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Scott N. Kersey, *Georgia Southern University*

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INVERTIBILITY OF SUBMATRICES OF PASCAL'S MATRIX AND BIRKHOFF INTERPOLATION

SCOTT N. KERSEY*

Abstract. The infinite (upper triangular) Pascal matrix is $T = [\binom{j}{i}]$ for $0 \le i, j$. It is easy to see that that submatrix $T(0:n, 0:n)$ is triangular with determinant 1, hence in particular, it is invertible. But what about other submatrices $T(r, x)$ for selections $r = [r_0, \ldots, r_d]$ and $x = [x_0, \ldots, x_d]$ of the rows and columns of T? The goal of this paper is provide a necessary and sufficient condition for invertibility based on a connection to polynomial interpolation. In particular, we generalize the theory of Birkhoff interpolation and Pölya systems, and then adapt it to this problem. The result is simple: $T(r, x)$ is invertible iff $r \le x$, or equivalently, iff all diagonal entries are nonzero.

Key words. Pascal's Matrix, Birkhoff Interpolation

AMS subject classifications. 05A10, 15A15.

1. Introduction.

The infinite and order n upper triangular Pascal matrices are

$$T = \left[\binom{j}{i}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad T(0:n, 0:n) = \begin{bmatrix} 1 & 1 & 1 & \cdots & \binom{n}{0} \\ 0 & 1 & 2 & \cdots & \binom{n}{1} \\ 0 & 0 & 1 & \cdots & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n} \end{bmatrix},$$

for $i, j = 0, 1, \ldots$, with $\binom{j}{i} := 0$ if $i > j$. Note that $T(0:n, 0:n)$ can be viewed as a submatrix, or *truncation*, of T determined by choosing the first $n+1$ rows and columns. General truncations are determined by selecting arbitrary (finite) combinations of the rows and columns of T. These can be represented as

$$T(r, x) = \left[\binom{x_j}{r_i}\right] = \begin{bmatrix} \binom{x_0}{r_0} & \binom{x_1}{r_0} & \cdots & \binom{x_d}{r_0} \\ \binom{x_0}{r_1} & \binom{x_1}{r_1} & \cdots & \binom{x_d}{r_1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{x_0}{r_d} & \binom{x_1}{r_d} & \cdots & \binom{x_d}{r_d} \end{bmatrix}$$

for some selections $r = [r_0, \ldots, r_d]$ and $x = [x_0, \ldots, x_d]$ of the rows and columns of T, respectively.

*Department of Mathematical Sciences, Georgia Southern Univ., Statesboro, GA 30460-8093, skersey@georgiasouthern.edu

The main issue taken up in this work is the invertibility of truncated Pascal matrices. It is trivial to see that T_n is invertible, with determinant 1. But what about the others? For example, consider

$$T([0,1,2],[1,2,5]) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 0 & 1 & 10 \end{bmatrix} \quad \text{and} \quad T([1,3,4],[1,2,5]) = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 10 \\ 0 & 0 & 5 \end{bmatrix}.$$

The second matrix, a triangular matrix with a zero on the main diagonal, clearly has a zero determinant, and so is not invertible. However, the first matrix is invertible, but is not obvious to see. More precisely, what we can say is the following:

THEOREM 1.1. *Truncated Pascal matrices are invertible iff the following equivalent conditions hold:*

- $r \leq x$ *(i.e.,* $r_i \leq x_i$ *for all* i*).*
- *There is no zero diagonal entry.*

The primary goal of this paper is to verify this result. As we do, we develop some connections between univariate polynomial interpolation. Following this, we derive expressions for the determinant of these truncated matrices, which involves a connection to multivariate polynomials. The paper is organized as follows:

1. In section (2), we use elementary matrix methods to show that $r \leq x$ is a necessary condition for invertibility.
2. In section (3), we show that the invertibility of truncated pascal matrices is equivalent to a particular two-point polynomial interpolation problem.
3. In section (4), we investigate the two-point polynomial interpolation problem in more detail. For this, we connect the problem of invertibility to the theories of Birkhoff interpolation and Pölya systems. In doing so, we develop a new construction of incidence matrices using a Boolean "sum-dot" operation for building up and trimming well-poised incidence matrices.
4. In section (5), we establish that invertibility conditions derived considered above is both necessary and sufficient. To do so, we use the machinery previously in section (4).

The connection between truncated Pascal matrices and polynomial interpolation described in sections (3) and (4) is known as Birkhoff interpolation. These latter matrices are invertible iff they satisfy a certain condition, known as the Pölya condition. This reference goes back to a paper of G. Pölya in [3], as used by J.M. Whittaker in [4], and later generalized by D. Ferguson in [2]. As shown, it turns out that the Pascal matrix is fundamentally connected to Hermite polynomial interpolation, while truncated Pascal matrices are fundamentally connected to 2-point Birkhoff interpolation problem studied by Pölya. The more general Birkhoff interpolation problem

(with more than two interpolation points) was presented by G. D. Birkhoff in [1].

2. Necessary Conditions for Invertibility of Truncated Pascal Matrices. In this section we derive very simple necessary conditions for the invertibility of truncated pascal's matrix. The proof is rather straight-forward as well. Later we show that these conditions are sufficient as well.

LEMMA 2.1. *Let A be a $n \times n$ square matrix. If $A[k:n, 1:k]$ is a zero block for some $k = 1:n$, then A is not invertible.*

Proof. The proof is by induction. If A is a 1×1 matrix with a zero on a diagonal element, i.e., $A = [0]$, then it is clearly singular. The same is obviously true for 2×2 matrices – here the possibilities are for $k = 1$ and $k = 2$, respectively:

$$\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \text{ or } \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix},$$

where $*$ indicates any number (zero or nonzero). Now, suppose that the the result is true for all $(n-1)\times(n-1)$ matrices, and consider an $n \times n$ matrix A with $A[k:n, 1:k]$ is a zero block. Let M_k be the minor deleting the first row and k-th column. Then, M_k is a matrix of size $(n-1) \times (n-1)$ with a zero block to the left and below a diagonal element. Hence, expanding the determinant along the first row gives a zero determinant. Hence, the result is true for all n. ∎

For example, consider:

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & \mathbf{0} & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Since the diagonal element $A(3,3)$ is zero, as well as the block to the left and below this element, then the following minors (expanding along the first row) each have a zero block below and to left of a diagonal element:

$$\begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, the determinant of these minors (and original matrix) is zero, by induction.

Now, we can derive a simple necessary condition for invertibility of truncated Pascal matrices.

THEOREM 2.2. *Let $r = [r_0, \ldots, r_d]$ and $x = [x_0, \ldots, x_d]$ be selections (strictly increasing sequences with $r_0 \geq 0$ and $x_0 \geq 0$) of the rows and columns, respectively, of the Pascal matrix T. If $T(r, x)$ is invertible, then $r \leq x$. I.e., $r_k \leq x_k$ for $k = 0:d$.*

Proof. Recall that the truncated Pascal matrix can be written

$$T(r,x) = \left[\binom{x_j}{r_i}\right],$$

with i ranging over the rows and j over the columns, for $i,j = 0:d$. Assume that $r_k > x_k$ for some k. Since the sequence r and x are strictly increasing, we have:

$$r_d > r_{d-1} > \cdots > r_k > x_k > x_{k-1} > \cdots > x_0.$$

Therefore, for $j \leq k \leq i$ we have $r_i > x_j$. It follows that $\binom{x_j}{r_i} = 0$ for these i and j, which gives a lower zero block below and to the left the k-th diagonal element. By lemma 2.1, $T(r,x)$ is not invertible. ∎

Noting that a diagonal element $\binom{x_k}{r_k} = 0$ is zero for some k iff $r_k > x_k$, we have the following corollary.

Corollary 2.3. *If $T(r,x)$ is invertible, then all diagonals elements are nonzero.*

In the next sections, we will prove that these conditions are sufficient for invertibility.

3. Truncated Pascal Matrices and Polynomial Interpolation. Let Λ be the sequence of functionals (data map)

$$\Lambda_\alpha = [\frac{\delta_\alpha}{0!}, \frac{\delta_\alpha D}{1!}, \ldots, \frac{\delta_\alpha D^i}{i!}, \ldots]$$

with action

$$\delta_\alpha D^i : f \mapsto f^{(i)}(\alpha),$$

and let V be the sequence of monomials

$$V = [1, x, x^2, \ldots].$$

We will be interested in the maps Λ_0 and Λ_1, when $\alpha = 0$ and $\alpha = 1$, respectively. In particular, it is straight-forward to see that $T = \Lambda^T V$. It follows that, for any selection of rows and columns of T,

$$T(r,x) = (\Lambda_1^T V)(r,x) = \Lambda_1(r)^T V(x).$$

And so, invertibility of truncated Pascal matrices is equivalent to correctness in polynomial interpolation. We state this formally in the following lemma.

Lemma 3.1. *Let $r = [r_0, \ldots, r_d]$ and $x = [x_0, \ldots, x_d]$ be selections of the rows and columns of T, respectively, with $0 \leq r_i < r_{i+1}$ and $0 \leq x_i < x_{i+1}$. Then, $T(r,x) = \Lambda_1(r)^T V(x)$.*

Let $\overline{x}$ be the complement of x in $[0:n]$, with $n \geq \max\{r(d), x(d)\}$. Typically, we choose equality. Now, consider the data map

$$\tilde{\Lambda} := [\Lambda_1(r), \Lambda_0(\overline{x})]$$

and basis

$$\tilde{V} := V(x, \overline{x}) = [t^{x_0}, \ldots, t^{x_d}, t^{\overline{x}_0}, \ldots, t^{\overline{x}_{n-d}}].$$

Then, we have the following:

LEMMA 3.2. *$T(r,x)$ is invertible iff $\tilde{\Lambda}^T \tilde{V}$ is invertible.*

Proof. As illustrated below, $\tilde{\Lambda}^T \tilde{V}$ is the block matrix

$$\tilde{\Lambda}^T \tilde{V} = [\Lambda_1(r), \Lambda_0(\overline{x})]^T V(r, \overline{x}) = \left[\begin{array}{c|c} \Lambda_1^T(r)V(x) & \Lambda_1^T(r)V(\overline{x}) \\ \hline \Lambda_0^T(x)V(x) & \Lambda_0^T(x)V(\overline{x}) \end{array}\right] = \left[\begin{array}{c|c} T(r,x) & B \\ \hline 0 & I \end{array}\right],$$

for some matrix B. Therefore, $\det(\tilde{\Lambda}^T \tilde{V}) = \det(T(r,x))$, and so $\tilde{\Lambda}^T \tilde{V}$ is invertible iff $T(r,x)$ is invertible. □

For example, let $r = [0,2,4,7]$ and $x = [1,2,5,8]$ in $[0:8]$. Then, with $\overline{x} = [0,3,4,6,7]$, we get

$$\tilde{\Lambda} = [\Lambda_1(r), \Lambda_0(\overline{c})] = \begin{bmatrix} \frac{\delta_1}{0!} & \frac{\delta_1 D^2}{2!} & \frac{\delta_1 D^4}{4!} & \frac{\delta_1 D^7}{7!} & \frac{\delta_0}{0!} & \frac{\delta_0 D^3}{3!} & \frac{\delta_0 D^4}{4!} & \frac{\delta_0 D^6}{6!} & \frac{\delta_0 D^7}{7!} \end{bmatrix}$$

and

$$\tilde{V} = V(x, \overline{x}) = \begin{bmatrix} t & t^2 & t^4 & t^8 & 1 & t^3 & t^4 & t^6 & t^7 \end{bmatrix}.$$

And so, $\tilde{\Lambda}^T \tilde{V}$ becomes:

$$\begin{array}{c} \\ \frac{1}{0!}\delta_1 \\ \frac{1}{2!}\delta_1 D^2 \\ \frac{1}{4!}\delta_1 D^4 \\ \frac{1}{7!}\delta_1 D^7 \\ \frac{1}{0!}\delta_0 \\ \frac{1}{3!}\delta_0 D^3 \\ \frac{1}{4!}\delta_0 D^4 \\ \frac{1}{6!}\delta_0 D^6 \\ \frac{1}{7!}\delta_0 D^7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} t & t^2 & t^5 & t^8 & 1 & t^3 & t^4 & t^6 & t^7 \end{array} \\ \left[\begin{array}{cccc|ccccc} \binom{1}{0} & \binom{2}{0} & \binom{5}{0} & \binom{8}{0} & \binom{0}{0} & \binom{3}{0} & \binom{4}{0} & \binom{6}{0} & \binom{7}{0} \\ \binom{1}{2} & \binom{2}{2} & \binom{5}{2} & \binom{8}{2} & \binom{0}{2} & \binom{3}{2} & \binom{4}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{1}{4} & \binom{2}{4} & \binom{5}{4} & \binom{8}{4} & \binom{0}{4} & \binom{3}{4} & \binom{4}{4} & \binom{6}{4} & \binom{7}{4} \\ \binom{1}{7} & \binom{2}{7} & \binom{5}{7} & \binom{8}{7} & \binom{0}{7} & \binom{3}{7} & \binom{4}{7} & \binom{6}{7} & \binom{7}{7} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

.

The connection established in the previous lemma, will be used to determine necessary and sufficient conditions for the invertibility of truncated Pascal triangles by considering a two-point interpolation problem. We describe the interpolation problem in the next section.

4. Birkhoff Interpolation and the Pölya condition. As it turns out, the kind of interpolation needed here to solve this problem is what is called (2-point) Birkhoff interpolation. In general, Birkhoff interpolation concerns when functionals of the form $\delta_{x_i} D^{n_j}$ are linearly independent on polynomial spaces. For example, we can interpolate a quadratic polynomial with the data $f(0)$, $f'(0)$ and $f(1)$, but not with $f(0)$, $f'(1)$ and $f(2)$. Here, we are interested in the two point Birkoff interpolation problem, with $x_0 = 0$ and $x_1 = 1$. To solve this problem (in the more general setting) Ferguson ([2]) used *incidence matrices.*

Definition 4.1. *An* incidence matrix E *for 2-point interpolation problems on* Π_n *is a* $2 \times (n+1)$ *matrix*

$$E := \begin{bmatrix} e_{ij} \end{bmatrix} = \begin{bmatrix} e_{00} & e_{01} & \cdots & e_{0m} \\ e_{10} & e_{11} & \cdots & e_{1m} \end{bmatrix}$$

of ones and zeros, with exactly $n+1$ *ones. I.e.,* $e_{ij} = 1$ *or* 0 *for all* $i = 0:1$ *and* $j = 0:n$*, and* $\sum_{i=0}^{1} \sum_{j=0}^{n} e_{ij} = n+1$.

For example, the following are incidence matrices, each of dimension 2×6 with exactly 6 ones:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Here, $n = 5$. As in [2], we define M_j to be the cumulative column sum

$$M_j := \sum_{i=0}^{1} \sum_{k=0}^{j} e_{ik},$$

for $j = 0:n$. In the previous examples, $M = [1, 2, 3, 5, 5, 6]$ and $M = [0, 2, 3, 3, 5, 6]$. Note that $M_5 = 6$ is the total number of ones in both examples. In general, $M_n = n+1$ for any incidence matrix. The problem of 2-point Birkhoff interpolation problems was studied by Pölya. With respect to the incidence matrices (of any size), the following defintion is used.

Definition 4.2 ([2]). *The incidence matrix* E *satisfies the* Pölya condition *if* $M_j > j$ *for* $j = 0:m$.

So, in the above two examples, the first matrix is Pölya, however the second is not because $M_0 = 0 < 1$ and $M_3 = 3 < 4$. The following result, proved independently by Pölya and Whittaker (as also described in [2]), gives necessary and sufficient conditions for correct interpolation.

Theorem 4.3 (adapted from [3], [4]). *Let* E *be a* $2 \times (n+1)$ *incidence matrix with entries* e_{ij}*, ones or zeros. Let*

$$\Lambda := [\delta_{t_i} D^j : e_{ij} = 1]$$

and $V^n = [1, t, \ldots, t^n]$. Then, the system $\Lambda_n^T V^n$ is invertible iff E satisfies the Pölya condition.

We now establish some results concerning the boolean algebra of incidence matrices. The Boolean operations of multiplication (intersection), addition (union) and conjugate (complement) are defined by

$$\begin{aligned} 1 \cdot 1 = 1 \text{ and } 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \\ 0 + 0 = 0 \text{ and } 1 + 0 = 0 + 1 = 1 + 1 = 1, \\ \overline{0} = 1 \text{ and } \overline{1} = 0, \end{aligned}$$

respectively. Then, for Boolean matrices E_1 and E_2, we define

$$E_1 \dot{+} E_2 = \begin{bmatrix} E_1(1,:) + E_2(1,:) \\ E_1(2,:) \cdot E_2(2,:) \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \dot{+} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The identity I such that $E = E \dot{+} I = I \dot{+} E$ for all $2 \times (n+1)$ Boolean matrices is:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Moreover, for any Boolean matrix $E = E \dot{+} E$.

Lemma 4.4. *Let E_1 and E_2 be two $2 \times (n+1)$ incidence matrices. Then $E = E_1 \dot{+} E_2$ is an incidence matrix iff*

$$|E_1(1,:) \cdot E_2(1,:)| = |\overline{E_1(2,:)} \cdot \overline{E_2(2,:)}|.$$

Moreover, E is Pölya if both E_1 and E_2 are Pölya.

Proof. Since E_1 is an incidence matrices, we have $(n+1) = |E_1| = E_1(1,:)| + |E_1(2,:)|$, and likewise for E_2. Then,

$$\begin{aligned} |E| &= |E_1(1,:)| + |E_2(1,:)| - |E_1(1,:) \cdot E_2(1,:)| + (n+1) \\ &\quad - \big(|\overline{E_1(2,:)}| + |\overline{E_2(2,:)}| - |\overline{E_1(2,:)} \cdot \overline{E_2(2,:)}|\big) \\ &= |E_1(1,:)| + |E_2(1,:)| + (n+1) - \big(|\overline{E_1(2,:)}| + |\overline{E_2(2,:)}|\big) \quad \text{iff hypothesis} \\ &= |E_1(1,:)| + |E_2(1,:)| + (n+1) - \big((n+1) - |E_1(2,:)| + (n+1) - |E_2(2,:)|\big) \\ &= |E_1(1,:)| + |E_1(2,:)| + |E_2(1,:)| + |E_2(2,:)| - (n+1) \\ &= (n+1) + (n+1) - (n+1) = n+1. \end{aligned}$$

For the second part, we first note that if E_1 and E_2 are both Pölya, then with

$$M_0 = M_0^1 + M_0^2 = |e_{00}^1 + e_{00}^2| + |e_{10}^1 \cdot e_{10}^2|$$

either one of e_{00}^k is 1, or otherwise both of e_{10}^k are 1 since E_1 and E_2 are Pölya. Proceeding inductively... $\square$

The converse to the last statement is not true. That is, E Pölya does not necessarily imply that E_1 and E_2 are Pölya. For example, consider

$$\begin{aligned} E &= \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \dot{+} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \dot{+} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

In the first decomposition of E, the two matrices are Pölya, and in the second they are not. Therefore, there are at least some Pölya matrices that can be broken down into a sum-dot of two incidence matrices, and these may or may not be Pölya. But the question here can one always express incidence matrices as the sum of two incidence matrices with less ones in the first row, and can one do it with Pölya matrices. We will answer this question with a constructive proof. Suppose that E is an incidence matrix, and suppose also that $|E(1,:)| \geq 2$ (i.e., there are at least two ones in the first row. Then define E_1 and E_2 as follows:

Construction: Let $d := |E(1,:)|$. Take E_1 and E_2 equal to E for all elements except the following:

1. For E_1, exchange the first one in row one of E with the first zero in row two.
2. For E_2, exchange the last $d-1$ ones in row one of E with the last $d-1$ zeros in row two.

In the extreme case, when $d = 0$, we get $E_1 = E_2 = I$. Consider two other examples:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \boxed{0} & 0 & 0 & 0 \\ 1 & 1 & 1 & \boxed{1} & 1 & 1 & 1 \end{bmatrix} \dot{+} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & \boxed{1} & 0 & 1 & 0 \end{bmatrix} \dot{+} \begin{bmatrix} 1 & 0 & 0 & 0 & \boxed{0} & \boxed{0} & 0 \\ 1 & 1 & 1 & 0 & \boxed{1} & 1 & \boxed{1} \end{bmatrix}. \blacksquare$$

All matrices are Pölya. The entries that were exchanged from the original E are indicated in boxes. In the first example, since $d = 1$ we get $E_1 = I$ and $E_2 = E$ (i.e.,

there are $d-1=0$ entries exchanged in the second). Note also that $|E(1,:)| = |E_1(1,:)| + |E_2(1,:)|$, which will be useful to us later. In general, we can say the following:

PROPOSITION 4.5. *Let E be an incidence matrix. Suppose that E_1 and E_2 are constructed as above. Then,*

1. $E = E_1 \dot{+} E_2$
2. *E_1 and E_2 are incidence matrices.*
3. *If E is Pölya, then E_1 and E_2 are Pölya.*
4. $|E(1,:)| = |E_1(1,:)| + |E_2(1,:)|$
5. $|\overline{E(2,:)}| = |\overline{E_1(2,:)}| + |\overline{E_2(2,:)}|$
6. *If $m=0$, $E = E_1 = E_2 = I$.*
7. *If $m=1$, $E_1 = I$ and $E_2 = E$.*
8. *If $m>1$, E, E_1 and E_2 are distinct.*

THEOREM 4.6. *Every incidence (Pölya) matrix can be written as a sum-dot of two incidence (Pölya) matrices. If $m \geq 2$, the sum can be non-trivial (E, E_1 and E_2 are different matrices), and one can have $|E_1(1,:)| < |E(1,:)|$ and $|E_2(1,:)| < |E(1,:)|$.*

Let $\overline{x}$ be the complement of x in $[0:n]$. That is, if $x = [2,4,5]$ with $n=6$, then $\overline{x} = [0,1,3,6]$. We define $E_{r,\overline{x}}$ the $2 \times n+1$ matrix with $E_{r,\overline{x}}(0,j) = 1$ if $r_i = 1$, and $E_{r,\overline{x}}(2,j) = 1$ if $\overline{x}_i = 1$, with all other entries 0.

LEMMA 4.7. *Let $E_{r,\overline{x}}$ be a $2 \times (n+1)$ matrix as defined above. Then, it is an incidence matrix.*

Proof. We need to show that there are exactly $n+1$ ones (hence, $n+1$ zeros). We denote this as $|E_{r,x}|$. But this is easy, since

$$|E_{r,x}| = |r| + |\overline{x}| = |r| + (n+1) - |c| = (d+1) + (n+1) - (d+1) = n+1.$$

□

Next, we want to consider combining incidence matrices in certain ways, i.e., by addition. That is

5. Sufficient Conditions for Invertibility of Truncated Pascal Matrices. We are now ready to determine conditions for the invertibility of truncated Pascal matrices. We provide two proofs, both involving Pölya systems. The first using an inductive argument along with the sum-dot decomposition described in the previous section, and the second is longer but more direct, using a counting argument.

THEOREM 5.1. *Let $r = [r_0, \ldots, r_d]$ and $x = [x_0, \ldots, x_d]$ be selections (increasing, 1-1 maps $[0:d] \to [0:n]$) of the rows and columns, respectively, of the Pascal matrix T, for some $d \leq n$. The matrix $A := T(r,x)$ is invertible iff $r \leq x$ (i.e., $r_i \leq x_i$ for $i = 0:d$).*

Proof. Proof 1: Fix n. Assume by way of induction that the result is true up to $d-1$ (i.e., $0+d-1$). Let $r_0, \ldots, r_d$ and $x_0, \ldots, x_d$ as described in the theorem. Define the matrix with first row $E(1, r(i)) = 1$ for $i = 0 : d$, all other entries zero, and second row $E(2, x(i)) = 0$, all other entries 1. Then, there are $d+1+(n-d) = n+1$ ones. Hence, this is an incidence matrix. The first row corresponds to $\delta_1 D^j$, and the second row $\delta_0 D^j$. Now, define incidence matrices E_1 and E_2 as in the construction given earlier, so that $E = E_1 \dot{+} E_2$, with $|E_1(1,:)| = d$ and $|E_2(1,:)| = 1$. Assume that $r_k \leq x_k$ for all k. Then, the same is true of the submatrices. By the inductive hypothesis, E_1 and E_2 are Pölya. By Lemma 4.4, E is Pölya. Therefore $\tilde{\Lambda}_n V_n$ is invertible iff $r \leq x$. By lemma 3.2, $T(r, x)$ is also invertible iff $r \leq x$

Proof 2: Our strategy in proving this theorem is to expand $T(r, x)$ to an $(n+1) \times (n+1)$ matrix B by adding diagonal entries where we want the coefficient of t^j to be zero. Then, we show that $B(r, x) = T(r, x)$, and that B is invertible iff $B(r, x)$ is invertible.

Let $\overline{x}$ be the complement of x in $[0 : n]$, a vector of length $n-d$, and let E be the $2 \times (n+1)$ incidence matrix with $E(0, \overline{x}_j) = 1$ for $j = 0 : n-d$ and $E(1, r_j) = 1$ for $j = 0 : d$, with all other entries 0. For example, suppose that $n = 6$, $x = [1, 3, 5]$ and $r = [1, 5, 6]$. Then with $\overline{x} = [0, 2, 4, 6]$, we get

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In particular, since $\#r = \#c$, the total number of ones is

$$M_n = \#r + \#\overline{x} = (d+1) + (n+1) - (d+1) = n+1,$$

as required. Now, we have

$$\sum_{j=0}^{r_k - 1} e_{1j} = k,$$

and

$$\sum_{j=0}^{c_k} e_{0j} = x_k - k.$$

Therefore, if $r_k > x_k$ for some k, then we have

$$\begin{aligned} M_{r_k-1} &= \sum_{j=0}^{r_k-1} e_{0j} + \sum_{j=0}^{r_k-1} e_{1j} \\ &\leq \sum_{j=0}^{c_k} e_{0j} + (r_{k-1} - x_k) + \sum_{j=0}^{r_k-1} e_{1j} \\ &= (x_k - k) + (r_{k-1} - x_k) + k \\ &= r_{k-1}. \end{aligned}$$

However, the Pölya condition requires that $M_{r_k-1} \geq r_k$. Hence, the system is not Pölya when $r_k > x_k$ for some k.

Now, suppose that $r_k \leq x_k$ for all k. Let $j \in [0:n]$. Then, let k_0 be the highest k such that $x_k \leq j$. Then, $r_{k_0} \leq x_{k_0} \leq j$. And so,

$$\begin{aligned} M_j &= \sum_{k=0}^{j} e_{0k} + \sum_{k=0}^{j} e_{1k} \\ &\geq \sum_{k=0}^{j} e_{0k} + \sum_{k=0}^{r_{k_0}} e_{1k} \\ &= \sum_{k=0}^{x_{k_0}} e_{0k} + (j - x_{k_0}) + \sum_{k=0}^{r_{k_0}} e_{1k} \\ &= (x_{k_0} - k_0) + (j - x_{k_0}) + (k_0 + 1) \\ &= j + 1. \end{aligned}$$

From this we conclude that E is Pölya iff $r_k \leq x_k$ for all k.

Now, let

$$\Lambda_n := [\delta_i D^j : e_{ij} = 1].$$

By the above theorem, the system

$$H_n = \Lambda_n^T V^n$$

is invertible iff E satisfies the Pölya condition. That is, we can say $H_n\alpha = 0 \implies \alpha = 0$ iff E is Pölya. But if $\lambda_i = \delta_0 D^j$ for some functional in Λ, this implies that $\alpha_j = 0$. Such coefficients correspond to the first row of E where there ones, i.e., those entries in $\overline{x}$, Therefore, we conclude that $H_n\alpha = 0 \implies \alpha = 0$ iff $H_n[r,x]\beta = 0 \implies \beta = 0$. Therefore, H_n is invertible iff $H_n[r,x]$ is invertible. Since $H_n = T_n D_n$, we get that

$T(r,x)$ is invertible iff H_n is invertible, which is true iff E is Pölya, which is true iff $r_k \leq x_k$ for all k. This proves the claim. $\square$

COROLLARY 5.2. *The matrix $T(r,x)$ is invertible iff all diagonal elements are nonzero.*

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