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Abstract. In the paper the problems of sliding modes simulation and sliding mode digital control design are considered. The simulation problem of the dynamic systems with discontinuous right-hand side is a nontraditional one. The principle difficulty is in the fact that in such systems there is a special kind of motions - sliding modes. For the purposes of sliding mode simulations a definition of a discrete sliding mode is introduced, which enables the design of discrete-time control algorithms with properties similar to those in continuous time systems with sliding-mode control algorithms.

Keywords. Sliding mode, abstract dynamic system, semigroup of operators, discrete-time system, discontinuous control algorithms.

INTRODUCTION

The sliding mode control algorithms are an efficient tool to provide the invariance of the closed loop systems to the bounded noises and their robustness with respect to parameter and structural disturbances. In addition to invariance and robustness the algorithms enable decoupling of the design problems into a set of independent subproblems of lower dimensions (Utkin, 1981).

Consider a continuous-time system

$$\dot{x} = f(x,u), x \in R^n, u \in R^m \quad (1)$$

to demonstrate the design principle for sliding mode control systems. The control components u_i are discontinuous on the surfaces in the system state space $\sigma_i = \{x : s_i(x) = 0\}$:

$$u_i = \begin{cases} u_i^+(x), & if \ s_i(x) > 0 \\ u_i^-(x), & if \ s_i(x) < 0, \ i=1,\dots,m. \end{cases} \quad (2)$$

The functions u_i^+, u_i^- and s_i are chosen in such a way that the system state reaches the intersection of the discontinuity surfaces σ_i at some time T and then state trajectories belong to this manifold.

Several problems arise when using a digital computer as a controller and simulating the continuous-time sliding modes. They are caused by the possibility to change the control signal only at the sampling instants $\delta \cdot k$ (k = 0,1,2,...), where δ is a sampling period. This leads to high frequency oscillations in the neighbourhood of the sliding manifold (socalled "zig-zag" mode).

The paper treats discrete-time implementation and simulation of sliding mode algorithms. The concept "discrete-time sliding mode" is introduced and the discrete-time dynamic systems are designed with the trajectories in the intersections of discontinuity surfaces.

DISCRETE-TIME SLIDING MODE DEFINITION

For mathematical description of continuous time sliding modes different methods of solution expantion within the set $\bigcup_i \sigma_i$ are used (Utkin, 1981).

In real life systems this motion actually takes place in some neighbourhood of $\bigcup_i \sigma_i$ and as a rule is characterized by high frequency oscillations caused by different nonidealities of the switching device delays, histeresis, unmodelled dynamics and so on. Such kind of motion is usually called real-life sliding mode. Evidently that "zig-zag" mode can be considered as a real-life sliding mode. Increasing sampling interval may lead to an inadmissibly high amplitude of the high-frequency component.

A class of discrete-time systems with continuous functions in motion equations is known to have a manifold consisting of the system trajectories with finite time for the state to reach it from some neighbourhood. (Generally speaking for continuous-time systems a manifold consisting of state trajectories may be reached only asymptotically with $t \to \infty$. Let's introduce a following definition.

Definition. We say that in the discrete-time dynamical system

$$x(k+1) = F(x(k)), x \in R^n \quad (3)$$

a discrete-time sliding mode (DSM) takes place on the subset M of a manifold

$$\sigma = \{x : s(x) = 0\} = \bigcap_i \sigma_i, \; s \in R^m \text{ if}$$

there exists an open neighbourhood U of this subset such that from $x \in U$ it follows $F(x) \in M$.

From the first sight the definition seems to be quite unnatural because the sliding mode can exist in the systems with a continuous right hand side. But the properties of that motion are similar to the sliding mode in (1), (2).

Suppose that the set $M \subset \sigma$ is the sliding set for (1), (2) (Utkin, 1981), and the expansion of the Cauchy problem solution is unique in some neighbourhood U_0.

Let x(k+1) be a solution value at the moment δ when the initial condition is equal to x(k). Thus we have a discrete-time system of the form (3) which corresponds to (1), (2). Since the solution of (1), (2) is a continuous function of its initial values - the function F(x) is continuous as well. Then as one can easily see for δ sufficiently small in such discrete-time system discrete-time sliding mode takes place should a continuous-time sliding mode occur in (1), (2).

Now let's look at the sliding modes from the theory of abstract dynamic system point of view. In order to include the systems with sliding modes to a class of dynamic ones the definition of a dynamic system should be widened. Let's remind this definition (Zubov V.I., 1973).

Definition. Let T be a numerical set such that $0 \in T$ (we associate it with a time scale).

The dynamical system in R^n is a parameter family of operators $F_t(\cdot)$ in R^n, $t \in T$ with the following properties

a) for each $x \in R^n$, $t \in T$ $F_t(x) \in R^n$ and $F_0(x) \equiv x$;

b) the value of the operator $F_t(x)$ is a continuous function of (t,x) if $t \in T$;

c) for every $t_1, t_2 \in T$ such that $t_1 + t_2 \subset T$ and $x \in R^n$

$$F_{t_1}(F_t(x)) = F_{t_1+t_2}(x) .$$

Usually it is assumed that T = R for the continuous-time systems, and T = Z (the set of integers) for the discrete-time systems. Conditions a) and c) means that $F_t(x)$ is a parameter group of transformations R^n into itself.

This definition excludes the systems described by the equations of the type (1), (2). Actually, as a rule solutions of the systems with discontinuous right hand-side do not admit time inversion (Fig. 1).

If we suppose that $T = [0, \infty)$ then the family of operators $F_t(\cdot)$ is a semigroup rather than a group. Therefore to include the systems with sliding modes into a class of dynamic systems operators $F_t(\cdot)$ should be supposed to constitute a semigroup.

In the case of discrete-time sliding modes we have the same situation. If DSM exists the function $F_t(x)$ is singular hence it does not have the inverse. In order to include such kind of systems into a class of dynamic we should suppose that T is a set of nonnegative integers (0, 1, ...) and $F_t(\cdot)$ is a semigroup.

Example. Consider the discrete-time system

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = u(k) . \tag{4}$$

If $u(k) = -c \cdot x_2(k)$, where $c > 0$ then DSM exists on the surface $s(x) = c \cdot x_1 + x_2 = 0$ in R^2. It follows from the fact that for every s (k) the value of s(k+1) = = 0. We note that the system (4) is linear and existence of DSM is equivalent to the existence of a zero pole.

If in the system (4) $u(k) = -\varphi(c \cdot x_2(k))$,

$$\varphi(z) = \begin{cases} z, & \text{if } |z| < M \\ \operatorname{sign} z, & \text{if } |z| \geq M \end{cases}$$

the DSM also takes place on the surface $s = c \cdot x_1 + x_2 = 0$ in the system with bounded control $|u| \leq M$ after finite number of steps.

SLIDING MODES SIMULATION

One of the aspects of our studies is the problem of sliding modes simulation by digital computers. We start with an example to dermonstrate that the conventional methods may prove to be inefficient.

Consider the system of the form

$$\dot{x} = -\operatorname{sign} x + f, \qquad x(0) = x_0 , \tag{6}$$

where f = const, $|f| < 1$. When $t \geq |x_0|/(1-|f|)$ there exists sliding mode in (6) and $x(t) \equiv 0$.

Using Euler procedure with the time step to simulate a solution of (6) we obtain a discrete-time system

$$x(k+1) = x(k) - \delta \cdot \operatorname{sign} x(k) + \delta \cdot f . \tag{7}$$

The plot of one of the system solution is in Fig. 2-3. It shows that the Euler procedure leads to oscillations in the origin's neighbourhood with amplitude proportional to δ.

We modify the procedure substituting sign x(k+1) for sign x(k) in the right-hand side of (7):

$$x(k+1) = x(k) - \delta \cdot \operatorname{sign} x(k+1) + \delta \cdot f .$$

Denote $x + \delta \cdot \operatorname{sign} x$ as $\alpha(x)$. The domain of $\alpha(x)$ is $R \setminus \{0\}$ and correspondingly an inverse of $\alpha(x)$ is the continuous function

$$\beta(y) = \begin{cases} y-\delta, & \text{if } y > \delta \\ 0, & \text{if } |y| \le \delta \\ y+\delta, & \text{if } y < -\delta \end{cases}$$

$$(\beta(\alpha(x)) \equiv x, \quad x \in R \setminus \{0\}).$$

Thus

$$x(k+1) = \beta(x(k) + \delta \cdot f). \qquad (8)$$

Any solution x (k) of the discrete-time system (8) with $|f| < 1$ in a finite number of steps K reaches the origin and then $x(k) \equiv 0$ for $k \geqslant K$ (Fig. 4-5).

According to the introduced above definition there exiists DSM in the system (8) and the system is adequate to (6).

The simulation of multivariable systems with scalar control does not differ essentially from this example.

Let's consider a linear system in the canonic form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\cdots\cdots\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= a_1 x_1 + \ldots + a_n x_n + bu + f \end{aligned} \qquad (9)$$

where $u = -M \cdot \operatorname{sign} s$, $s = c_1 x_1 + \ldots + c_n x_n$ $(c_n \in R \setminus \{0\})$.

Rewriting (9) with respect to variables $(x_1, \ldots, x_{n-1}, s)$ and applying the above modified Euler procedure we obtain:

$$\begin{aligned} x_1(k+1) &= x_1(k) + \delta \cdot x_2(k) \\ &\cdots\cdots\cdots\cdots\cdots \\ x_{n-1}(k+1) &= x_{n-1}(k) - \delta \cdot (c_1 x_1 + \ldots + c_{n-1} x_{n-1}) / c_n + \delta \cdot s(k)/c_n \\ s(k+1) &= (1 + \delta c_{n-1}/c_n + \delta a_n) s(k) + \delta((c_0 + c_n a_1 - c_{n-1} c_1/c_n - a_n c_1) x_1(k) + \\ &\ldots + (c_{n-2} + c_n a_{n-1} - c^2_{n-1}/c_n - a_n c_{n-1}) \times x_{n-1}(k)) - \delta c_n b M \operatorname{sign} s(k+1) + \delta c_n f, \quad (c_0 = 0). \end{aligned} \qquad (10)$$

Applying function β solve the last equation in (10) with respect to s (k+1):

$$\begin{aligned} s(k+1) = {} & \beta(s(k) + \delta \cdot c_{n-1}/c_n + a_n) s(k) + \delta((c_0 + c_n \cdot a_1 - c_{n-1} \cdot c_1/c_n - a_n \cdot c_1) \times \\ & x_1(k) + \ldots + (c_{n-2} + c_n a_{n-1} - c^2_{n-1}/c_n - a_n - c_{n-1}) \cdot x_{n-1}(k)) + \delta c_n f). \end{aligned} \qquad (11)$$

The first n-1 equations (10) and equation (11) form a discrete-time system where DSM takes place on the manifold s=0. Besides the difference equations describing this motion are exactly the Euler discretization of the sliding equations obtaining from (9) by means of the equivalent control method (Utkin, 1981).

DSM CONTROL ALGORITHMS

Let in the motion equation of a linear time-invariant plant

$$\dot{x} = A_0 x + b_0 u + f_0, \quad x \in R^n, \; u \in R^1 \qquad (12)$$

a scalar control be bounded, $|u| \le M$ and a disturbance vector $f_0 \in \operatorname{span} b_0$.

Corresponding to (12) discrete-time system (under the assumption that u is constant in the intervals $(k \cdot \delta, (k+1) \cdot \delta)$) if of form

$$x(k+1) = A \cdot x(k) + b \cdot u(k) + f(k), \qquad (13)$$

where $A = \exp\{\delta \cdot A_0\}$,

$$b = \int_0^{\delta} \exp\{A_0 \xi\} d\xi \cdot b_0, \quad f(k) = \int_{\delta \cdot k}^{\delta \cdot (k+1)} \exp\{A_0 (\delta \cdot k + \delta - \xi)\} \cdot f_0(\xi) d\xi.$$

Similarly to continuous-time systems (Utkin, 1981) let the surface $s = c' x = 0$ be chosen such that the state trajectories of (13) being confined to this surface have desirable dynamic characteristics.

The control providing the motion in the manifold s = 0 is of form:

$$u(k) = \begin{cases} M, & \text{if } v(k) > M \\ v(k), & \text{if } |v(k)| \le M \\ -M, & \text{if } v(k) < M \end{cases} \qquad (14)$$

where v(k) is a solution of s (k+1) = 0 with respect to u(k) on the trajectories of system (13)

$$v(k) = -(c'b)^{-1} \cdot (A \; x(k) + f(k)). \qquad (15)$$

In the system (13), (14), (15) there is DSM on the surface s=0. Note that due to the lack of information on the disturbance f_0 we have to use the control (14, 15) with f=0, which of course leads to the deviations from the surface s=0. These deviations are caused by the nature of discrete-time controls and can be considered as real-life DSM, but what is important that the motion is free of a high-frequency component. The solution of (12) under piecewise constant controls (14, 15) tends to the solution of the system with continuous-time sliding mode on s=0 when $\delta \longrightarrow 0$, which is invariant to disturbances if the condition $f_0 \in \text{span } b_0$ is fulfilled.

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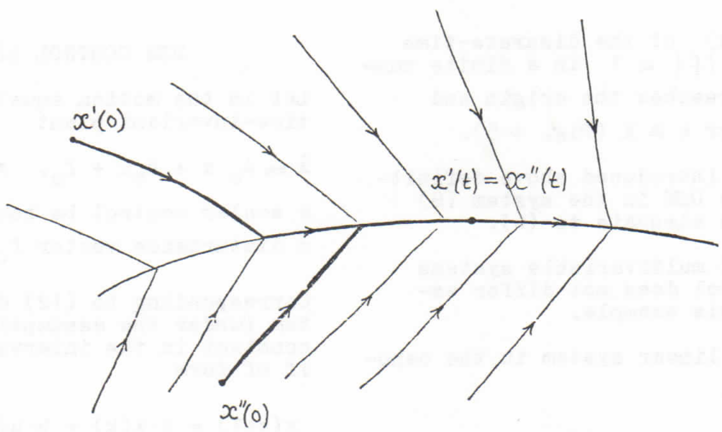

Fig. 1

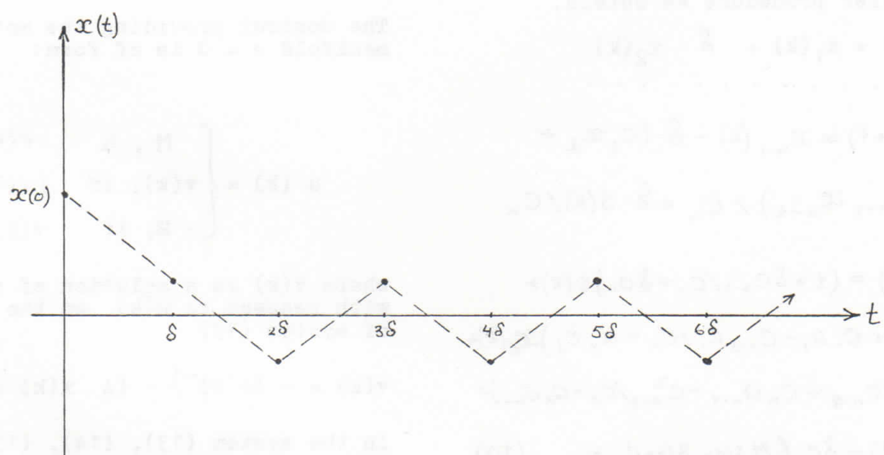

Fig. 2

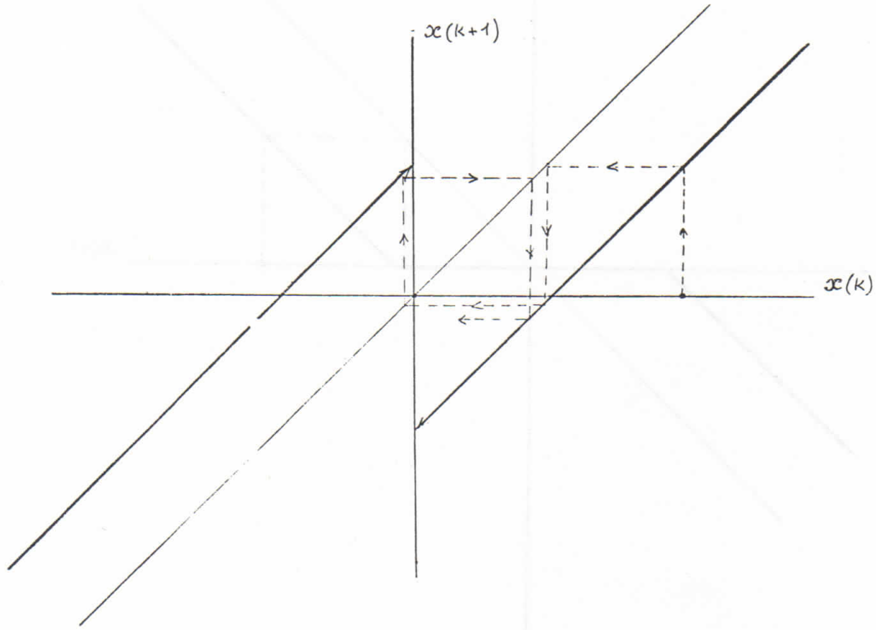

Fig. 3.

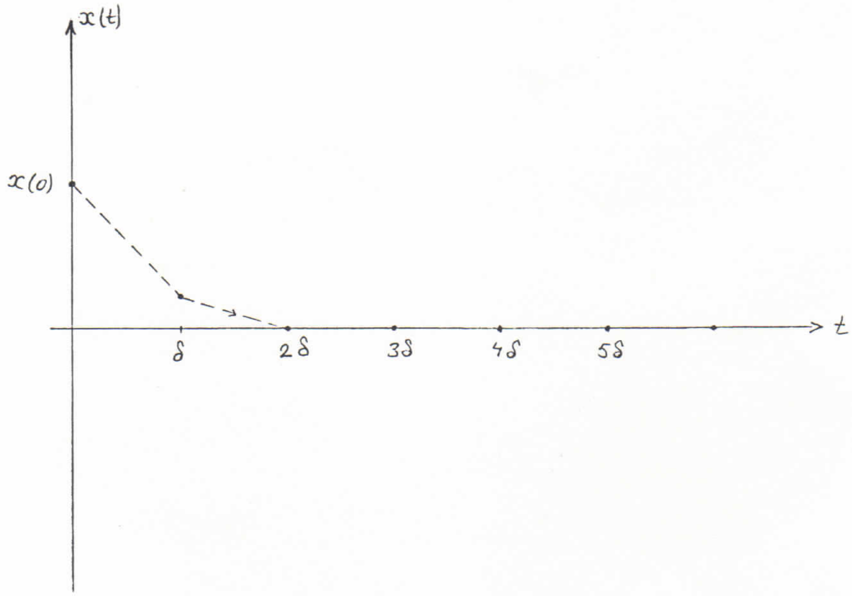

Fig. 4

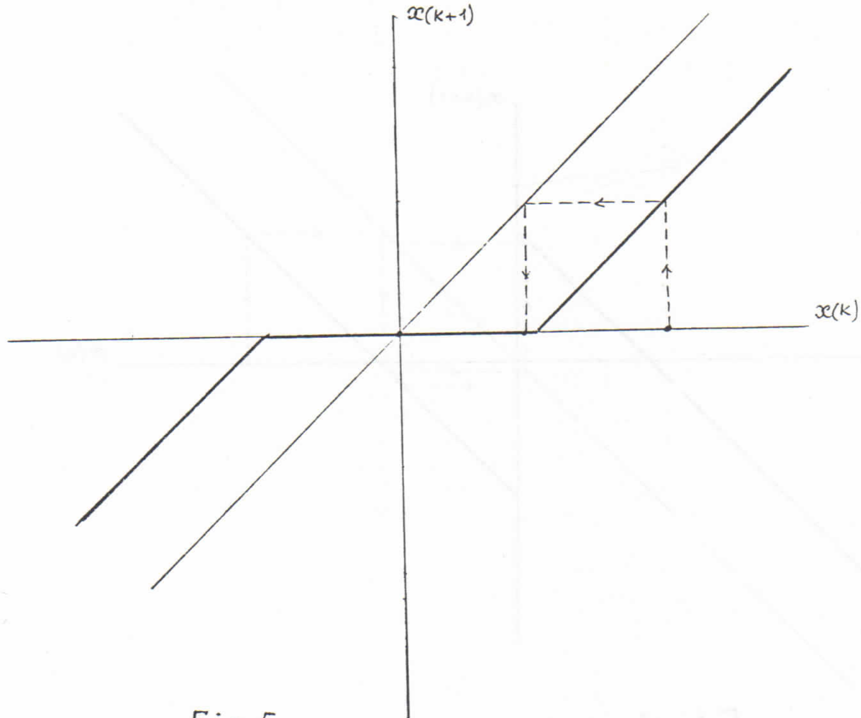

Fig. 5