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Existence of double Walsh series universal in weighted $L^1_\mu[0,1]^2$ spaces

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In this paper we construct a weighted $L^1_\mu[0,1]^2$ space and a series $\sum\limits_{n,k=1}^{\infty} c_{n,k} W_n(x) W_k(y)$ with $\sum\limits_{n,k=1}^{\infty} |c_{n,k}|^q < \infty$, $\forall q > 2$, by Walsh system, which is universal in $L^1_\mu[0,1]^2$ with respect to subseries.

1. Introduction

Let $\mu(x)$ be a measurable on $[0,1]$ function with $0 < \mu(x) \leq 1, x \in [0,1]$ and let $L^1_\mu[0,1]$ be a space of mesurable functions $f(x)$, $\ x \in [0,1]$ with

$$\int_0^1 |f(x)|\mu(x)dx < \infty.$$

and $\{W_n(x))\}_{n=0}^{\infty}$ be a Walsh-Paly system.

Definition 1.1 A functional series

$$\sum_{k=1}^{\infty} f_k(x), \quad f_k(x) \in L^1_\mu[0,1] \tag{1}$$

is said to be universal in weighted spaces $L^1_\mu[0,1]$ with respect to rearrangements, if for any function $f(x) \in L^1_\mu[0,1]$ the members of (1) can be rearranged so that the obtained series $\sum\limits_{k=1}^{\infty} f_{\sigma(k)}(x)$ converges to the function $f(x)$ in the metric $L^1_\mu[0,1]$, i.e.

$$\lim_{n\to\infty} \int_0^1 \left| \sum_{k=1}^{n} f_{\sigma(k)}(x) - f(x) \right| \cdot \mu(x)dx = 0.$$

Definition 1.2 The series (1.1) is said to be universal in weighted spaces $L^1_\mu[0,1]$ concerning subseries, if for any function $f(x) \in L^1_\mu[0,1]$ it is possible to choose a partial series $\sum\limits_{k=1}^{\infty} f_{n_k}(x)$ from (1.1), which converges to the $f(x)$ in the metric $L^1_\mu[0,1]$.

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The above mentioned definitions are given not in the most general form and only in the generality, in which they will be applied in the present paper.

Note, that in one-dimensional case many papers are devoted (see [1]- [5]) to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure.

In [6] and [7] are proved the following results.

Theorem 1.1 There exists a series of the form

$$\sum_{k=1}^{\infty} c_k W_k(x) \quad with \quad \sum_{k=1}^{\infty} |c_k|^q < \infty, \quad \forall q > 2 \tag{2}$$

such that for any number $\epsilon > 0$ a weighted function $\mu(x)$ with

$$0 < \mu(x) \leq 1, |\{x \in [0,1] : \mu(x) \neq 1\}| < \epsilon \tag{3}$$

can be constructed, so that the series (2) is universal in $L^1_\mu[0,1]$ with respect to rearrangements.

Theorem 1.2 There exists a series of the form (2) such that for any number $\epsilon > 0$ a weighted function $\mu(x)$ with (3) can be constructed, so that the series (2) is universal in $L^1_\mu[0,1]$ concerning subseries.

In this paper we proved that the Theorems 1.1 and 1.2 can be transferred from one-dimensional case to two-dimensional one.

Moreover, the following are true.

Theorem 1.3 There exists a double series of the form

$$\sum_{n,k=1}^{\infty} c_{n,k} W_n(x) W_k(y) \quad with \quad \sum_{n,k=1}^{\infty} |c_{n,k}|^q < \infty, \quad \forall q > 2 \tag{4}$$

with the following property: for any number $\epsilon > 0$ a weighted function $\mu(x,y)$ with

$$0 < \mu(x,y) \leq 1, |\{(x,y) \in T : \mu(x,y) \neq 1\}| < \epsilon \tag{5}$$

can be constructed, so that the series (4) is universal in $L^1_\mu(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums.

Theorem 1.4 There exists a double series of the form (4) with the following property: for any number $\epsilon > 0$ a weighted function $\mu(x,y)$ with (5) can be constructed, so that the series (4) is universal in $L^1_\mu(T)$ concerning rearrangements with respect to convergence by both spherical and rectangular partial sums.

2. Notation and basic lemma

First we will give a definition of Walsh-Paly system (see [6]).

$$W_0(x) = 1, \quad W_n(x) = \prod_{s=1}^{n} r_{m_s}(x), \quad n = \sum_{s=1}^{k} 2^{m_s}, \quad m_1 > m_2 > ... > m_s \tag{6}$$

where $\{r_k(x)\}_{k=0}^{\infty}$ is the system of Rademacher:

$$r_0(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}); \\ \\ -1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

$r_0(x+1) = r_0(x), \quad r_k(x) = r_0(2^k x), \quad k = 1, 2, ...$

The rectangular and spherical partial sums of the double series

$$\sum_{k,\nu=1}^{\infty} c_{k,\nu} W_k(x) W_\nu(y)$$

will be denoted by

$$S_{n,m}(x,y) = \sum_{k=1}^{n} \sum_{\nu=1}^{m} c_{k,\nu} W_k(x) W_\nu(y)$$

and

$$S_R(x,y) = \sum_{n^2+k^2 \le R^2} c_{k,\nu} W_k(x) W_\nu(y).$$

The following lemma plays a central role in the proof of our theorems (see [7]).

Lemma . For any numbers $\epsilon > 0$, $N > 1$ and function $f(x,y) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \cdot \chi_{\Delta_\nu}(x,y)$ there exists a measurable set $E \subset T$ and a polynomial $P(x,y)$ of the form

$$P(x,y) = \sum_{k,s=N}^{M} c_{k,s} W_k(x) \cdot W_s(y),$$

which satisfy the following conditions:

$$P(x,y) = f(x,y) \quad for \quad (x,y) \in E,$$

$$|E| > 1 - \epsilon,$$

$$\sum_{k,s=N}^{M} |c_{k,s}|^{2+\epsilon} < \epsilon,$$

$$\max_{N \le \overline{n},\overline{m} \le M} \left[\int\int_e \left| \sum_{k,s=N}^{\overline{n},\overline{m}} c_{k,s} W_k(x) \cdot W_s(y) \right| dxdy \right] +$$

$$+ \max_{\sqrt{2}N \le R \le \sqrt{2}M} \left[\int\int_e \left| \sum_{2N^2 \le k^2+s^2 \le R^2} c_{k,s} W_k(x) \cdot W_s(y) \right| dxdy \right] \le$$

$$\le 2 \cdot \int\int_e |f(x,y)| dxdy + \epsilon,$$

for every measurable subset e of E.

3. Proofs of the Theorems

Proof of the Theorem 1.3

Let

$$\{f_s(x,y)\}_{s=1}^{\infty}, \quad (x,y)\in T \tag{7}$$

be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers.Applying Lemma consecutively, we can find a sequence $\{E_s\}_{s=1}^{\infty}$ of sets and a sequence of polynomials

$$P_s(x,y)=\sum_{k,\nu=N_{s-1}}^{N_s-1} c_{k,\nu}^{(s)} W_k(x)W_\nu(y), \tag{8}$$

$$1=N_0<N_1<...<N_s<...., \quad s=1,2,....,$$

which satisfy the conditions:

$$P_s(x,y)=f_s(x,y), \quad (x,y)\in E_s, \tag{9}$$

$$|E_s|>1-2^{-2(s+1)}, \quad E_s\subset T, \tag{10}$$

$$\sum_{k,\nu=N_{s-1}}^{N_s-1}\left|c_{k,\nu}^{(s)}\right|^{2+2^{-2s}}<2^{-2s}, \tag{11}$$

$$\max_{N_{s-1}\le\overline{n},\overline{m}\le N_s}\left[\int\int_e\left|\sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)}W_k(x)\cdot W_\nu(y)\right|dxdy\right]+$$

$$+\max_{\sqrt{2}N_{s-1}\le R\le\sqrt{2}N_s}\left[\int\int_e\left|\sum_{2N_{s-1}^2\le k^2+\nu^2\le R^2} c_{k,\nu}^{(s)}W_k(x)\cdot W_\nu(y)\right|dxdy\right]\le$$

$$\le 2\cdot\int\int_e|f(x,y)|dxdy+2^{-2(s+1)}, \tag{12}$$

for every measurable subset e of E_s.

Denote

$$\sum_{k,\nu=1}^{\infty}c_{k,\nu}W_k(x)W_\nu(y)=\sum_{s=1}^{\infty}\left[\sum_{k,\nu=N_{s-1}}^{N_s-1}c_{k,\nu}^{(s)}W_k(x)W_\nu(y)\right], \tag{13}$$

where

$$c_{k,\nu}=c_{k,\nu}^{(s)}, \quad for \quad N_{s-1}\le k,\nu<N_s, \quad s=1,2,...$$

For an arbitrary number $\epsilon>0$ we set

$$\begin{cases} \Omega_n=\bigcap_{s=n}^{\infty}E_s, \quad n=1,2,....; \\ E=\Omega_{n_0}=\bigcap_{s=n_0}^{\infty}E_s, \quad n_0=[\log_{1/2}\epsilon]+1; \\ B=\bigcup_{n=n_0}^{\infty}\Omega_n=\Omega_{n_0}\bigcup\left(\bigcup_{n=n_0+1}^{\infty}\Omega_n\setminus\Omega_{n-1}\right) \end{cases} \tag{14}$$

It is obvious (see (10), (15)) that $|B| = 1$ and $|E| > 1 - \epsilon$.

We define a function $\mu(x, y)$ in the following way:

$$\mu(x,y) = \begin{cases} 1, & for \quad (x,y) \in E \cup (T \setminus B); \\ \mu_n, & for \quad (x,y) \in \Omega_n \setminus \Omega_{n-1}, \quad n \geq n_0 + 1, \end{cases} \tag{15}$$

where

$$\begin{cases} \mu_n = \left[2^{2n} \cdot \prod_{s=1}^{n} h_s \right]^{-1}; \\ h_s = ||f_s||_C + \max_{N_{s-1} \leq \overline{n}, \overline{m} \leq N_s} \left|\left| \sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x) \cdot W_\nu(y) \right|\right|_C + \\ + \max_{\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s} \left|\left| \sum_{2N_{s-1}^2 \leq k^2+\nu^2 \leq R^2} c_{k,\nu}^{(s)} W_k(x) \cdot W_\nu(y) \right|\right|_C + 1, \end{cases} \tag{16}$$

where $g(x, y)$ is a continuous function on T and

$$||g(x,y)||_C = \max_{(x,y) \in T} |g(x,y)|.$$

From (11),(13)-(16) we obtain

$(A) - 0 < \mu(x, y) \leq 1, \mu(x, y)$ is a measurable function and

$$|\{(x,y) \in T : \mu(x,y) \neq 1\}| < \epsilon.$$

$$(B) - \sum_{k,\nu=1}^{\infty} |c_{k,\nu}|^q < \infty, \forall q > 2.$$

Hence, obviously we have (see (11) and (13))

$$\lim_{k,\nu \to \infty} c_{k,\nu} = 0. \tag{17}$$

It follows from (14)-(16) that for all $s \geq n_0$ and $N_{s-1} \leq \overline{n}, \overline{m} \leq N_s$

$$\int\int_{T \setminus \Omega_s} \left| \sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x,y) dxdy =$$

$$= \sum_{n=s+1}^{\infty} \left[\int\int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu_n dxdy \right] \leq$$

$$\leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[\int\int_T \left| \sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| h_s^{-1} dxdy \right] < \frac{1}{3} 2^{-2s}. \tag{18}$$

Analogously for all $s \geq n_0$ and $\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s$ we have

$$\int\int_{T \setminus \Omega_s} \left| \sum_{2N_{s-1}^2 \leq k^2+\nu^2 \leq R^2} c_{k,\nu}^{(s)} W_k(x) \cdot W_\nu(y) \right| \mu(x,y) dxdy < \frac{1}{3} 2^{-2s}. \tag{19}$$

By (9), (14)-(16) for all $s \geq n_0$ we have

$$\int\int_T |P_s(x,y) - f_s(x,y)|\,\mu(x,y)dxdy =$$

$$\int\int_{\Omega_s} |P_s(x,y) - f_s(x,y)|\,\mu(x,y)dxdy +$$

$$+\int\int_{T\setminus\Omega_s} |P_s(x,y) - f_s(x,y)|\,\mu(x,y)dxdy =$$

$$= \sum_{n=s+1}^{\infty}\left[\int\int_{\Omega_n\setminus\Omega_{n-1}} |P_s(x,y) - f_s(x,y)|\,\mu_n dxdy\right] \leq$$

$$\leq \sum_{n=s+1}^{\infty} 2^{-2n}\left[\int\int_T \left(|f_s(x,y)| + \sum_{k,\nu=N_{s-1}}^{N_s-1} c_{k,\nu}^{(s)} W_k(x)\cdot W_\nu(y)\right) h_s^{-1} dxdy\right]$$

$$< \frac{1}{3}2^{-2s} < 2^{-2s}. \tag{20}$$

By (12), (14) - (16) and (18) for all $N_{s-1} \leq \overline{n}, \overline{m} \leq N_s$ and $s \geq n_0 + 1$ we obtain

$$\int\int_T \left|\sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x)\cdot W_\nu(y)\right| \mu(x,y)dxdy =$$

$$\int\int_{\Omega_s} \left|\sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x)\cdot W_\nu(y)\right| \mu(x,y)dxdy +$$

$$\int\int_{T\setminus\Omega_s} \left|\sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x)\cdot W_\nu(y)\right| \mu(x,y)dxdy <$$

$$< \sum_{n=n_0+1}^{s}\left[\int\int_{\Omega_n\setminus\Omega_{n-1}} \left|\sum_{k,\nu=N_{s-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(s)} W_k(x)\cdot W_\nu(y)\right|\cdot \mu_n dxdy\right] + \frac{1}{3}2^{-2s} <$$

$$< \sum_{n=n_0+1}^{s}\left(2^{-2(s+1)} + 2\cdot\int\int_{\Omega_n\setminus\Omega_{n-1}} |f_s(x,y)|dxdy\right)\mu_n + \frac{1}{3}2^{-2s} =$$

$$= 2^{-2(s+1)}\cdot\sum_{n=n_0+1}^{s}\mu_n + +\int\int_{\Omega_s} |f_s(x,y)|\mu(x,y)dxdy + \frac{1}{3}2^{-2s} <$$

$$< 2\cdot\int\int_T |f_s(x,y)|\mu(x,y)dxdy + 2^{-2s}. \tag{21}$$

Analogously for all $s \geq n_0$ and $\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s$ we have (see (19))

$$\int\int_T \left|\sum_{2N_{s-1}^2 \leq k^2+\nu^2 \leq R^2} c_{k,\nu}^{(s)} W_k(x)\cdot W_\nu(y)\right| \mu(x,y)dxdy < \frac{1}{3}2^{-2s}.$$

$$< 2 \cdot \int\int_T |f_s(x,y)|\mu(x,y)dxdy + 2^{-2s}. \tag{22}$$

Now we'll show that the series (13) is universal in $L^1_\mu(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums.

Let $f(x,y) \in L^1_\mu(T)$, i. e.

$$\int\int_T |f(x,y)|\mu(x,y)dxdy < \infty.$$

It is easy to see that we can choose a function $f_{n_1}(x,y)$ from the sequence (7) such that

$$\int\int_T |f(x,y) - f_{n_1}(x,y)|\, \mu(x,y)dxdy < 2^{-2}, \quad n_1 > n_0 + 1. \tag{23}$$

Hence, we have

$$\int\int_T |f_{n_1}(x,y)|\, \mu(x,y)dxdy < 2^{-2} + \int\int_T |f(x,y)|\mu(x,y)dxdy. \tag{24}$$

From (20) and (22) we get

$$\int\int_T |f(x,y) - P_{n_1}(x,y)|\, \mu(x,y)dxdy \le \int\int_T |f(x,y) - f_{n_1}(x,y)|\, \mu(x,y)dxdy+$$

$$+ \int\int_T |f_{n_1}(x,y) - P_{n_1}(x,y)|\, \mu(x,y)dxdy < 2 \cdot 2^{-2}. \tag{25}$$

Assume that numbers $n_1 < n_2 < ... < n_{q-1}$ are chosen in such a way that the following condition is satisfied:

$$\int\int_T \left| f(x,y) - \sum_{s=1}^{j} P_{n_s}(x,y) \right| \mu(x,y)dxdy < 2 \cdot 2^{-2j}, \quad 1 \le j \le q-1. \tag{26}$$

Now we choose a function $f_{n_q}(x,y)$ from the sequence (7) such that

$$\int\int_T \left| \left(f(x,y) - \sum_{s=1}^{q-1} P_{n_s}(x,y) \right) - f_{n_q}(x,y) \right| \mu(x,y)dxdy < 2 \cdot 2^{-2q}, \quad n_q > n_{q-1}. \tag{27}$$

This with (25) imply

$$\int\int_T \left| f_{n_q}(x,y) \right| \mu(x,y)dxdy < 2^{-2q} + 2 \cdot 2^{-2(q-1)} = 9 \cdot 2^{-2q}. \tag{28}$$

Hence and from (8), (20) - (22) we obtain

$$\int\int_T \left| f_{n_q}(x,y) - P_{n_q}(x,y) \right| \mu(x,y)dxdy < 2^{-2n_q}, \tag{29}$$

where

$$P_{n_q}(x,y) = \sum_{k,\nu=N_{n_q-1}}^{N_{n_q}-1} c_{k,\nu}^{(n_q)} W_k(x) W_\nu(y),$$

$$\max_{N_{n_q-1}\le\overline{n},\overline{m}<N n_q}\left[\int\int_T\left|\sum_{k,\nu=N_{n_q-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(n_q)}W_k(x)\cdot W_\nu(y)\right|\mu(x,y)dxdy\right]<19\cdot 2^{-2q}. \tag{30}$$

Analogously we have

$$\max_{\sqrt{2}N_{n_q-1}\le R\le\sqrt{2}N_{n_q}}\left[\int\int_T\left|\sum_{2N_{n_q-1}^2\le k^2+\nu^2\le R^2} c_{k,\nu}^{(n_q)}W_k(x)\cdot W_\nu(y)\right|\mu(x,y)dxdy\right]<19\cdot 2^{-2q}. \tag{31}$$

In quality subseries of Theorem we shall take

$$\sum_{q=1}^{\infty}P_{n_q}(x,y)=\sum_{q=1}^{\infty}\left[\sum_{k,\nu=N_{n_q-1}}^{N_{n_q}-1} c_{k,\nu}^{(n_q)}W_k(x)W_\nu(y)\right], \tag{32}$$

From (27) and (29) we have

$$\int\int_T\left|f(x,y)-\sum_{s=1}^{q}P_{n_s}(x,y)\right|\mu(x,y)dxdy\le$$

$$\le\int\int_T\left|\left(f(x,y)-\sum_{s=1}^{q-1}P_{n_s}(x,y)\right)-f_{n_q}(x,y)\right|\mu(x,y)dxdy+$$

$$+\int\int_T\left|f_{n_q}(x,y)-P_{n_q}(x,y)\right|\mu(x,y)dxdy<2\cdot 2^{-2q}. \tag{33}$$

Let $\overline{n}$ and $\overline{m}$ be arbitrary natural numbers. Then for some natural number q we have $N_{n_q-1}\le\min\{\overline{n},\overline{m}\}<N_{n_q}$.

Taking into account (30) and (33) for rectangular partial sums $S_{\overline{n},\overline{m}}(x,y)$ of (32) we get

$$\int\int_T\left|S_{\overline{n},\overline{m}}(x,y)-f(x,y)\right|\mu(x,y)dxdy\le\int\int_T\left|f(x,y)-\sum_{s=1}^{q}P_{n_s}(x,y)\right|\mu(x,y)dxdy+$$

$$+\max_{N_{n_q-1}\le\overline{n},\overline{m}<N n_q}\left[\int\int_T\left|\sum_{k,\nu=N_{n_q-1}}^{\overline{n},\overline{m}} c_{k,\nu}^{(n_q)}W_k(x)\cdot W_\nu(y)\right|\mu(x,y)dxdy\right]<21\cdot 2^{-2q}. \tag{34}$$

Analogously for $\sqrt{2}N_{n_q-1}\le R\le\sqrt{2}N_{n_q}$ we have

$$\int\int_T\left|S_R(x,y)-f(x,y)\right|\mu(x,y)dxdy<21\cdot 2^{-2q}, \tag{35}$$

where $S_R(x,y)$ the spherical partial sums of (32).

From (34) and (35) we conclude that the series (13) is universal in $L^1_\mu(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums (see Definition 1).

The Theorem 1.3 is proved.

Remark. Using the same as those in proof of Theorem 1.3 we obtain the proof of Theorem 1.4.

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