=4$ [Figs. 2(c) and 2(d)]. We analyze the behavior of the PDFs at different values of the parameter β. We present the plots of the photon distributions for very low temperature, $\beta^{-1}=10^{-3}$ [Figs. 2(a) and 2(c)] and for larger temperature, $\beta^{-1}=1.25$ [Figs. 2(b) and 2(d)]. As is pointed out in Sec. III, the thermal state at low temperature is close to the CVS having the PDF vanishing for odd sums n_1+n_2 [14,30], and the behavior of the PDFs depicted in Figs. 2(a) and 2(c) shows that those almost vanish for odd sums n_1

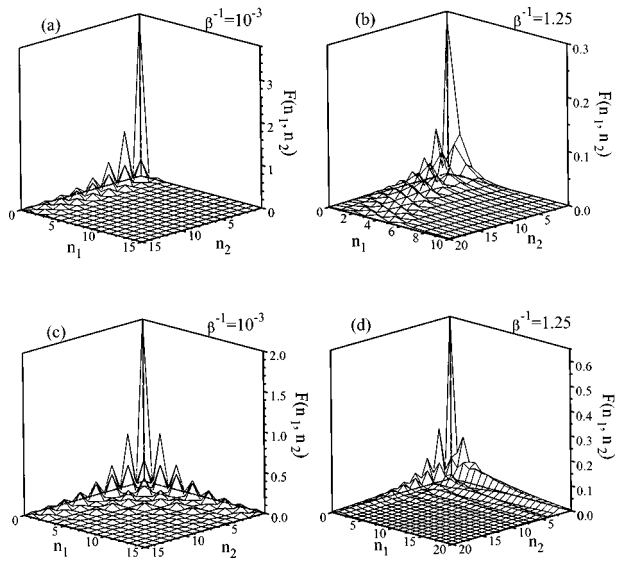

FIG. 2. Photon distribution function $F(n_1,n_2)$ in the thermal state for $\omega=2$: (a), (b) $\omega_0=1.3$, $t_0=3\pi/4$ (strong step-function coupling), and (c), (d) $\lambda_0=4$ (the δ-function coupling). In (a) and (c) $\beta^{-1}=10^{-3}$, in (b) and (d) $\beta^{-1}=1.25$. Figure shows $F(n_1,n_2)$ in units 10^{-1}.

$+n_2$. The distribution functions in Fig. 2 exhibit an oscillatory behavior, which is less pronounced for higher temperature. The suppression of the oscillations is demonstrated in Fig. 3, where the cut $F(0,n_2)$ is plotted as a function of β for $\lambda_0=4$. The probability to have $(0,n_2)$ photons at high temperature decreases proportionally to β. This is reminiscent of the photon statistics of the squeezed correlated mixed state of a single-mode system [31].

One should note the essential difference of plots in Figs. 2(a) and 2(b) from Figs. 2(c) and 2(d). The distributions of photons in the strongly coupled modes have a pronounced asymmetric character, the maxima of the distributions being concentrated along the n_2 axis, which corresponds to the lower-frequency mode [Eq. (16) gives $\langle n_1(t_0)\rangle/\langle n_2(t_0)\rangle \approx 1/4$]. The excitation of the relatively large photon numbers in the ''red'' mode occurs exclusively under the conditions

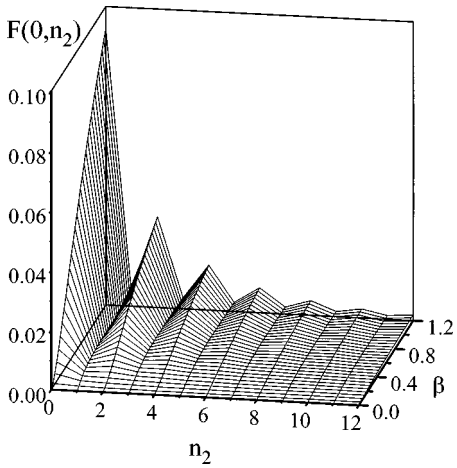

FIG. 3. The cut $F(0,n_2)$ for $\lambda_0=4$ as a function of β. The probability of counting $(0,n_2)$ photons decreases proportionally to β.

of strong coupling, and this anomaly of photon statistics is not observed under the δ-function coupling conditions, as is clear in Figs. 2(c) and 2(d). We describe the features of the PDFs in Figs. 2(c) and 2(d) using the deformed Planck distribution formula obtained from Eq. (15),

$$\langle n_j(t\geqslant+0)\rangle=\overline{n_j}+\left(\frac{\lambda_0}{2}\right)^2\coth\left(\frac{\beta}{2\omega^{\delta_j}}\right), \tag{18}$$

where $\overline{n_j}=[\exp(\beta\omega^{\delta_j})-1]^{-1}$ is the usual Planck distribution formula corresponding to the initial state (14). When the temperature tends to zero ($\beta\to\infty$), and $\lambda_0\sim O(1)$, one has $\langle n_j(t\geqslant+0)\rangle\sim(\lambda_0/2)^2$. This is the case for the parameters of Fig. 2(c) displaying the PDF symmetric with respect to the main diagonal $n_1=n_2$. For larger temperature, say, $\beta\sim O(1)$, the symmetry of the distribution is broken, as is seen in Fig. 2(d). [The deformed Planck formula (18) gives $\langle n_1\rangle\approx 2.5\langle n_2\rangle$.] Therefore, unlike the strong step-function mode coupling, the δ-function coupling excites the larger number of photons in the higher-frequency mode. To examine this tendency, we proceed to higher temperatures, $\beta\ll 1$, and retain ω of the order of unity. Then, Eq. (18) yields

$$\langle n_j(t\geqslant+0)\rangle\sim\frac{\omega^{-\delta_j}}{\beta}\left(1+\frac{\omega^{2\delta_j}\lambda_0^2}{2}\right)\gg 1. \tag{19}$$

When $\lambda_0\gg 1$, the ratio of the mean numbers (19) becomes

$$\frac{\langle n_1(t\geqslant+0)\rangle}{\langle n_2(t\geqslant+0)\rangle}\sim\omega^2\left[1-\frac{2}{\lambda_0^2}\left(\omega^2-\frac{1}{\omega^2}\right)\right],$$

which, for $\lambda_0=\infty$, is nothing but the inverse of the ratio $\langle n_1\rangle/\langle n_2\rangle=1/\omega^2$ corresponding to the unperturbed initial state (14).

V. PHOTON STATISTICS OF CORRELATED COHERENT STATE OF COUPLED MODES

In the present section the distribution of the numbers of photons in the CCS $|\boldsymbol{\alpha},t\rangle$ (see Sec. III) is investigated. The two-mode PDF, which follows from a generic polymode expression [23,30], reads

$$F(\mathbf{n},t)=\frac{F(\mathbf{0},t)}{\mathbf{n}!}|H_{\mathbf{n}}^{\{\zeta^{-1}\eta\}}(\eta^{-1}\boldsymbol{\alpha})|^2,$$
$$F(\mathbf{0},t)=\frac{\exp(-|\boldsymbol{\alpha}|^2)}{|\det\zeta|}\left|\exp\left(\frac{1}{2}\boldsymbol{\alpha}^T\eta^*\zeta^{-1}\boldsymbol{\alpha}\right)\right|^2, \tag{20}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)^T$ denotes a two-dimensional column vector with complex entries $\alpha_j=|\alpha_j|e^{i\theta_j}$, and $\mathbf{n}$ is a discrete vector variable; the 2×2 complex matrices ζ and η are the blocks of the symplectic matrix $\boldsymbol{\Omega}$ [see Eq. (8)]. At the initial time $t=0$, the PDF (20) is a usual two-dimensional Poissonian distribution with both means and dispersions of the photon numbers equal to $|\alpha_j|^2$. For $t>0$, the phases θ_j become additional parameters, on which the distribution of the photon numbers depends. The PDF (20) can be expanded into a finite sum of classical (single-variable) Hermite polynomials [32], and it can be shown that under the conditions of δ-function mode coupling the distribution (20) is not depen-

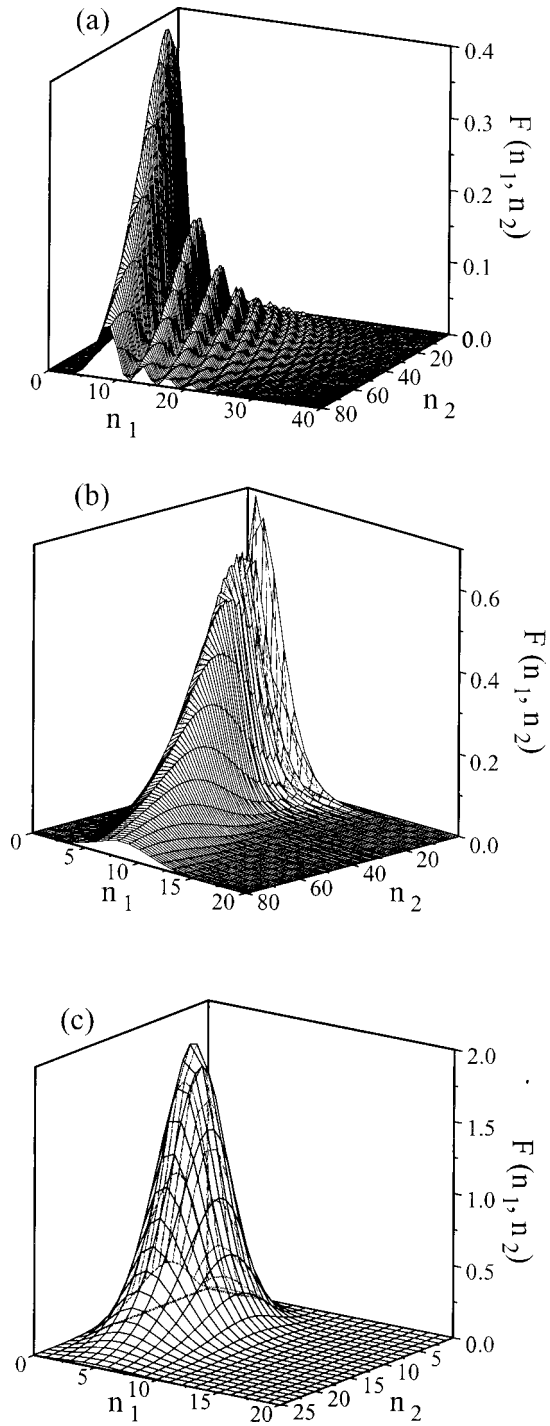

FIG. 4. Photon distribution function $F(n_1,n_2)$ in the correlated coherent state: (a) $\lambda_0=4$ (δ-function coupling), (b) $\omega_0=1.3$, $t_0=3\pi/4$ (strong step-function coupling). In (c), we assume the same parameters as in (b), except that $\omega_0=0.9$ (weak coupling) instead of $\omega_0=1.3$. Figure shows $F(n_1,n_2)$ in units 10^{-2}, and the values of parameters are $\omega=2$, $\alpha_1=2$, $\alpha_2=2i$.

dent on ω. Thus the effect of the phases θ_j is the most clear under the conditions of δ-function mode coupling.

We consider below the distributions (20) computed for $\alpha_1=2=-i\alpha_2$. The plot in Fig. 4(a) corresponds to the conditions of δ-function mode coupling, $\lambda_0=4$. The PDF in Fig. 4(a) undergoes strong oscillations and has a pronounced asymmetric character. For $t\geqslant+0$, the mean numbers of photons are

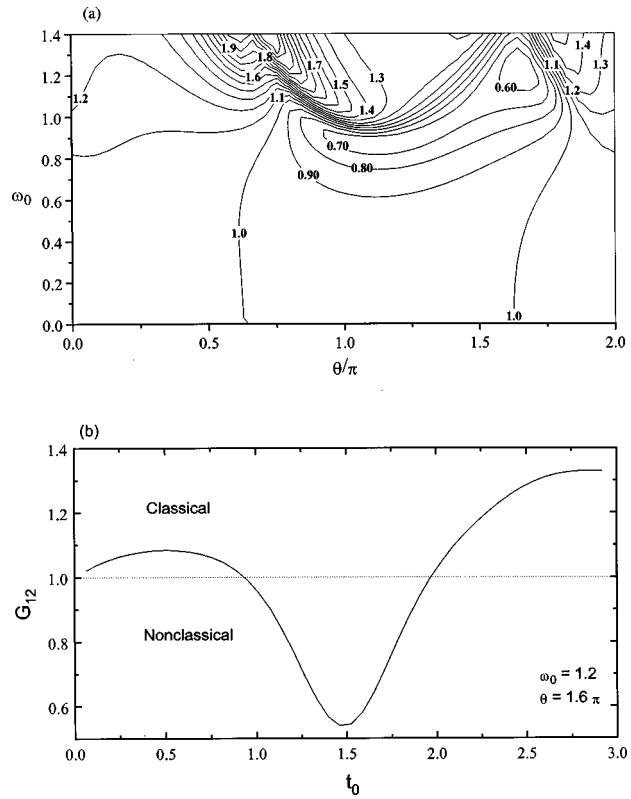

FIG. 5. The Glauber coherence function $G_{12}^{(2)}$ in a correlated coherent state with $\alpha_1=2$, $\alpha_2=\alpha_1 e^{i\theta}$ for $\omega=2$. In (a), the contour plot of $G_{12}^{(2)}$ versus ω_0 and θ is shown for fixed $t_0=3\pi/2$. The nonclassical behavior is seen for $\omega_0\approx1$ and $\theta\approx1.7\pi$. In (b), $G_{12}^{(2)}$ is shown as a function of t_0 for fixed $\omega_0=1.2$ and $\theta=1.6\pi$.

$$\langle n_j\rangle=\left(\frac{\lambda_0}{2}\right)^2+(\mathrm{Im}\ \alpha_j-\lambda_0\ \mathrm{Re}\ \alpha_s)^2+\mathrm{Re}^2\ \alpha_j, \qquad (21)$$

where $s(j)=2/j$, $\mathrm{Re}\ \alpha_j$ is the real part, and $\mathrm{Im}\ \alpha_j$ is the imaginary part of α_j. For the parameters of Fig. 4(a), Eq. (21) gives $\langle n_1\rangle/\langle n_2\rangle=2/11$. For $\lambda_0\gg1$, the ratio of the mean numbers is governed by the real parts of α's,

$$\frac{\langle n_1(t\geqslant+0)\rangle}{\langle n_2(t\geqslant+0)\rangle}\sim\frac{1+4\,\mathrm{Re}^2\ \alpha_2}{1+4\,\mathrm{Re}^2\ \alpha_1}\left[1+\frac{8}{\lambda_0}\left(\frac{\mathrm{Im}\ \alpha_2\ \mathrm{Re}\ \alpha_1}{1+4\,\mathrm{Re}^2\ \alpha_1}-\frac{\mathrm{Im}\ \alpha_1\ \mathrm{Re}\ \alpha_2}{1+4\,\mathrm{Re}^2\ \alpha_2}\right)\right],$$

and the larger mean number of photons corresponds to the mode with the imaginary α_j. Also, we calculate the PDF under the conditions of strong, $\omega_0=1.3$ [Fig. 4(b)], and weak, $\omega_0=0.9$ [Fig. 4(c)], step-function coupling; $t_0=3\pi/4$ is chosen. Figure 4(b) shows the distortion of the PDF at low n_1 and n_2. Unlike that, the distribution in Fig. 4(c) is a smooth function. Both the distributions are not symmetric with respect to the main diagonal $n_1=n_2$. Excitation of the large photon numbers occurs under the conditions of strong coupling, and Eq. (16) gives the ratio of the mean numbers $\langle n_1(t_0)\rangle/\langle n_2(t_0)\rangle\approx1/4$.

The coupling of the modes causes the intermode correlations of the photon numbers. To study them, we employ the Glauber coherence function

$$G_{12}^{(2)}=1+\frac{\langle n_1(t_0)n_2(t_0)\rangle-\langle n_1(t_0)\rangle\langle n_2(t_0)\rangle}{\langle n_1(t_0)\rangle\langle n_2(t_0)\rangle}.$$

Motivation for this definition can be found in Ref. [33]. The classical values of the Glauber function are restricted from below by unity. We calculate $G_{12}^{(2)}$ for $\alpha_1=2$, and $\alpha_2=\alpha_1 e^{i\theta}$. The contour plot of the Glauber function versus ω_0 and θ for fixed $t_0=3\pi/2$ is seen in Fig. 5(a). The nonclassical values $G_{12}^{(2)}<1$ are seen near the threshold of strong coupling $(\omega_0\approx 1)$ in the neighborhood of $\theta=3\pi/2$. We take the values $\theta=1.6\pi$ and $\omega_0=1.2$ corresponding to the minimum value of $G_{12}^{(2)}$ in Fig. 5(a), and calculate the coherence function for various times [see Fig. 5(b)]. Figure 5(b) demonstrates that both classical and nonclassical values can be achieved by $G_{12}^{(2)}$ for fixed ω_0 and θ in the course of the evolution of the system.

VI. CONCLUSION

We have discussed how the nonstationary mode coupling affects the quantum-statistical properties of the two-mode light. The effect of both δ- and step-function coupling between the quadrature components $\hat{q}_1$ and $\hat{q}_2$ of the light modes is investigated in detail. The conditions of weak and strong step-function coupling are distinguished. The excitation of the quantum fluctuations in the strongly coupled modes is attributed to the instability of the classical two-dimensional strongly coupled oscillator, which is the classical counterpart of the quantum system under consideration. Also, the exponentially large mean number of photons corresponds to either of the strongly coupled modes, and the mode with a lower frequency possesses the greater mean number. On the contrary, the quadrature squeezing is established in the thermal state under the conditions of weak coupling for moderate temperature; the squeezing disappears when the temperature is of order or exceeds $\hbar(\omega_1\omega_2)^{1/2}$. The two-mode PDF in the thermal state reveals oscillations, which decrease while the temperature grows. It is found that the PDF in the CCS of the coupled modes exhibits both oscillating and nonoscillating behavior depending on the concrete values of parameters. The nonclassical intermode correlations of the photon numbers are detected and briefly studied.

ACKNOWLEDGMENTS

The authors express sincere appreciation to Professor V. I. Man'ko for his kind attention and fruitful discussions that made this research possible. M.E.V. expresses his particular appreciation to ISSEP for the financial support.

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