Strongly coupled large-angle stimulated Raman scattering of short laser pulse in plasma-filled capillary

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(Received 2 April 2004; accepted 5 January 2005; published online 7 April 2005)

Strongly coupled large-angle stimulated Raman scattering (LA SRS) of a short intense laser pulse develops in a plane plasma-filled capillary differently than in a plasma with open boundaries. Coupling the laser pulse to a capillary seeds the LA SRS in the forward direction (scattering angle smaller than \( \pi/2 \)) and can thus produce a high instability level in the vicinity of the entrance plane. In addition, oblique mirror reflections off capillary walls partly suppress the lateral convection of scattered radiation and increase the growth rate of the SRS under arbitrary (not too small) angle. Hence, the saturated convective gain falls with an angle much slower than in an unbounded plasma and even for the near-forward SRS can be close to that of the direct backscatter. At a large distance, the LA SRS evolution in the interior of the capillary is dominated by quasi-one-dimensional leaky modes whose damping is related to the leakage of scattered radiation through the walls. © 2005 American Institute of Physics. [DOI: 10.1063/1.1862628]

I. INTRODUCTION

The technique of chirped-pulse amplification \(^1\) made sub-picosecond laser pulses of high power \((P > 10^{12} \text{ W})\) available for generation of coherent x-rays, \(^2\) high harmonics of radiation, \(^3\) and laser wakefield acceleration (LWFA) of electrons \(^4–6\) in rarefied plasmas \(\{\omega_0 > \omega_{pe}\}, \omega_0 \text{ is a laser frequency, } \omega_{pe} = (4 \pi e^2 n_0 / m_e)^{1/2} \text{ is an electron plasma frequency, } n_0 \text{ is a background electron density, } m_e \text{, and } [-e] \text{ are the electron mass at rest and charge}\). Full potential of these applications can be realized with the laser-plasma interaction length increased beyond the Rayleigh diffraction length, \(z_R = \pi \sigma_0^2 / \lambda_0 \), by means of external optical guiding \(^6\) (here and later, \(\lambda_0 \approx 2 \pi / \omega_0 \text{ is a laser wavelength and } \sigma_0 \text{ is a laser beam waist radius}\). One of the guiding options is using a dielectric capillary, \(^7–9\) where the oblique mirror reflections suppress the laser beam diffraction. Inside a capillary, plasma can be created by an optical field ionization of the filling gas \(^10–12\) or by a laser ablation of the walls. \(^13\) Then, the large-angle stimulated Raman scattering (LA SRS) starts to challenge transportation of a laser beam over a long distance. \(^14,15\)

In the standard SRS process, \(^16\) the pump electromagnetic wave (EMW) is scattered off spontaneous fluctuations of electron density, which, in turn, can be amplified by the ponderomotive beat wave of pump and scattered light. Appropriate phase matching of the waves results in a positive feedback loop with the onset of a spatiotemporal instability. \(^17\) When the plasma extent is much larger than a laser pulse length and no reflections off plasma boundaries occur, both scattering electron plasma waves (EPW) and scattered EMW quit the region of amplification, and the convective gain saturates within a time interval of the order of pulse duration. \(^18–21\) Given the pulse length, the maximum possible gain remains the same for all scattering angles, and whether it is achieved or not for a given angle is determined by the laser pulse aspect ratio only. \(^18–21\) Convection of scattered radiation out of the laser waist may result in a strong pulse depletion. \(^22\) Even when the full depletion does not occur, the LA SRS can produce considerable pulse erosion, \(^23\) suppression of the relativistic self-focusing, \(^24\) heating and preacceleration of plasma electrons, \(^25\) and seeding the forward SRS. \(^26,27\) Thereby, knowing the details of the LA SRS evolution in various physical conditions is a matter of high importance for applications.

Confining plasma by reflecting surfaces deeply modifies the SRS process. In the one-dimensional (1D) geometry, the Raman backscatter changes its nature from convective to absolute: \(^28\) reflections trap the unstable radiation modes inside plasma and give rise to the continuous amplification. When the laser beam is confined between the mirror-reflecting partly transparent walls and propagates collinearly to them, reflections reduce the sideward convection of scattered light. If the reflective modes dominate in plasma, the LA SRS gain tends to that of the direct backscatter and thus reveals a dramatic increase in comparison with the open-boundary system. \(^14\) The LA SRS in this geometry has been considered so far in the regime of weak coupling, \(^14\) when the scattering EPW is similar to the plasma natural mode \(^17\) and temporal growth rate is well below the electron plasma frequency. This regime requires fairly low amplitude of a laser pulse, i.e., \(\alpha_0 \ll \omega_{pe} / \omega_0 \ll 1 \) \([\alpha_0 = \varepsilon E_0 / (m_e \omega_0) \text{ is a normalized amplitude of the laser electric field}]\). However, for the efficient LWFA, \(^4–6\) the plasma density has to be reduced in order to increase the \(\gamma \) factor of the laser wakefield, \(\gamma_R = \omega_0 / \omega_{pe} \) and, hence, the electron energy gain. With \(1 / \gamma_R \).
<a>^2<1, the LA SRS becomes strongly coupled: its temporal growth rate exceeds \( \omega_{pe} \) and the scattering EPW differs from the natural mode of plasma oscillations.\(^{18,21,26,29,30}\) In a plasma-filled capillary, the strongly coupled LA SRS acquires new specific features: in a wide range of parameters relevant to the self-modulated LWFA,\(^5\) our particle-in-cell (PIC) simulations [using the code WAKE (Ref. 31)] discovered a vast enhancement of the near-forward SRS in the immediate vicinity of the capillary entrance aperture. The unstable plasma modes were primarily transverse and therefore useless for the longitudinal electron acceleration. For the same range of parameters, this effect has never been significant in an open-boundary plasma. Independent fluid modeling\(^32\) verified these observations.

Making a step to understanding this phenomenon we propose a two-dimensional (2D) linear theory of strongly coupled SRS of a short laser pulse under a given angle \( \alpha \) in a slab of rarefied plasma laterally confined between the mirror-reflecting partly transparent flat walls (flat capillary). Boundary conditions for the scattered radiation describe the oblique mirror reflections and the electromagnetic (EM) seed at the entrance plane. We associate the latter with the signal formed of the high-order capillary eigenmodes produced by the laser beam coupling to the capillary (the coupling process is described elsewhere\(^8,11\) and outlined in Appendix A). The forward (\( \alpha < \pi/2 \)) and backward (\( \alpha > \pi/2 \)) SRS proceed differently. Forward Raman amplification of the EM seed can be dominant within a finite distance from the entrance plane and be responsible for the instability enhancement observed in the modeling. On the other hand, the backward SRS is affected by the reflections only. As the LA SRS of a finite-length laser pulse preserves the convective nature (backward and, partly, sideward convection of radiation is allowed), the gain saturation occurs within a finite distance from the entrance plane. Reflections give the unstable modes additional rise time in any transverse cross section of a plasma, and, even for relatively small scattering angles (such as \( \alpha = \pi/6 \) taken for numerical examples of this paper), the saturated convective gain can approach that of the backward SRS (BSRS). The field structure is then approximated by a quasi-1D lossy mode whose damping is produced by the leakage of radiation through the walls.

The paper is organized as follows. Section II presents a theoretical model for the strongly coupled LA SRS in a 2D slab geometry. The laser pulse entrance into a plasma and oblique mirror reflections of scattered light are expressed in terms of appropriate boundary-value conditions for the coupled-mode equations. General solution of the boundary-value problem is presented (derivation is given in Appendix B). Section III discuses the spatiotemporal evolution of instability. The case of fully transparent lateral boundaries is considered in Sec. III A. Enhancement of the LA SRS in the generic reflective case is considered, and the maximum gain factors are evaluated in terms of appropriate asymptotic solutions in Sec. III B. Section IV summarizes the results.

In the rarefied plasma, both \( k_0 \) and \( k_{\pm \alpha} \) obey the same dispersion relation \( \omega_{pe} = \omega_{pe} + c^2 k^2 / \varepsilon^2 \). Hence, \( |k_{\pm \alpha}| = k = k_0, k_{\pm \alpha} = k_0 \text{cos } \alpha, \pm k_{\pm \alpha} = k_0 \text{sin } \alpha, \) and the amplitudes \( a_{\pm \alpha} \) vary slowly in time and space on the scales \( \omega_{pe}^{-1}, k_{e 1}^{-1}, \) and \( k_{e 1}^{-1}. \) Ions form a homogeneous positive background; this assumption holds for a laser pulse shorter than an ion plasma period, \( \omega_{pe}^2 \approx 2 \pi \omega_{pe}^{-1}. \) The beat wave of incident and scattered radiation excites perturbations of electron density,

\[
\frac{n_e - n_0}{n_0} = \sum_{\alpha = \pm} N_0'(r,t) e^{i[k_{\pm \alpha} \cdot r]} + c.c.,
\]

where wave vectors obey the matching conditions \( k_{\pm \alpha} = k_0 - k_{\pm \alpha}; \) hence, \( |k_{\pm \alpha}| = k = 2 k_0 \text{sin } (\alpha/2), k_{\pm } = k_0 (1 - \text{cos } \alpha), k_{\pm \alpha} = \pm k_0 \text{sin } \alpha. \) Wave vector diagram of the LA SRS is shown in Fig. 1. The longitudinal component of phase velocity of the scattering EPW is small if compared with the speed of light, i.e., \( k_v > k = \omega_{pe} / c. \) This restriction eliminates the near-forward Raman scattering,\(^{26,33}\) and resonant modulational instability (RMI).\(^{34}\) In the rarefied plasma, the amplitudes \( N_0 \) vary slowly in space on the scales \( k_{e 1}^{-1}, k_{e 2}^{-1}. \)

The amplitudes of up- and down-going scattered EMW and scattering EPW obey the coupled-mode equations:
laser pulse enters a semi-infinite plasma-filled gap between pulse leading front $z = 0$, and the walls $x = 0$ and $x = L_x$. Rear edge of the pulse $z = c(t - t_0)$ is a free boundary through which the waves quit the region of amplification. The boundary-value problem is solved in the area $c(t - t_0) < z < ct$, $0 < x < L_x$.

Figure 2 shows the interaction area. At $z = 0$, $t = 0$, the laser pulse enters a semi-infinite plasma-filled gap between flat mirror-reflecting walls $z \geq 0$, $0 \leq x \leq L_x$ and propagates towards positive $z$. The pulse leading front, $\xi = 0$, encounters the stationary level of electron density perturbations with a constant amplitude $N_0$,

$$N_p(x, z, 0) = N_0,$$

$$\partial N_p / \partial \xi (x, z, 0) = 0,$$

fluctuations of radiation in fresh plasma being neglected,

$$a_{sk}(x, z, 0) = 0.$$ (5)

At the capillary entrance plane, the transverse profile of radiation can have a significant content of the high-order capillary eigenmodes (coupling the incident laser beam to the capillary is discussed elsewhere and is outlined in Appendix A). The resonant condition for the wave vectors selects the modes that can be amplified by the forward SRS

$$a_{sk}(x, z) = a_{0k}[1 - (1 - 2x/L_x)^2].$$ (6)

Inside the capillary, oblique mirror reflections couple up-going and down-going EMW: each reflection converts an up-going wave into a down-going one and vice versa,

$$a_{sk}(0, z, \xi) = r(a)a_{sk}(0, z, \xi),$$ (7a)

$$a_{sk}(L_x, z, \xi) = r(a)a_{sk}(L_x, z, \xi).$$ (7b)

The conditions (7) set up a quasi-1D exponential behavior of waves at large $z$. The reflectivity coefficient is a known function of scattering angle, $r(a) = |\sin \alpha - [\delta_p / \delta_s]^2 - \cos^2 \alpha|^{1/2} / |\sin \alpha + [\delta_p / \delta_s]^2 - \cos^2 \alpha|^{1/2}$, where $\delta_s$ and $\delta_p$ are the refraction indexes of walls and plasma. Figure 3 shows $r(\alpha)$ for a glass capillary with $\delta_s \approx 1.5$ and $\delta_p \approx 1$.

The temporal increment of strongly coupled LA SRS exceeds the electron plasma frequency. Hence, we neglect $k^2 N_p$ in comparison with $\partial^2 N_p / \partial \xi^2$ in the left-hand side (LHS) of Eq. (3b). With the allowance for not very tight capillary, the pump field envelope $a_{0k}(x, \xi)$ represents a portion of laser radiation coupled to the capillary which experiences mostly paraxial propagation, $k_0 / k_0 \ll \omega_{pe} / \omega_0$. Assuming that the pump field evolution at a given point $(x, z)$ takes much longer than the SRS growth $(z_p/c \gg t_0)$, and in order to enable the analytic progress, we approximate $a_{0k}(x, \xi)$ with a fixed flat profile at any position $z$ in a capillary of the width $L_x$, $a_{0k}(x, \xi) = a_{0k}H(x)H(L_x - x)H(\xi)H(c t_0 - \xi)$, using the effective pulse duration $t_0$ and amplitude $a_{0k}$. Here and below, $H(\cdot)$ is the Heaviside step function. Solution of the boundary-value problem,
\[ N_i(\mathbf{R}; r) = \frac{N_0}{3} \sum_{j=1}^{3} e^{j\beta}[1 - \Phi_{1D}(\mathbf{R}; r, c_j) - \Phi_{2D}(\mathbf{R}; r, c_j)] \]

\[ + a_{d,j}(\sqrt{\pi/4})g_2(\mathbf{R}; r)(\xi - z/V_z)^2 \left( \frac{2/3}{2;i\xi} \right) \]

is then obtained via the 2D Laplace transform; here, \( \mathbf{R} = (r, \xi), c_j^2 = iG^3, \xi = (G^3/4)(\xi - z/V_z)^2/V_z \), \( g_2(h_1, h_2; i\xi) \) is the regularized generalized hypergeometric function,

\[ \Phi_{1D} = \Phi_{2D} \left( \begin{array}{c} x_n \\ H(V_z \xi - V_z x_n) \end{array} \right) \]

\[ \mathcal{F} = \left( \frac{(V_z x - V_z z)(V_z x - V_z z - V_z L_x)}{(V_z L_x/2)^2} \right) H(V_z x - V_z z) \]

\[ + \sum_{n=1}^{\infty} r^n \left( \frac{V_z x_n - V_z z - V_z L_x}{V_z L_x/2} \right) H(V_z x - V_z z) \]

\[ \Rightarrow \Phi_{1D} = 0, \quad \Phi_{2D} = 0 \] (10)

at the wall \( x=0 \) (range II),

\[ x < z/V_z \quad \Rightarrow \Phi_{1D} = 0, \quad \Phi_{2D} \neq 0 \] (11)

or at the entrance plane \( z=-0 \) (range III),

\[ x > z/V_z \quad \Rightarrow \Phi_{1D} \neq 0, \quad \Phi_{2D} = 0. \] (12)

The principal feature that makes the LA SRS of a short pulse different from the case of semi-infinite laser beam is the gain saturation within a finite distance from the entrance plane. At some point, \( z_{sat} < +\infty \), the scattered radiation arriving from the plasma boundary \( z=-0 \) drops behind the laser pulse, and in all the points \( z > z_{sat} \) neither part of the pulse belongs to the range III. Then, the evolution of waves and the gain do not alter with \( z \). The gain saturates differently for the “forward” \( (\alpha < \pi/2) \) and “backward” \( (\alpha > \pi/2) \) scattering.

When \( \alpha < \pi/2 \), and the distance from the entrance plane is not too large, i.e., \( z < \min(V_z c \tau\phi, L_x(V_z / V_y)) \), all three areas (10)–(12) are available in the pulse body (that is, within a rectangle \( 0 < \xi < c \tau\phi, 0 < x < L_x \)). Given the point \( (x, z) \), waves fall initially within a range I, where they grow in time exponentially with an angle-independent increment

\[ \gamma_0 = (\sqrt{3}/2)G \approx (\sqrt{3}/2) c^2 \Pi \omega_0 \omega \] (13)

Note that \( \kappa = \gamma_0/c \) is the known “spatial” increment of the strongly coupled BSRS in the comoving frame. The evolution of waves is strictly 1D in space on this stage. Later, information from the boundaries \( x=0, z=-0 \) reaches the point \( \Phi(\xi, \tau\phi) \), and the spatial dependence becomes either 2D for \( \xi > x/V_z, x < z/V_z \) (range II) or remains 1D for \( \xi > z/V_z, x > z/V_z \) (range III). The waves are not exponentially growing at this time. In the range III, the entrance effect dominates: vanishing the scattered EMW at the boundary \( z=-0 \) determines the behavior of 1D amplitudes. Deeply enough in plasma, \( z \approx \min(V_z c \tau\phi, L_x(V_z / V_y)) \), the entrance effect vanishes as the pulse terminates sooner than the scattered EMW from the entrance plane can reach the observer at given \( z \). The pulse body is then divided between the ranges I and II, and the evolution of LA SRS is the same through the rest of the plasma.

For \( \alpha > \pi/2 \), the boundary-value condition posed for radiation at \( z=-0 \) can only produce the scattered EMW converging outwards (the EMW characteristic \( \xi = z/V_z \) recasts in the lab frame variables \( z = -c\tau\phi \cos \theta_\alpha \) and corresponds to the wave propagating towards negative \( z \)). Thus, the entrance effect does not change the solution at a positive \( z \), and the boundary-value condition (6) becomes excessive (this is also valid in the reflective case). The spatiotemporal evolution of the LA SRS remains the same at any \( z \geq 0 \).

Given the pulse duration, the maximum of the saturated gain, \( a_{s,z} \), \( N_s \approx -e^{b_0/6} \), does not depend on the scattering angle, and whether it is achieved or not is determined solely by the pulse aspect ratio. If the pulse is wide, or the scattering
angle is sufficiently large, $\alpha > \alpha_0 = 2 \arctan(ct_0/L_z)$, the maximum gain is achieved at the pulse rear edge $\xi = ct_0$ for $ct_0 \cot(\alpha/2) < x < L_z$. Hence, the angular spectrum of scattered light is prescribed by the pulse aspect ratio rather than the angular dependence of the increment. For $L_z \gg ct_0$, scattering within a broad range of angles $2ct_0/L_z < \alpha < \pi$ proceeds with the maximum gain. Otherwise, for $L_z \ll ct_0$, the highest gain corresponds to the near-backward scattering only, $\pi - L_z/(ct_0) < \alpha < \pi$ (Ref. 21). To estimate the pulse energy depletion due to the LA SRS, it is sufficient to neglect the radiation scattered under angles smaller than $\alpha_0$ (Ref. 36).

B. LA SRS evolution in the reflective case

In a capillary, the oblique mirror reflections of scattered light off the walls [the boundary condition (7)] contribute to the LA SRS evolution over the whole laser path in plasma, but become actually dominating later, when the entrance effect vanishes ($z > V_c ct_0$). The reflections establish a long-distance asymptotic state of the LA SRS—amplification of a quasi-1D radiation and plasma modes with a temporal increment close to that of the direct backscatter (13). Besides, the forward SRS is seeded by the EM signal $a_s(x, -0, \xi)$ at the capillary entrance plane [the boundary condition (6)]. If the signal amplitude $a_{s0}$ exceeds the amplitude of the electron density noise $N_0$, the unstable waves will achieve a large amplitude (by virtue of the high seed level) at the transient stage of laser propagation, $0 < z < V_c ct_0$. In this case, contrary to the SRS in the unbounded plasmas described in the preceding section, the near-forward SRS exhibits a much higher level of amplification than the backward SRS ($\alpha > \pi/2$), and its contribution to the dynamics of electron density perturbations can be dominating. Nonlinearities of plasma response that can then appear are worth investigating and will be addressed in future publications.

The asymptotic $\langle \delta F(\tilde{b}_1, b_2; z, \xi) \rangle \sim (2\pi^3/3)\{1/\xi(1-b_1+b_2)\} \times e^{3\xi^2/2} + O(1/\xi^2)$ at $\xi \to \infty$ (Ref. 35) helps to evaluate the level of the EM seed amplification on the interval $0 < z < V_c ct_0$.

$$\langle a_s(z < V_c ct_0) \rangle \sim a_{s0} \frac{F}{2\sqrt{3\pi}} e^{3\xi^2/2} \xi^{-1/6}, \quad (14)$$

where $\xi > 1$. Then, $\langle N_s(z < V_c ct_0) \rangle \sim (N_s/G_s) \langle F \xi^{15/2} \rangle |a_{s0}|$. At the pulse trailing edge, $\xi = ct_0$, the argument of the asymptotic reaches the maximum $\xi_{\text{max}} = \langle Gc t_0/3 \rangle$ at $z_{\text{max}} = V_c ct_0/3$. At this point, the scattered EPW reaches the amplitude

$$\langle N_s(\xi = \xi_{\text{max}}) \rangle \sim a_{s0} \frac{G \sqrt{3\pi}}{8\xi_{\text{max}}^{15/2} e^{\gamma_{\text{max}}}}, \quad (15)$$

and $\langle a_s(\xi = \xi_{\text{max}}) \rangle \sim (g_1/G) \langle N_s(\xi = \xi_{\text{max}}) \rangle$. Equation (15) shows that the maximum gain on the transient stage is determined by the 1D temporal increment (13) independent on $\alpha$ (the angular dependence is retained in the preexponential factors only). For the minimally allowed scattering angle, $\alpha_{\text{min}} = \sqrt{2} \omega_p \omega_0 / \omega_0$, the point of the maximal gain is $\xi_{\text{max}}(\alpha_{\text{min}}) = \omega_0 / \omega_p (ct_0/3) \gg ct_0$. Parameters of the following example (Fig. 4) give $\xi_{\text{max}}(\alpha_{\text{min}})$ $\approx 3$ mm, which is shorter than a typical capillary length used in experiments ($\approx 1$ cm). 8–13,15

For $z > z_{\text{max}}$ the entrance effect becomes less pronounced and finally vanishes at the point $z = V_c ct_0 = 3\xi_{\text{max}}$, where the EM signal arriving from $z = 0$ drops behind the laser pulse and the instability growth saturates. Evolution of both forward and backward scattering is then determined by the lateral reflections only and, given the scattering angle, remains the same in any $x$-$y$ cross section for $z = V_c ct_0$. We show in Appendix C that at $z = V_c ct_0$ and $\gamma_{\text{g0}} > 1$, a cumbersome exact solution (8) admits a simple asymptotic evaluation in the form of quasi-1D damped mode with a temporal growth rate close to (13),

$$N_s(x, \xi) \sim (N_{\text{g0}}[(1 - r)\ln r]) e^{i\xi - (\xi L_x) \ln r}, \quad (16)$$

and $a_{s0}(x, \xi)$ is given by Eq. (C2). Here, $s_0 = (\gamma_{\text{g0}} - \Delta \gamma)/\Delta \gamma = V_c \ln r/(3L_x)$. Equations (16) and (C2) are valid under the “low leakage” condition, $r > \exp (3L_x/V_c)$, which indicates that the scattered light is mostly trapped inside a plasma slab: the energy leakage through the wall at one reflection (that produces an effective decrement $\Delta \gamma < \gamma_{\text{g0}}$ is less than the energy gain on the way between the walls. The lateral growth of the asymptotic (16) is much slower than the growth with $\xi$. Asymptotic solution (16) displays the basic result of the reflective theory of the LA SRS: at large distances from the entrance plane and large coefficients of amplification, the amplitudes of unstable waves tend to quasi-1D leaky modes exponentially growing in time with the increment tending to that of the BSRS (13).

FIG. 4. Spatiotemporal evolution of the up-going EPW in the field of transversely limited laser pulse of finite duration; the pulse aspect ratio is $ct_0/L_z = 0.5$. The scattering angle is $\alpha = \pi/6$. Temporal evolution of the amplitudes is traced at the longitudinal positions (a) $z = V_c ct_0/3 = 2.2ct_0$ and (c), (d) $z = V_c ct_0 = 6.5ct_0$. The left column shows the SRS evolution inside the glass capillary ($r = 0.42$) with the EM seed amplitude at the entrance plane $a_{s0} = 0.4 \times 10^{-5}$. Plot (b) shows the saturated solution for an open-boundary plasma, $r = a_{s0} = 0$, for $z = (V_c/V_s)ct_0 = 3.4ct_0$. In the plot (d), the long-term asymptotic (16) of the reflective problem (dashed line) is compared with the exact solution (solid line) and with nonreflective (dotted line) and BSRS (dash-dotted line) solutions at the point $z = V_c ct_0, z = L_x/4$. 

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Comparison of Eqs. (15) and (16) shows that the maximum amplitude of the scattering EPW on the transient stage, \( z < V_c t_0 \), differs from the final asymptotic level of density perturbations roughly by a factor of \( (G / g_1) \times (\gamma_0 - 1 / 2) (a_0 / N_0) \). When this factor is larger than unity, and \( N_\epsilon (z) = V_c t_0 / \xi, t_0) \approx 1 \), the plasma response can become nonlinear in the vicinity of \( z = V_c t_0 / 3 \). This could be avoided by keeping the ratio of the seed amplitudes \( a_0 / N_0 \) below \( (g_1 / G) (\gamma_0 - 1 / 2) \). The amplitude of the electron density noise is difficult to control in experiment; however, as shown in Appendix A, the content of the high-order eigenmodes in the laser radiation coupled to the capillary (and, hence, the amplitude \( a_0 \) of the EM seed signal) can be effectively reduced by increasing the capillary radius versus the radius of the incident laser beam.

Figure 4 shows the spatiotemporal evolution of an up-going EPW [Eq. (8b)] for the SRS under the angle \( \alpha = \pi / 6 \). The laser and plasma parameters are \( a_0 = 0.7, \lambda_0 = 0.5 \mu m, t_0 = 5.5 \times 6^3, \omega_{pl} / c_0 = 0.007 \), which give \( n_0 = 2.2 \times 10^3 c m^{-3} \), the pulse duration \( t_0 = 230 fs \), and the maximum increment \( \gamma_0 = 2.5 \omega_{pe} \). The level of EM seed is chosen as \( a_0 = 0.05 \times 10^{-5} a_0 \) (according to Appendix A, it corresponds to a capillary by a factor of 2 wider than in the case of perfect matching; by the definition, in axisymmetric geometry, the perfect matching condition provides coupling 98\% of energy of an incident Gaussian laser pulse to the fundamental eigenmode EH1 of a capillary). The level of the plasma noise evaluated in Appendix A is \( N_\epsilon = 1.5 \times 10^{-6} \). The plots (a) and (c) correspond to the plasma confined in a glass capillary with the reflection coefficient \( r = 0.42 \) (see Fig. 3), and (b) corresponds to the unbound plasma \( (r = 0 \) and \( a_0 = 0 \) ); plot (d) shows the long-term asymptotic behavior of the reflective solution. The plasma cross sections are set at (a) \( z = V_c t_0 / 3 \approx 2.2 c t_0 \), and (c), \( z = V_c t_0 = 6.5 c t_0 \). Given the calculation parameters, the nonreflective solution saturates at \( z = L_c (V_c / V) \approx 3.4 c t_0 \) and is exactly the same in any plasma cross section \( x-y \) beyond that point. Figure 4(b) shows this solution. The ranges of influence of boundary conditions are shown in Fig. 5. In range I (see Eq. (10)), the electron density noise from \( \xi = 0 \) is amplified and the waves do not experience reflections. In range IIa the instability is seeded by the free-plasma noise, and yet is enhanced by the reflections; the Raman amplified signal arriving from the entrance plane is added to these waves in range IIb The EM seed from the entrance plane \( z = -0 \) is amplified by the forward SRS in range III.

In a capillary, the forward SRS is considerably enhanced at \( z \approx V_c t_0 \) (roughly by a factor of 70 in amplitude) versus the case of unbound plasma [compare Figs. 4(a) and 4(b)]. Despite the EM seed amplified in range IIb is nonexponentially growing, the high ratio of seed amplitudes, \( a_0 / N_0 = 2.7 \times 10^9 G (\gamma_0 - 1 / 2) = 0.026 \), makes the entrance effect rather pronounced [Fig. 4(a)]. For \( N_0 = 1.5 \times 10^6 \) and \( F \sim 1 \), Eq. (15) gives \( \log_{10} N_\epsilon (\xi_0, z_\text{max}) / N_0 = 7.15 \), which agrees with Fig. 4(a). Hence, parameters of the numerical example lay at the border of validity of the linear approach, and increase in \( a_0 \) will result in the nonlinearity of the plasma response on the transient stage [e.g., perfect matching gives \( a_0 = 0.012 a_0 \), hence, according to Eq. (15), \( N_\epsilon (\xi_0, z_\text{max}) \approx 10^6 \); this burst of the forward SRS has been the regular feature in our numerical experiments and in the fluid simulations]. On the other hand, reducing \( a_0 \) to the level \( 7 \times 10^6 a_0 \) (which in the axisymmetric case would correspond to a capillary tube by a factor 2.5 wider than in the case of perfect matching, see Appendix A) will make the forward Raman amplification of the EM seed almost negligible and thus tolerable on the transient stage.

Figure 4(c) shows a quasi-1D saturated solution [compare with Fig. 4(b)] thus shaped by the contribution from two reflections [according to Fig. 5(b)], which, in full agreement with the long-scale asymptotic (16), demonstrates the growth rate close to that of BSRS. Difference between Figs. 4(a) and 4(c) shows that, under the parameters of our example, the forward scattering \( (\alpha < \pi / 2) \) is characterized at \( z < V_c t_0 \) by a much higher gain than the SRS in the backward direction \( (\alpha > \pi / 2) \). This situation is completely reverse of the SRS in an unbounded plasma, where the gain can only fall as an angle drops. So, the higher the EM seed level produced by the laser beam coupling (that is, the tighter the capillary in a numerical or real-scale experiment), the more important becomes the forward SRS. In such case, a high amplification level of waves may be observed within quite a long distance in plasma, \( z < V_c (a_{min} / a_0) c t_0 \) [see also the discussion following Eq. (15)]. This effect is adverse for such applications as the self-modulated LWFA in capillaries. The way of reducing the excessive forward SRS enhancement can be found in using a wider capillary (the laser focal spot fixed) than the perfect matching requires.

Figure 4(d) traces the temporal evolution of the up-going EPW at \( x = L_c / 4 \) (near the capillary wall). The asymptotic (16) perfectly approximates (and, for applications, can be used instead of) the exact reflective solution (8b). The dash-dotted and dotted lines in Fig. 4(d) correspond to the BSRS
solution $|N_o(\xi)| = (N_0/3)\exp(\gamma_0\xi/c)$ and the exact nonreflec-
tive solution, respectively, providing the upper and lower
limits of the convective gain variation. In this example, only
in the vicinity of the border $x=0$ the coefficients of ampli-
fractional and nonreflecctive cases are consider-
differently [compare Figs. 4(b) and 4(c)]. Contribution
from the reflections increases the wave amplitude at $x=
L_{x}/4$ by roughly an order of magnitude (compare solid
and dotted lines at $k_p\xi = 6$). For the parameters chosen, the
scattering EPW remains linear in the convective saturated regime
($\zeta > V_{c}t_{0}$): the plasma noise level, $N_0 \approx 1.5 \times 10^{-6}$,
substituted into Eq. (16), gives $|N_o| \approx 0.2$ throughout the whole
time interval $0 < t < t_0$.

In the limit $r \rightarrow 1$ the total suppression of the lateral con-
vection occurs. The up- and down-going amplitudes (8) be-
come purely one dimensional for $\zeta > V_{c}t_{0}$ and completely
identical. These amplitudes grow in time exponentially with
the BSRS increment (13).

IV. CONCLUSION

We have proposed a 2D nonstationary linear theory of
strongly coupled LA SRS of a short laser pulse in a flat
plasma slab confined between mirror-reflecting walls (flat
capillary). In a capillary, the lateral convection of scattered
light is partly suppressed by the oblique reflections, and the
instability experiences an enhancement. Additional enhance-
ment of the SRS in forward direction ($\alpha > \pi/2$) is produced by
the amplification of the electromagnetic seed signal that is
formed of the high-order capillary eigenmodes at the en-
trance plane (formation of the signal is a consequence of the
laser beam coupling to the capillary). The convective nature of
LA SRS does not change. The asymptotic behavior of the waves
demonstrates the transition from the set of 2D modes to
the dominant quasi-1D damped mode. Even for near-
forward scattering the convective gain of the dominant
 quasi-1D mode may be close to the BSRS gain.

ACKNOWLEDGMENTS

The authors wish to acknowledge useful conversations
with N. E. Andreev, B. Cros, L. M. Gorbunov, G. Matthieu-
sen, and J. Meyer-ter-Vehn. S.Y.K. sincerely appreciates
hospitality of the Ecole Polytechnique and the Max-Planck-
Institut für Quantenoptik and their financial support in the
form of postdoctoral fellowships.

APPENDIX A: SEED SOURCES FOR LA SRS

The LA SRS under arbitrary angle in strongly rarefied
plasmas ($\omega_{pe} < \omega_{0}$) is seeded by spontaneous electron density
fluctuations ahead of the pulse. A root-mean-square (rms)
amplitude of these fluctuations is represented in the equa-
tions by the quantity $N_0$, which gives the amount of seed
corresponding to the element of solid angle $d\Omega_k$ in the
direction of the wave vector $k$, of scattering EPW, and can be
expressed as $N_0(k) = d\Omega_k \int n_{e}^{3D}k^2dk = n_{e}^{3D}(k)k^2\Delta_k d\Omega_k$, where
$n_{e}^{3D}(k)$ is a spectral density of electron fluctuations
integrated over frequencies. The integral is taken over the
area of maximal spectral density of scattering EPW $[k = k_e$, 
the amplification bandwidth $c\Delta_k = 4\sqrt{\gamma_0/T_0}$ is estimated at
$\gamma_0T_0 \gg 1$ with taking account of the gain narrowing (Ref.
36). Using the ratio of the phase volumes $|dk/d\Omega|$
$= 2\sin^2(\alpha/2)$ (Ref. 38), we express the seed amplitude
through the element of solid angle $d\Omega_k$, in the direction of
detector, $N_0 \approx (8/c)n_{e}^{3D}(k)k^2\gamma_0^{1/2}/\alpha(\alpha/2)d\Omega_k$. Our theo-
retical formalism based on the assumption of quasiplane in-
teracting waves requires small variation of the scattered
wave amplitude across the direction $k$, in the transversely
limited area, $0 < x < L_x$, which gives an estimate of the an-
gular spread $\Delta \alpha = \alpha/L(k_0)$. The element of solid angle
then evaluated as $\Delta\Omega_k = 2\pi \sin \Delta \alpha = 2 \pi \sin 2 \alpha/(L_xk_0)$ gives

$$|N_o| \approx \left[ \frac{8\pi}{n_0} + \frac{(k_e/\alpha)^2}{2 + (k_e/\alpha)^2} \left( \omega_0/\omega_{pe} \right)^{1/2} k_0L_x \right]^{1/2},$$

where the spectral density of low-frequency electron fluctua-
tions $n_{e}^{3D}(k_e)$ is evaluated using formula (11.2.6.6) of Ref. 37.
Parameters of Fig. 4 and $k_e\alpha \approx 1$ give $|N_o| \approx 1.5 \times 10^{-6}$.

For the strongly coupled SRS in the forward direction
($\alpha < \pi/2$), coupling the laser beam to a capillary creates an
additional source of instability. The radial profile of an inci-
dent beam with the wings cut off by the edges of the entrance
aperture is approximated with an expansion through an in-
nite number of radial eigenmodes (having the same fre-
quency $\omega_0$). The high-order eigenmodes (characterized by
the frequency $\omega_e = \omega_0$ and high transverse wave numbers,
k_{n+1} = k_{n},$ where the integer $n$ is the mode order) form the
seed signal that is further amplified in plasma by the SRS [in
the model form, the transverse profile of this signal is given by
Eq. (6)]. It should be emphasized that these modes pro-
vide no seed for the weakly coupled SRS [including near-
forward SRS and RMI (Ref. 34) that correspond to small
scattering angles, $\alpha < \omega_{pe}/\omega_{0}$], as this process requires the
frequency matching $\omega_e = \omega_0 - \omega_{pe}$ between the seed and the
pump.

Exact functional form of the capillary eigenmodes de-
}pends on the geometry chosen. Despite the LA SRS in a flat
capillary is considered in the paper, we suppose that an esti-
}mation of the EM seed level will be more useful for applica-
tions if inferred from the axisymmetric theory of the laser
beam propagation in a dielectric tube. The theory repre-
sents an electric field profile at the entrance aperture of the
radius $r_0$ as an infinite sum $a(r) = a(r)r J_0(k_0 r) r$ (Ref. 11),
where $C_k = 2[r_0J_1(k_0 r)]^2 \int [a(r) r J_0(k_0 r) r] dr$ is the over-
lap integral of the hybrid capillary eigenmode $E_{1n}$ with
the incident laser profile $a(r) = \exp(-r^2/\sigma_0^2)$ (Fig. 6). Here,
$u_n$ is the $n$th zero of the zero-order Bessel function of the first
kind, $J_0(u_n) = 0$; for $k_0 r_0 \gg 1$ and $n \gg 1$, $u_n \approx \pi n/(n + 1)/2$. Figure 6 shows that about 98% of laser energy is
coupled to the fundamental mode for $\sigma_0 = 0.645 r_0$ (the
perfect matching condition); however, the overlap integral de-
cays very slowly as $n$ grows [the analytic fit, $C_n \approx 1 - 0.325 r_0$ (n
+ 1)/2], is almost exact for $n > 10$. When the ratio
$\sigma_0 \approx \pi r_0$ drops, the larger number of lower-order eigenmodes is
effectively excited (up to 5 for $\sigma_0 = 0.258 r_0$), but contribution
from the higher-order modes into the radiation profile at
$\zeta = 0$ drops sharply [e.g., analytic fit $C_n \approx 3 \times 10^{-6} r_0$, $n$
> 10, holds for $\sigma_0 = 0.258r_0$). Therefore, choosing wider capillary is the way of reducing the effect of laser coupling on the forward strongly coupled SRS.

The numerical example of Sec. III B shows how the SRS under the angle $\alpha = \pi/6$ develops in the capillary with an EM seed amplitude characteristic of the axisymmetric capillary tube by a factor of 2 wider than in the case of perfect matching. For the parameters of Fig. 4, with the value $L_s/2 = 860m^{-1}$ assigned to $r_0$, equalizing the effective radial wave number $k_{n,1} = n\pi/r_0$ to the resonant wave number $k_{n,1} = k_0 sin \alpha$ gives the resonant mode order $n' = (k_0r_0/\pi)sin \alpha = 136$ corresponding to $C_{136} = 0.58 \times 10^{-5}$ under the condition $\sigma_0 = 0.3225r_0$ (analytic fit $C_n = 0.7 \times 10^{-3}F_n$ was used; see also Fig. 6). One can expect that several modes with $n = n'$ can contribute to the effective amplitude $a_0$ of the seed signal. The difference in the mode numbers $\Delta n = n' - n$ causes the angular spread $\Delta \alpha$ around the given scattering angle; this spread would not exceed the admissible value established above, $\Delta \alpha < \cos \alpha/(L_s)k_0 = \cos \alpha/(2r_0k_0)$. For $|\Delta n| < n'$, one has $\Delta n = (k_0r_0/\pi)\cos \Delta \alpha < \cos^2 \alpha/(2 \pi) < 1$, so that only one capillary mode with $n = n'$ can contribute to the scattering under the given angle. Therefore, the seed amplitude used for the numerical demonstration of Fig. 4(a) is evaluated as $a_0 = C_{136}a_0 = 0.4 \times 10^{-5}$.

**APPENDIX B: DERIVATION OF THE EXACT REFLECTIVE SOLUTION**

On omitting $k_{p}^2$ in the LHS of Eq. (3b), the Laplace transform (LT) of Eqs. (3) with respect to $\xi$ (LT variable $s$) and with respect to $z$ (LT variable $p$) gives the set of ordinary differential equations (ODE) for the Laplace images $\tilde{a}_{s,x}(x,p,z;r)$,

$$\frac{\partial}{\partial x} \tilde{a}_{s,x} = \pm K(x),$$

where $\Omega = (V_c/V_s)(\Gamma_s - p)$, $\Gamma_s = (iG^3/s^2 - s)/V_s$, $K(x) = K_1/(s^p)$, and $+(K_2/s)[1-2x/(L_s-1)^2]$, $K_1 = g_1N_0/(iV_s)$, and $K_2 = (V_c/V_s)a_0$. The boundary conditions are $\tilde{a}_{s,x}(0,p,s) = r\tilde{a}_{s,x}(0,p,s)$ and $\tilde{a}_{s,x}(L_s,p,s) = r\tilde{a}_{s,x}(L_s,p,s)$. Equations (B1) admit the solution

$$\tilde{a}_{s,x}(x,p,s;r) = -\left[\frac{K_1 + K_2}{s}\right] \left[1 - \frac{(1-r)e^{\Omega x}}{1 - re^{s\Omega x}}\right] \frac{1}{s\Omega} \left[1 + \left(\Omega \frac{L_s}{2} - 1\right)^2\right] + \frac{K_2}{(L_s/2)^2} \left[1 + \left(\Omega \frac{L_s}{2} - 1\right)^2\right] \frac{1}{s\Omega^2} \left[1 + \left(\Omega \frac{L_s}{2} + 1\right)^2\right] - \frac{1}{s\Omega^3}, \quad (B2)$$

where $x_s = x + nL_s$. Lateral symmetry gives $\tilde{a}_{s,x}(x) = \tilde{a}_{s,x}(L_s - x)$. The Laplace transform inversion of Eq. (B2) with respect to $p$ reads

$$\tilde{a}_{s,x}(x,z,s;r) = -\left[\frac{K_1}{s^{1/2}} + \left[\frac{K_1 e^{Gz}}{s^{1/2}} - \frac{K_2 e^{Gz}}{s^{1/2}}(x - x)(x - x - L_s)\right]\left(\frac{L_s}{2}\right)^2 \right] H(x - z)$$

$$+ \frac{K_1(1-r)\sum_{n=1}^{\infty} \rho^n e^{(V_c/V_s)\rho_n}}{s^{1/2}} \int(H(z - x))$$

$$+ \frac{\sum_{n=1}^{\infty} n\rho^n e^{(V_c/V_s)\rho_n}}{s^{1/2}} \int\left[H(z - x_{n-1}) - H(z - x_{n})\right], \quad (B3)$$

where, in accordance with Ref. 14, the expansion $[1 - r \exp(\Omega L_s)]^{-1} = \sum_{n=0}^{\infty} \rho^n \exp(n\Omega L_s)$ is used, and $\zeta = (V_c/V_s)\zeta$, $\tilde{K}_{1,2} = (V_c/V_s)\tilde{K}_{1,2}$. Equation (B3) includes Laplace images of the three types: $1/(s\Gamma_s)$, $e^{x\xi}/(s\Gamma_s)$, and $e^{x\xi}/s$. Their inversions read

$$L^{-1}_{\zeta} \left[\frac{1}{s\Gamma_s}\right] = \frac{V_c}{3} \sum_{n=1}^{\infty} e^{x\xi}$$

$$\Gamma_s$$

$$L^{-1}_{\zeta} \left[\frac{\rho^n}{s^{1/2}}\right] = -\frac{V_c}{3} \sum_{n=1}^{\infty} e^{x\xi}$$

$$\Gamma_s$$

which is expressed through the fundamental solutions

$$F_s(\mu,\nu,\xi) = e^{-\nu r} \sum_{n=0}^{\infty} \frac{(\mu\nu)^n}{n! (2n)!} \gamma(n+1, \mu(\xi - v)) \Gamma(\xi - v),$$

where $\gamma(m, \theta) = \int_{\theta}^{\infty} e^{-t} t^{m-1} dt = \theta^m (\Gamma_{n+m} - \theta)^m [n! (n+m)]$ is the incomplete gamma function of order $m$ of a complex variable $\theta$ (Ref. 35); and

$$\gamma(m, \theta) = \int_{\theta}^{\infty} e^{-t} t^{m-1} dt = \theta^m (\Gamma_{n+m} - \theta)^m [n! (n+m)]$$

$\gamma(m, \theta) = \int_{\theta}^{\infty} e^{-t} t^{m-1} dt = \theta^m (\Gamma_{n+m} - \theta)^m [n! (n+m)]$
expressed through the regularized generalized hypergeometric function of variable $i\xi=i(\gamma+i\xi)z/(V_c/L)$ and $V_c\gamma$. Combining expressions (B4), (B5), and (B7) in Eq. (B3) gives the envelope (8a) of an up-going EMW. The amplitude (8b) of the scattering EPW is derived in the similar fashion.

APPENDIX C: ASYMPTOTIC REFLECTIVE SOLUTION

The asymptotic can be found by applying the inversion formula, $\tilde{a}_\alpha(x,z,s) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\beta \pi/4} \tilde{a}_\alpha(x,p,s) dp$, to the expression (B2). The asymptotic behavior at $z \to \infty$ is determined by the singularities of the integrand at $p = \Gamma_\alpha$, and $p = p_\alpha$, where $\Gamma_\alpha$ and $p_\alpha$ are the origin of the asymptotic solution, so as the distance $z$ from the entrance plane exceeds $V_c t_0$ or for $\alpha > \pi/2$. (Actually, the Laplace image singularities at $p = \Gamma_\alpha$, and $p = p_\alpha$ determine the entrance effect, i.e., the waves produced by the seed at the entrance aperture amplified in the pump field in plasma; these waves inevitably drop behind the laser pulse and their effect therefore vanishes at $z \to \infty$.) The contribution from $p = 0$, 

$$a_\alpha(x,s) \sim \frac{1}{\pi^3} \left( 1 - \frac{1}{r} \exp \left( \frac{V_c V_x}{\Gamma_\alpha L_x} \right) \right),$$

thus determines the long-term evolution of instability in plasma, which is dominated by the lateral reflections of scattered EMW. Neither $s = s_c$, where $\Gamma_\alpha = 0$, nor $s = 0$ are the singularities of the image (C1), so contributions to the asymptotic originate from the singular points $s_n$ only, which are the solutions of the equation $\Gamma_\alpha = \frac{V_c}{s_n}$ where $\frac{V_c}{s_n} = (V_c / L_x) \times (-\ln r + 2 \pi n)$. $s_n$ is an integer. The fundamental specific point $s_0$ with the maximum real part is the root of the cubic equation $s^3 + n V_c s - G^2 = 0$, where $n$ is integer. For arbitrary $s_0$, admits quite a cumbersome explicit expression. We address to the physically interesting limit of low leakage, i.e., $s_n < \exp \left( \frac{V_c V_x}{\Gamma_\alpha L_x} \right)$ or $s > -3 G / V_c$, which means that the scattered EMW gains more energy between two reflections than losses due to leakage through the wall at one reflection. In this case the lateral convection is mostly suppressed. Solution obtained via the perturbation approach reads $s_n = (i + 3)^2 G / (2 \nu_0 V_c / s_c)$. Contribution to the asymptotic from that point is of the order of $e^{-\nu_0 s_c}$, which grows in time with an increment $Re s_n$. However, the absolute value of $\nu_0 s_c$ grows with $n$, so an analysis has to be done of the contribution from the points $s_n$ with large $n$. We again use the perturbation approach with the small parameter $\mu_n = G / (s_n V_c)$, that is, $n > G / (2 \nu_0 V_c)$, and find the solutions $s_n(i) \approx -i G / (1 + n)$. Obviously, $s_n \to \infty$ for $\mu_n \ll 1$, so that contribution from these points to the asymptotic is negligible compared to that from the points $s_n$ with $n > G / (2 \nu_0 V_c)$, which is of the order of $e^{-\nu_0 s_n}$. Expanding the image (C1) in the vicinity of $s = s_0$, we arrive at the asymptotic valid for $G \xi \gg 1$,

$$a_\alpha(x,\xi) \sim \frac{V_c}{G} \left( 1 - r \frac{i}{\sqrt{3}} \right) \frac{e^{i \sqrt{3} \xi}}{6 \exp \left( \frac{s_0 - \ln r}{L_x} \right)},$$

which represents the unstable solution as a quasi-1D exponentially growing mode.


32 N. E. Andreev (private communication).


