Winter December 30, 2017

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PROGRESSIVE HEDGING LOWER BOUNDS FOR TIME CONSISTENT RISK-AVERSE MULTISTAGE STOCHASTIC MIXED-INTEGER PROGRAMS

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Abstract

Risk-averse models have attracted attention in stochastic programming in situations where the decision maker is more concerned about large losses than average performance. When the risk-averse stochastic program is multistage, however, one key issue of time consistency arises. While definitions of time consistency vary, overall this property multistage stochastic programs is not guaranteed and depends on how the risk measure is computed. Expected conditional risk measures, which extend single-period risk measures to multiple stages, have been proved to be time consistent by Homem-de-Mello and Pagnoncelli (2016) according to their definition and, for some risk measures, allow for risk-neutral reformulations. . We propose scenario-decomposed versions of these risk-neutral formulations for a variety of risk measures and present an approach to obtain convergent and tight lower bounds from the Progressive Hedging (PH) algorithm. For mixed-integer programs where convergence is not guaranteed, this method can assess the quality of PH solutions and also integrate with exact algorithms that rely on lower bounds. We report computational results on financial portfolio optimization, lot sizing and a realistic-scale generation expansion planning instance and show that convergent and tight lower bounds are found.

Keywords: Risk-averse stochastic optimization; Scenario decomposition; Progressive Hedging algorithm; Time consistency; Expected conditional risk measures; Lower bounding

1. Introduction

Traditional stochastic programming is risk-neutral in the sense that it is concerned with the optimization of an expectation criterion. This may yield solutions that are good in the long run over repeated instances. But for non-repetitive decision making problems under uncertainty, the classical stochastic programming approach may perform poorly under certain realizations of the uncertain parameters. Thus, risk-averse models have attracted attention in the stochastic programming literature.

The two-stage risk-averse stochastic program can extend from the risk-neutral model in a straightforward way (Schultz 2006). In the multistage case, however, the picture is quite different and there is no natural or obvious way of measuring risk (Homem-de-Mello and Pagnoncelli 2016). Time consistency is an important issue in modeling multistage risk-averse models. Risk-neutral stochastic programs are time consistent, which means the solutions for later stages found originally remain optimal if the probem is resolved in the later stages (Pflug and Pichler 2016). In general, time consistency for multistage risk-averse stochastic programs does not hold true.

Significant efforts have been made to achieve time consistency for multistage risk-averse problems. One popular way is to adopt the nested conditional risk measure proposed by Shapiro and Ruszczynski (2006), which has a drawback that the problem must be solved according to the recursive Bellman equations. Homem-de-Mello and Pagnoncelli (2016) extend single-period coherent risk measures to a class of multi-period risk measures called expected conditional risk measures (ECRMs) and show they are time consistent according to their broader definition, which allows for multiple optimal solutions. One advantage of ECRMs is that its resulting risk-averse problem can be reformulated as a risk-neutral model with some additional variables and constraints.

In this paper, we show that the risk-neutral reformulations of several ECRMs are scenario decomposable. The resulting scenario formulations enable the use of existing scenario decomposition approaches such as the progressive hedging (PH) algorithm to efficiently solve the risk-averse problems.

The PH algorithm is a scenario decomposition method developed by Rockafellar and Wets (1991) for stochastic programs with continuous decision variables. It has been explored by Watson and Woodruff (2011) as an effective heuristic for solving stochastic mixed-integer programs. Gade et al. (2016) presented a lower bounding technique for the PH algorithm and showed that, in two-stage convex problems, the best lower bound obtained from PH algorithm is as tight as the lower bound obtained from using Dual Decomposition developed by Caroe and Schultz (1999). In many applications where computational efficiency is valued, near-optimal solutions are desired within a reasonable amount of computation time. This lower bounding approach can assess solution quality in any iteration of the PH algorithm and can also integrate with exact algorithms that rely on lower bounds (Guo et al. 2015). This lower bounding technique, however, is restricted to risk-neutral models.

In this paper, we show how to obtain PH lower bounds for time-consistent multistage risk-averse stochastic integer programs with scenario-decomposable ECRMs. In the case of expected conditional value-at-risk, the optimization problems solved to obtain the bounds may be unbounded. . To overcome this hindrance, we find bounds for the optimal values of the additional decision variables introduced to obtain the risk-neutral reformulation, which also help speed up the convergence of PH algorithm. Our numerical results show that convergent and tight lower bounds are found.

2. Multistage stochastic mixed-integer program

Suppose T is the number of stages. We denote the uncertain parameters by $\xi = (\xi_2, ..., \xi_T)$, whose probability distributions are known. The decision vectors are represented as $x = (x_1, x_2, ..., x_T)$. The realization of ξ_t at stage $t = 2, ..., T$ is known only when decisions x_{t-1} have been made. The history of the data process up to stage t is denoted as $\xi_{[t]} \equiv (\xi_2, ..., \xi_t)$. The

decisions and realizations are sequenced as
$x_1, \xi_2, x_2(x_1, \xi_2), \xi_3, x_3(x_1, x_2, \xi_{[3]}), \ldots, x_T(x_1, \ldots, x_{T-1}, \xi_{[T]})$.

We write the risk-neutral multi-stage stochastic mixed-integer program as

$$z = \min_{x_1, \ldots, x_T} \left\{ c_1^T x_1 + \mathbb{E}_{\xi_2}[Q_2(x_1, \xi_2)] : Ax_1 = b, x_1 \in \mathbb{Z}_+^{p_1} \times \mathbb{R}_+^{n_1 - p_1} \right\} \tag{1}$$

For $t = 2, \ldots T$, $Q_t(x_{t-1}, \xi_{[t]})$ is defined recursively as

$$\begin{aligned} &Q_t(x_{t-1}, \xi_{[t]}) = \min_{x_t} \left\{ c_t^T(\xi_{[t]}) x_t + \mathbb{E}_{\xi_{t+1}|\xi_{[t]}}[Q_{t+1}(x_t, \xi_{[t+1]})] \right\} \\ &T_t(\xi_{[t]}) x_{t-1} + W_t(\xi_{[t-1]}) x_t = h_t(\xi_{[t]}) \\ &x_t \in \mathbb{Z}_+^{p_t} \times \mathbb{R}_+^{n_t - p_t} \end{aligned}$$

Here $c_1 \in \mathbb{R}^{n_1}, b \in \mathbb{R}^{m_1}$ and $c_t(\xi) \in \mathbb{R}^{n_t}, h_t(\xi) \in \mathbb{R}^{m_t}$ are given vectors, while $A \in \mathbb{R}^{m_1 \times n_1}$ and $T_t(\xi) \in \mathbb{R}^{m_t \times n_{t-1}}, W_t(\xi) \in \mathbb{R}^{m_t \times n_t}$ are given matrices. The sets $x_t \in \mathbb{Z}_+^{p_t} \times \mathbb{R}_+^{n_t - p_t}$ denote the integer requirements on the variables at each time stage. The decisions are non-anticipative in the sense that a decision can depend on information revealed before the stage but not after.

The notation $\mathbb{E}_{\boldsymbol{\xi}}$ denotes expectation with respect to the distribution of random variable $\boldsymbol{\xi}$. To avoid complications when computing the integral behind $\mathbb{E}_{\boldsymbol{\xi}}$ we assume that we have only a finite number of realizations ξ with corresponding probabilities p_ξ. Let n_t be a scenario node that belongs to the set of all scenario tree nodes N_t at stage $t \in T$. Let $\xi(n_t)$ be a scenario that belongs to the set of scenarios $\Xi(n_t)$ that define the node $n_t \in N_t$. Let $n_t(\xi)$ be the corresponding tree node for scenario $\xi \in \Xi$ at stage $t \in T$. Let $\hat{x}(n_t(\xi))$ be the non-anticipative decision made at scenario tree node $n_t(\xi)$. Then problem (1) is decomposable by scenario and can be written as its so-called scenario reformulation of the multistage stochastic mixed-integer program:

$$\begin{aligned} &\min_{x_1, \ldots, x_T} \left\{ \sum_{\xi \in \Xi} p_\xi \left[c_1^T x_1(\xi) + \sum_{t=2}^{T} q_t(n_t(\xi))^T x_t(\xi) \right] : \right. \\ &\left. x_t(\xi) \in X_\xi, p_\xi x_t(\xi) - p_\xi \hat{x}(n_t(\xi)) = 0, \forall \xi \in \Xi, \forall t = 1, \ldots, T \right\} \end{aligned} \tag{2}$$

where

$$X_\xi = \left\{ \begin{array}{l} x_t(\xi) : Ax_1(\xi) = b, x_t(\xi), \hat{x}(n_t(\xi)) \in \mathbb{Z}_+^{p_1} \times \mathbb{R}_+^{n_1 - p_1}, \forall t = 1, \ldots, T \\ T_t(n_t(\xi)) x_{t-1}(\xi) + W_t(n_{t-1}(\xi)) x_t(\xi) = h_t(n_t(\xi)), \forall \xi \in \Xi, \forall t = 2, \ldots, T \end{array} \right\}$$

The above problem (2) can decompose into scenario sub-problems

$\min_{x_1,\ldots,x_T}\left\{c_1^T x_1(\xi)+\sum_{t=2}^{T} q_t\left(n_t(\xi)\right)^T x_t(\xi): x_t(\xi)\in X_\xi, \forall t=1,\ldots,T\right\}$ for scenarios $\forall \xi \in \Xi$ which are coupled by the non-anticipativity constraints $p_\xi x_t(\xi)-p_\xi \hat{x}\left(n_t(\xi)\right)=0$.

3. Risk measures

We distinguish between two classes of risk measures according to whether they are defined via quantiles or via deviation measures. Quantile risk measures are based on the quantiles of the probability distributions of the costs. Types of quantile based risk-measures include conditional value-at-risk (CVaR), which measures the expectation of worst outcomes for a given probability; and excess probability (EP), which measures the probability of exceeding a prescribed target level. Deviation risk measures are given by expectations of deviations of the relevant random variable from its mean or from some prescribed target. Examples of deviation based risk-measures include expected excess (EE), which measures the expected value of the excess over a given target; and semi-deviation (SD), which measures the expected value of the excess over the mean. We use the definitions and notations for two-stage problems from Schultz (2006) where $f(x(\xi),\xi)$ is the objective function for a two-stage stochastic program.

Definition 3.1. *The* α – conditional value-at-risk (α – *CVaR) reflects the expectation of the* $(1-\alpha)\cdot 100\%$ *worst outcomes for a given probability level* $\alpha \in (0,1)$*, and can be expressed by the following minimization formula:*

$$Q_{\alpha-CVaR}(x)=\min_{\eta\in\mathbb{R}} g(\eta,x), \tag{3}$$

where $g(\eta,x):=\eta+\frac{1}{1-\alpha}\mathbb{E}\left[\max\left\{f(x(\xi),\xi)-\eta,0\right\}\right]$.

Definition 3.2. *Excess probability (EP) is the probability of exceeding a prescribed target level* $\beta\in\mathbb{R}$ *, and is defined as:*

$$Q_{\mathbb{P}^\beta}(x)=\mathbb{P}\left[\xi\in\Xi: f(x(\xi),\xi)>\beta\right]. \tag{4}$$

Definition 3.3. *Expected excess (EE) reflects the expected value of the excess over a given target* $\gamma\in\mathbb{R}$ *, and is defined as:*

$$Q_{D^\gamma}(x)=\mathbb{E}\left[\max\left\{f(x(\xi),\xi)-\gamma,0\right\}\right]. \tag{5}$$

Definition 3.4. *Semi-deviation (SD) is similar in spirit to the expected excess, but with the prefixed target replaced by the mean, and is defined as:*

$$Q_{D^+}(x)=\mathbb{E}\left[\max\left\{f(x(\xi),\xi)-Q_{\mathbb{E}}(x),0\right\}\right]. \tag{6}$$

4. Time consistency of risk-averse multistage stochastic programs

4.1 Risk-averse multistage stochastic programs

Risk-averse models have attracted considerable attention in stochastic programming in situations where the decision maker is more concerned about large losses than average performance. Risk aversion is addressed by replacing the expectation in traditional stochastic programs with risk measures to identify the best decisions. Another way to handle risk is to include risk measures in the constraints with important applications, such as in portfolio optimization with CVaR constraints. Krokhmal, Palmquist, and Uryasev (2002) is the first paper to deal with optimization approach with CVaR constraints. Fabian (2008) studies and proposes solution schemes for two-stage CVaR-minimization and CVaR-constrained problems. Guigues and Sagastizábal (2013) propose a risk-averse rolling-horizon time consistent approach with CVaR constraints. In this paper, however, we focus on the stochastic programs to handle risk measures in the objective.

For two-stage stochastic programs, the risk-averse model extends immediately from the risk-neutral model by replacing the expectation of the second stage cost with some risk measure. Ahmed (2006) provides two classes of mean-risk two-stage stochastic linear programs involving the semi-deviation risk measure and the quantile deviation risk measure, and proposes a decomposition algorithm. Schultz and Tiedemann (2006) present a mixed-integer programming formulation of a two-stage stochastic mixed-integer model involving CVaR. Miller and Ruszczynski (2011) formulate a risk-averse two-stage stochastic linear programming problem with a composition of conditional risk measures and demonstrate improvement over the existing decomposition approaches. Noyan (2012) develops two Benders-based decomposition algorithms for a risk-averse two-stage stochastic linear programming model with a CVaR risk measure.

When it comes to multistage models, however, there is no natural way of measuring risk, asrisk measures can be applied at every stage additively or to the complete scenario path or be measured in a nested form (Homem-de-Mello and Pagnoncelli 2016). The challenges in extending risk measures to the multistage case have been discussed extensively. Collado and Papp (2011) introduce a scenario decomposition method for risk-averse multistage stochastic linear programs by using the dual properties of dynamic measures of risk.

4.2 Time consistency

The definitions of time consistency differ by their focus. Some focus on the sequences of random variables (Ruszczynski 2010; Kovacevic and Pflug 2014), some are defined for continuous time dynamic models (Detlefsen and Scandolo 2005; Cheridito et al. 2006; Bion-Nadal 2008), while others take the point of view of optimization and decision making at every stage (Shapiro 2009; Carpentier et al. 2012; Rudloff et al. 2014). Here, we are most interested in time consistency for multistage stochastic programs. Shapiro (2009) claims that for time consistency of a problem, the solution at a node in the scenario tree must not depend on children of other nodes. Carpentier et al. (2012) formulate the property of time-consistency such that the optimal strategies obtained

when solving the original problem remain optimal for all subsequent-stage problems. Pflug and Pichler (2016) consider a multistage stochastic decision problem to be time consistent if, when resolving the problem at later stages, the original solutions remain optimal for those stages. Homem-de-Mello and Pagnoncelli (2016) define time consistency in terms of an inherited optimality property. Here we use the same definition of time consistency as Homem-de-Mello and Pagnoncelli (2016) such that given the optimal solutions from previous stages, resolving the problem results in the same solutions for the later stages if the optimal solutions are unique. If the optimal solutions are not unique, resolving the problem at the later stages gives the same optimal objective as computed by the original optimal solutions.

4.3 Time consistency for risk-averse multistage stochastic programs

Risk-neutral and two-stage risk-averse stochastic programs are time consistent. For multistage risk-averse stochastic programs, however, time consistency is not guaranteed and depends on how the risk measure is computed. The risk-averse models with risks measured at every stage separately or measured for the complete scenario path are shown to be time inconsistent (Pflug and Pichler 2016). To enforce time consistency for decision problems, significant efforts and investigations have been initiated to identify classes of time consistent multistage risk measures. Shapiro and Ruszczynski (2006) propose a nested conditional risk measure for multistage optimization problems which proves to be time consistent. The nested conditional risk measure is formulated in recursive function which is not given in explicit form. Homem-de-Mello and Pagnoncelli (2016) address this drawback by proposing a class of expected conditional risk measures (ECRMs) which prove to be time consistent. One important advantage of ECRMs is that their resulting risk-averse problem can be formulated by a risk-neutral model for a modified problem with some additional variables and constraints. We will show in the next section that the risk-averse multistage stochastic program of ECRMs, based on a variety of single-period risk measures, can be decomposed by scenario.

5. Scenario reformulation for expected conditional risk measures

Here, we will use the notations from Homem-de-Mello and Pagnoncelli (2016). Consider a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_T$ be sub sigma-algebras of $\mathcal{F}$ such that each $\mathcal{F}_t$ corresponds to the information available up to stage t, with $\mathcal{F}_1 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Let $\mathcal{Z}_t$ denote a space of $\mathcal{F}_t$-measurable functions from Ω to $\mathbb{R}$, and let $\mathcal{Z} := \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_T$. A multi-period risk function $\mathbb{F}$ is defined as a mapping from $\mathcal{Z}$ to $\mathbb{R}$.

Homem-de-Mello and Pagnoncelli (2016) define the following multi-period risk measures $\mathbb{F}$ as expected conditional risk measures (ECRMs):

$$\mathbb{F}(Z_1, \ldots, Z_T) = Z_1 + \rho_2(Z_2) + \mathbb{E}_{\xi_{[2]}}\left[\rho_3^{\xi_{[2]}}(Z_3)\right] + \cdots + \mathbb{E}_{\xi_{[T-1]}}\left[\rho_T^{\xi_{[T-1]}}(Z_T)\right] \tag{7}$$

where the subscript in $\mathbb{E}$ indicates that the expectation is taken with respect to the corresponding variables. Homem-de-Mello and Pagnoncelli (2016) prove that any $\mathbb{F}$ defined as in (7) is time consistent, provided that each $\rho_t^{\xi_{[t-1]}}$ satisfies some basic properties that automatically hold, for example, for coherent risk measures.

Homem-de-Mello and Pagnoncelli (2016) use a particular case of ECRMs with $\rho_t = CVaR_{\alpha_t}$ denoted as $\mathbb{E}$-CVaR and show that $\mathbb{E}$-CVaR has the appealing property that any risk-averse multistage stochastic program defined with $\mathbb{E}$-CVaR can be written as a risk-neutral model with some additional variables. Thus, existing algorithms can be adopted to solve the $\mathbb{E}$-CVaR problems. The other risk measures, however, are not yet investigated. Besides, the scenario decomposition of risk-averse multistage stochastic programs are not explored. In the next section, therefore, we will discuss the scenario reformulation of multistage stochastic programs with ECRMs based on various risk measures.

Homem-de-Mello and Pagnoncelli (2016) write the optimization formulation for $\mathbb{E}$-CVaR as follows:

$$\begin{aligned}
&\min_{x_1,\ldots,x_T} \Big\{ c_1^T x_1 + CVaR_{\alpha_2}\left(c_2^T(\xi_{[2]})x_2\right) + \mathbb{E}_{\xi_2}\Big[CVaR_{\alpha_3}^{\xi_{[2]}}\left(c_3^T(\xi_{[3]})x_3\right) \\
&+\mathbb{E}_{\xi_3}\Big[CVaR_{\alpha_4}^{\xi_{[3]}}\left(c_4^T(\xi_{[4]})x_4\right) + \cdots \\
&+\mathbb{E}_{\xi_{T-1}}\Big[CVaR_{\alpha_T}^{\xi_{[T-1]}}\left(c_T^T(\xi_{[T]})x_T\right) \mid \xi_{[T-2]}\Big] \cdots \mid \xi_{[2]}\Big]\Big], \\
&x_t(\xi) \in X_\xi,\ p_\xi x_t(\xi) - p_\xi \hat{x}\left(n_t(\xi)\right) = 0, \forall \xi \in \Xi, \forall t = 1,\ldots,T \Big\}
\end{aligned} \tag{8}$$

By substituting the CVaR representation of Rockafellar and Uryasev (2000), we write the optimization formulation of $\mathbb{E}$-CVaR in the form of a dynamic program:

$$z_{E-CVaR_{\alpha_t}} = \min_{x_1,\eta_2} \left\{ \begin{array}{l} c_1^T x_1 + \eta_2 + \mathbb{E}_{\xi_2}\left[Q_2\left(x_1,\eta_2,\xi_2\right)\right]: \\ Ax_1 = b \end{array} \right\} \tag{9}$$

For $t = 2,\ldots T-1$, we have:

$$Q_t\left(x_{t-1},\eta_t,\xi_{[t]}\right) = \min_{x_t,\eta_{t+1},v_t} \left\{ \begin{array}{l} \dfrac{1}{1-\alpha_t} v_t + \eta_{t+1} + \mathbb{E}_{\xi_{t+1}}\left[Q_{t+1}\left(x_t,\eta_{t+1},\xi_{t+1}\right)\right]: \\ T_t\left(\xi_t\right)x_{t-1} + W_t\left(\xi_t\right)x_t = h_t\left(\xi_t\right) \\ \eta_t + v_t \geq c_t\left(\xi_t\right)^T x_t \\ v_t \geq 0 \end{array} \right\}$$

For the last period $t = T$ we have:

$$Q_T\left(x_{T-1},\eta_T,\xi_{[T]}\right)=\min_{x_T,v_T}\left\{\begin{array}{l}\frac{1}{1-\alpha_T}v_T:\\ T_T\left(\xi_T\right)x_{T-1}+W_T\left(\xi_T\right)x_T=h_T\left(\xi_T\right)\\ \eta_T+v_T\geq c_T\left(\xi_T\right)^T x_T\\ v_t\geq 0\end{array}\right\}$$

Given α_t for stage t, the auxiliary variable η_t is a "$(t-1)$-stage variable" to represent the value-at-risk (VaR_α); i.e., the minimum α_t-quantile such that the probability that the t-stage cost exceeds it is at least α_t. Another auxiliary variable v_t is a "t-stage variable" to represent the excess of t-stage cost of above η_t. As can be seen from formulation (9), the $\mathbb{E}$-CVaR optimization problem can be formulated as a risk-neutral model with two new variables (η_t and v_t) and two additional constraints in stages $t=2,...,T$.

For this formulation, existing algorithms can be readily adapted to solve the $\mathbb{E}$-CVaR optimization problem such as stochastic dual dynamic programming (SDDP) algorithm developed by Pereira and Pinto (1991) for continuous variables and stochastic dual dynamic integer programming (SDDIP) algorithm developed by Zou, Ahmed and Sun (2016) for integer variables. When the number of possible realizations are significant, however, the scenario decomposition algorithms display advantages in computational efficiency via parallel computing. In order to realize the scenario decomposition in $\mathbb{E}$-CVaR problem, we propose the scenario reformulation of $\mathbb{E}$-CVaR in the following Proposition 5.1.

Proposition 5.1. *Consider the case with finitely many realizations ξ and corresponding probabilities p_ξ. Let $\alpha\in(0,1)$. Then the scenario reformulation of $\mathbb{E}$-CVaR optimization can be represented as:*

$$z_{\mathbb{E}-CVaR_\alpha}=\min_{\substack{x_1,\ldots,x_T,\\ \eta_2,\ldots,\eta_T,\\ v_2,\ldots,v_T}}\left\{\begin{array}{l}\sum_{\xi\in\Xi}p_\xi\left[c_1\left(\xi\right)^T x_1\left(\xi\right)+\eta_2\left(\xi\right)+\frac{1}{1-\alpha_2}v_2\left(\xi\right)\cdots+\eta_T\left(\xi\right)+\frac{1}{1-\alpha_T}v_T\left(\xi\right)\right]:\\ \eta_t\left(\xi\right)+v_t\left(\xi\right)\geq c_t\left(\xi_t\right)^T x_t\left(\xi\right),\forall t=2,\ldots,T\\ x_t\left(\xi\right)\in X_\xi,\, p_\xi x_t\left(\xi\right)-p_\xi\hat{x}\left(n(\xi_t)\right)=0,\forall\xi\in\Xi,\forall t=1,...,T\end{array}\right\}\quad(10)$$

where

$$X_\xi=\left\{\begin{array}{l}x_t\left(\xi\right):Ax_1\left(\xi\right)=b,x_t\left(\xi\right),\hat{x}\left(n(\xi_t)\right)\in\mathbb{Z}_+^{p_1}\times\mathbb{R}_+^{n_1-p_1},\forall t=1,...,T\\ T_t\left(\xi\right)x_{t-1}\left(\xi\right)+W_t\left(\xi\right)x_t\left(\xi\right)=h_t\left(\xi\right),\forall\xi\in\Xi,\forall t=2,\ldots,T\end{array}\right\}$$

Similarly, we write the scenario reformulation for the risk-averse multistage stochastic program for the ECRM based on $\rho_t=EP_{\beta_t}$, denoted as $\mathbb{E}$-EP, as follows.

Proposition 5.2. *Consider the case with finitely many realizations* ξ *and corresponding probabilities* p_ξ*. Given a prescribed target level* $\beta_t \in \mathbb{R}$ *for each stage* $t = 1,\ldots,T$ *. Then there exists a constant* $M > 0$ *such that the scenario reformulation of* $\mathbb{E}$*-EP optimization is equivalent to the following program:*

$$z_{\mathbb{E}-EP_{\beta_t}} = \min_{x_1,\ldots,x_T} \left\{ \begin{array}{l} \sum_{\xi\in\Xi} p_\xi \sum_{t=1,\ldots,T} \theta_t(\xi): \\ \beta_t + M_t \cdot \theta_t(\xi) \geq c_t(\xi_t)^T x(\xi)_t, \forall t = 1,\ldots,T \\ x_t(\xi) \in X_\xi, p_\xi x_t(\xi) - p_\xi \hat{x}(n(\xi_t)) = 0, \forall \xi \in \Xi, \forall t = 1,\ldots,T \end{array} \right\}. \tag{11}$$

Note that the constant $M_t > 0$ can be selected as $\sup\left\{c_t(\xi)^T x_t(\xi) : x_t(\xi) \in X_\xi, \forall \xi \in \Xi\right\}$.

The scenario reformulation for ECRM based on $\rho_t = EE_{\gamma_t}$, denoted as $\mathbb{E}$-EE, is addressed in Proposition 5.3.

Proposition 5.3. *Consider the case with finitely many realizations* ξ *and corresponding probabilities* p_ξ*. Given a prescribed target level* $\gamma_t \in \mathbb{R}$ *for each stage* $t = 1,\ldots,T$ *. Then the scenario reformulation of* $\mathbb{E}$*-EE optimization is equivalent to the following program:*

$$z_{\mathbb{E}-EE_{\gamma_t}} = \min_{x_1,\ldots,x_T} \left\{ \begin{array}{l} \sum_{\xi\in\Xi} p_\xi \sum_{t=1,\ldots,T} e_t(\xi): \\ \gamma_t + e_t(\xi) \geq c_t(\xi_t)^T x(\xi)_t, \forall t = 1,\ldots,T \\ x_t(\xi) \in X_\xi, p_\xi x_t(\xi) - p_\xi \hat{x}(n(\xi_t)) = 0, \forall \xi \in \Xi, \forall t = 1,\ldots,T \end{array} \right\}. \tag{12}$$

For the ECRM with $\rho_t = SD$, denoted as $\mathbb{E}$-SD, we are able to formulate its risk-averse stochastic program as a risk-neutral problem as in Proposition 5.4. However, unlike the previous formulations of the $\mathbb{E}$-CVaR, $\mathbb{E}$-EP and $\mathbb{E}$-EE risk measures, the formulation of $\mathbb{E}$-SD optimization is not separable by scenario due to the presence of constraint $s_t(\xi) \geq \sum_{\xi\in\Xi} p_\xi \left[c_t(\xi_t)^T x(\xi)\right], \forall t = 1,\ldots,T$ and, therefore, is not eligible for scenario decomposition.

Proposition 5.4. *Consider the case with finitely many realizations* ξ *and corresponding probabilities* p_ξ*. Then the* $\mathbb{E}$*-SD optimization problem is equivalent to the following program:*

$$z_{\mathbb{E}-SD} = \min_{x_1,\ldots,x_T} \left\{ \begin{array}{l} \sum_{\xi\in\Xi} p_\xi \sum_{t=1,\ldots,T} \left[s_t(\xi) - c_t(\xi_t)^T x(\xi) \right]: \\ s_t(\xi) \geq c_t(\xi_t)^T x(\xi)_t, \forall t = 1,\ldots,T \\ s_t(\xi) \geq \sum_{\xi\in\Xi} p_\xi \left[c_t(\xi_t)^T x(\xi) \right], \forall t = 1,\ldots,T \\ x_t(\xi) \in X_\xi, p_\xi x_t(\xi) - p_\xi \hat{x}(n(\xi_t)) = 0, \forall \xi \in \Xi, \forall t = 1,\ldots,T \end{array} \right\}. \tag{13}$$

6. Lower bounding approach for risk-averse problems

6.1 Progressive Hedging (PH) algorithm

Proposed by Rockafellar and Wets (1991), the progressive hedging (PH) algorithm is a scenario decomposition method for stochastic programs motivated by augmented Lagrangian theory. By decomposing the extensive form into scenario sub-problems, the PH algorithm effectively reduces the computational burden by solving the scenario sub-problems in parallel instead of solving extensive forms directly, especially for large-scale instances. Solving scenario sub-problems separately can also take advantage of any special structures that are present.

For a two-stage stochastic mixed-integer program, a solution is said to be admissible in one scenario if it is feasible in this scenario; a solution is said to be implementable or non-anticipative if its first-stage decision is scenario-independent; and a solution is feasible if it is both admissible to all scenarios and implementable. The idea of the PH algorithm is to aggregate the admissible solutions of modified scenario subproblems which progressively causes the aggregated solution to be non-anticipative and optimal. The modified scenario subproblem comes from scenario decomposition of the augmented Lagrangian as a close approximation of problem (3). The modified cost function includes a penalty term relative to the non-anticipative constraint and a proximal term that measures the deviation of the scenario solution from the aggregated solution for first-stage decisions. The weight vector $w \in \mathbb{R}^{n_1 \times s}$ is updated by the penalty parameter (vector) $\rho > 0$ in each iteration. This weight update rule is essential to the proofs of the convergence theorems (Rockafellar and Wets 1991).

The PH algorithm has been proven to converge when all decision variables are continuous. It can also serve as a heuristic in the mixed-integer case. While convergence is not guaranteed for mixed-integer problems, computational studies have shown that the PH algorithm can find high-quality solutions within a reasonable number of iterations (Watson and Woodruff 2011). The PH algorithm for multistage stochastic mixed-integer programs is restated as follows (Gade et al. 2016):

STEP 1 Initialization: Let $v := 0$ and $w^v(n(\xi_t)) := 0, \forall \xi \in \Xi, t = 1,\ldots,T$. Compute for each $\xi \in \Xi$:

$$x_t^{v+1}(\xi) := \arg\min \left\{ c^T x_1 + \sum_{t=2}^{T} q_t(n(\xi_t))^T x_t : x_t \in X_\xi \right\}$$

STEP 2 Iteration update: $v \leftarrow v+1$

STEP 3 Non-anticipative policy: Compute for each $t = 1,\ldots,T-1$ and each $n(\xi_t) \in N_t$:

$$\hat{x}_t^v\left(n(\xi_t)\right) := \sum_{\xi(n_t)\in\Xi(n_t)} p_\xi x_t^v(\xi) \Big/ \sum_{\xi(n_t)\in\Xi(n_t)} p_\xi$$

STEP 4 Weight update: Compute for each $t = 1,\ldots,T-1$ and for each $\xi \in \Xi$:

$$w_t^v\left(n(\xi_t)\right) := w_t^{v-1}\left(n(\xi_t)\right) + \rho(x_t^v\left(n(\xi_t)\right) - \hat{x}_t^v\left(n(\xi_t)\right))$$

STEP 5 Decomposition: Compute for each $\xi \in \Xi$:

$$x_t^{v+1}(\xi) := \arg\min\left\{c^T x_1 + \sum_{t=2}^{T} q_t\left(n(\xi_t)\right)^T x_t + \sum_{t=1}^{T-1}\left[w_t^v\left(n(\xi_t)\right)^T x_t + \frac{\rho}{2}\left\|x_t - \hat{x}_t^v\left(n(\xi_t)\right)\right\|^2\right] : x_t \in X_\xi\right\}$$

STEP 6 Termination: If at each tree node, all the scenario solutions agree to within some tolerance, then stop. Otherwise, return to Step 2.

The performance of PH using various fixed, global values of penalty parameter ρ with a single scalar used for all variables has been extensively explored in the literature (Mulvey and Vladimirou 1991; Listes and Dekker 2005; Fan and Liu 2010). We denote the corresponding method by $FX(\cdot)$, where the FX stands for fixed and the argument provides the single value of ρ. Watson and Woodruff (2011) observe that the objective cost per decision variable may range in magnitude and an effective ρ values should be close in magnitude to the unit cost per decision variable in the objective. Specifically, they set $\rho(i)$ for each decision variable i to be a fixed multiple of the corresponding objective cost coefficient. The method is denoted by $CP(\cdot)$, where CP stands for cost-proportional and the argument gives the cost multiplier $k > 0$. Previous experience (Watson and Woodruff 2011; Gade et al. 2016) indicates that larger values of ρ can accelerate the convergence of PH while oscillation can occur when the weight vector is updated too aggressively by large values of ρ. While smaller values of ρ lead to slow changes in weight vector as well as little movement in convergence of PH, the quality of resulting solutions and lower bounds is improved. In addition to the strategies for choosing the PH ρ parameter, Watson and Woodruff (2011) introduced additional strategies such as variable fixing and slamming to break cycles and accelerate PH convergence.

6.2 Lower bounds from PH on multistage stochastic mixed-integer programs

Although the PH algorithm has been applied successfully as a heuristic to solve multistage stochastic mixed-integer programs, it is limited by the lack of a convergence guarantee as well as the lack of information to evaluate solution quality relative to the optimal objective value. Gade et al. (2016) addressed this deficiency by presenting a method to compute lower bounds in the

PH algorithm for multistage stochastic mixed-integer programs. The lower bounds not merely allow us to assess the quality of the solutions in each iteration, but also can provide lower bounds for solution methods like branch-and-bound that rely on lower bounds. We elaborate the lower bounding approach for two-stage stochastic mixed-integer programs proposed by Gade et al. (2016) to multistage cases and show that the weights w define implicit lower bounds, $D(w)$, on the optimal objective value denoted by z^*.

Proposition 6.1. *Let* $w = \left(w\left(n\left(\xi_t\right)\right)\right)_{\xi \in \Xi}$ *where* $w\left(n\left(\xi_t\right)\right) \in \mathbb{R}^{n_1}$ *satisfy* $\sum_{\xi(n_t) \in \Xi(n_t)} p_\xi w\left(n\left(\xi_t\right)\right) = 0$ *for each* $n_t \in N_t$. *Let*

$$D_\xi\left(w\left(n\left(\xi_t\right)\right)\right) := \min\left\{c^T x_1 + \sum_{t=2}^{T} q_t\left(n\left(\xi_t\right)\right)^T x_t + \sum_{t=1}^{T-1} w_t\left(n\left(\xi_t\right)\right)^T x_t : x_t \in X_\xi\right\} \tag{14}$$

Then $D(w) := \sum_{\xi \in \Xi} p_\xi D_\xi\left(w\left(n\left(\xi_t\right)\right)\right) \le z^*$.

It can be verified $\sum_{\xi(n_t) \in \Xi(n_t)} p_\xi w\left(n\left(\xi_t\right)\right) = 0$ is maintained in every iteration by the weight update rule. Proposition 6.1 indicates that one can compute a lower bound on z^* in any iteration of PH algorithm using the current weights with approximately the same effort as one PH iteration.

6.3 Scenario bundling in Progressive Hedging

Motivated by Wets' strategy of aggregating scenarios in stochastic optimization (Wets 1989), Gade et al. (2016) formalized the bundle version of PH algorithm, which allows Steps 1 and 5 of the PH algorithm to solve smaller extensive forms of the original problem. We extend the bundle version of lower bounding approach for two-stage cases introduced by Gade et al. (2016) to multistage cases in Proposition 6.2. Suppose the set of all the scenario tree nodes N_t at stage $t = 2, \ldots, T-1$ is partitioned into bundles, β_t, of K scenario tree nodes each. We denote the set of bundles by B_t, with $\beta_t \in B_t$. Let $P_{\beta_t} = \sum_{\xi \in \Xi(\beta_t)} p_\xi$.

Proposition 6.2. *Let* $w = \left(w\left(\beta_t\right)\right)_{\beta_t \in B_t}$ *where* $w(\beta) \in \mathbb{R}^{n_1}$ *satisfy* $\sum_{\beta_t \in B_t} P_{\beta_t} w\left(\beta_t\right) = 0$. *Let*

$$D_{\beta_t}\left(w\left(\beta_t\right)\right) := \min_x \left\{c^T x_1 + \sum_{t=2}^{T} \sum_{\xi \in \Xi(\beta_t)} \frac{p_\xi}{P_{\beta_t}} q_t\left(\xi\right)^T x_t + \sum_{t=1}^{T-1} w_t\left(\beta_t\right)^T x_t : x_t \in X_\beta\right\} \tag{15}$$

Then $D(w) := \sum_{\beta_t \in B_t} P_{\beta_t} D_{\beta_t}\left(w\left(\beta_t\right)\right) \le z^*$.

6.4 Lower bounds on E-CVaR stochastic mixed-integer problem

By applying the lower bounding approach in Proposition 6.1 to the optimization formulation of the $\mathbb{E}$-CVaR problem, we have the following Proposition 6.3.

Proposition 6.3. *Let* $w=\left(w(\xi)\right)_{\xi\in\Xi}$ *where* $w(\xi)\in\mathbb{R}^{n_1}$ *satisfy* $\sum_{\xi(n_t)\in\Xi(n_t)} p_\xi w(\xi)=0$, $w'=\left(w'(\xi)\right)_{\xi\in\Xi}$ *where* $w'(\xi)\in\mathbb{R}$ *satisfy* $\sum_{\xi(n_t)\in\Xi(n_t)} p_\xi w'(\xi)=0$, and $w''=\left(w''(\xi)\right)_{\xi\in\Xi}$ *where* $w''(\xi)\in\mathbb{R}$ *satisfy* $\sum_{\xi(n_t)\in\Xi(n_t)} p_\xi w''(\xi)=0$ *for each* $n_t\in N_t$. *Let*

$$D_\xi\left(w(\xi),w'(\xi),w''(\xi)\right):=\min\left\{\begin{array}{l} c_1^T x_1+\eta_2+\dfrac{1}{1-\alpha_2}v_2+\cdots+\eta_T+\dfrac{1}{1-\alpha_T}v_T \\ +w_1^T(\xi)x_1+\sum_{t=2}^{T} w_t'^T(\xi)\eta_t+\sum_{t=2}^{T-1} w_t''^T(\xi)v_t: \\ \eta_t+v_t\geq c_t(\xi_t)^T x_t, \forall t=2,\ldots,T \\ x_t(\xi)\in X_\xi, p_\xi x_t(\xi)-p_\xi\hat{x}\left(n(\xi_t)\right)=0, \forall\xi\in\Xi, \forall t=1,\ldots,T \end{array}\right\} \quad (16)$$

Then $D(w,w',w'':=\sum_{\xi\in\Xi} p_\xi D_\xi\left(w(\xi),w'(\xi),w''(\xi)\right)\leq z^*_{\alpha-CVaR}$.

However, the lower bounding approach for solving the $\mathbb{E}$-CVaR problem is not as straightforward as the one for solving risk-neutral problems. Take a 3-stage stochastic mixed-integer program for instance. The problem $D_\xi\left(w(\xi),w'(\xi),w''(\xi)\right)$ can be written as:

$$D_\xi\left(w(\xi),w'(\xi),w''(\xi)\right):=\min\left\{\begin{array}{l} c_1^T x_1+w_1^T(\xi)x_1+\left(w_2''^T(\xi)+\dfrac{1}{1-\alpha_2}\right)(\eta_2+v_2) \\ +\left(w_2'^T(\xi)-w_2''^T(\xi)-\dfrac{\alpha_2}{1-\alpha_2}\right)\eta_2+\left(w_3'^T(\xi)+1\right)\eta_3+\dfrac{1}{1-\alpha_3}v_3: x_t\in X_\xi \end{array}\right\}$$

This optimization problem is unbounded due to the unboundedness property of decision variables η_2,η_3 and η_2+v_2. To solve the optimization problem $D_\xi\left(w(\xi),w'(\xi),w''(\xi)\right)$, we must find valid upper and lower bounds for the decision variables η_2,η_3 and a valid upper bound for η_2+v_2. Such bounds as derived in Proposition 6.4 lead to tighter lower bounds of $z^*_{\alpha-CVaR}$ and can be obtained with little computational effort. In addition, the introduction of the bounds also speeds up the convergence of the PH algorithm.

Proposition 6.4. *Let* η_t^*,v_t^* *be optimal values of* $\eta_t,v_t,\forall t=2,\ldots,T$ *for the* $\mathbb{E}$*-CVaR stochastic program. Then*

$$U_t = \max_{\xi\in\Xi}\left\{\max_x\left\{c_t(\xi)^T x_t : x_t \in X_\xi\right\}\right\} \geq \eta_t^* + v_t^*,$$
$$L_t = \min_{\xi\in\Xi}\left\{\min_x\left\{c_t(\xi)^T x_t : x_t \in X_\xi\right\}\right\} \leq \eta_t^*. \tag{17}$$

Proof: Let $x_t^*(\xi), \eta_t^*(\xi), v_t^*(\xi)$ be optimal solutions for the $\mathbb{E}$-CVaR stochastic program.

(a) The definition of CVaR straightforwardly indicates that
$U_t = \max_{\xi\in\Xi}\left\{\max_x\left\{c_t(\xi)^T x_t : x_t \in X_\xi\right\}\right\} \geq \eta_t^* + v_t^*$.

(b) Based on the definition of CVaR (Schultz and Tiedemann 2006), we define the cumulative distribution function of η_t to be $\Psi(x_t^*(\xi), \eta_t) := P(\{\xi \in \Xi : c_t(\xi)^T x_t^*(\xi) \leq \eta_t\})$ and define $\alpha - VaR$ as $\eta_{t,\alpha}(x_t^*(\xi)) := \min\{\eta_t : \Psi(x_t^*(\xi), \eta_t) \geq \alpha\}$. Since the cumulative distribution function $\Psi(x_t^*(\xi), \eta_t)$ of η_t is a monotonically increasing function over η_t and a function defined as $\Psi'(x_t^*(\xi), \alpha) := \min\{\Psi(x_t^*(\xi), \eta_t) : \Psi(x_t^*(\xi), \eta_t) \geq \alpha\}$ is a monotonically increasing function over α, then $\eta_{t,\alpha}(x_t^*(\xi))$ is monotonically increases over α. Besides, since $\lim_{\alpha\to 0^+} \eta_{t,\alpha}\left(x_t^*(\xi)\right) = \min_{\xi\in\Xi}\left\{c_t(\xi)^T x_t^*(\xi)\right\}$, then we have $\min_{\xi\in\Xi}\left\{c_t(\xi)^T x_t^*(\xi)\right\} \leq \eta_{t,\alpha}\left(x_t^*(\xi)\right)$. Since $x_t^*(\xi), \forall \xi \in \Xi$ are feasible for the risk-neutral problem, then $c_t(\xi)^T x_t^*(\xi) \geq \min_{x_t}\left\{c_t(\xi)^T x_t : x_t \in X_\xi\right\}$ for each $\xi \in \Xi$. By taking the minimum for all $\xi \in \Xi$ on both sides, we have $\min_{\xi\in\Xi}\left\{c_t(\xi)^T x_t^*(\xi)\right\} \geq \min_{\xi\in\Xi}\left\{\min_{x_t}\left\{c_t(\xi)^T x_t : x_t \in X_\xi\right\}\right\}$. Thus, it is proved $L_t = \min_{\xi\in\Xi}\left\{\min_{x_t}\left\{c_t(\xi)^T x_t : x_t \in X_\xi\right\}\right\} \leq \eta_t^*$.■

Therefore, to compute the lower bounds for the $\mathbb{E}$-CVaR problem, one must compute both U_t and L_t by solving the minimization problem and the maximization problem of its risk-neutral model for each stage $t, \forall t = 2,...,T$ beforehand and then solve the modified problem of (16) with two additional constraints (17) in each stage $t, \forall t = 2,...,T$.

6.5 Lower bounds on risk-averse stochastic mixed-integer problem with E-EP

It is straightforward to apply the lower bounding approach in Proposition 6.1 to the multistage risk-averse stochastic mixed-integer problems with $\mathbb{E}$-EP, resulting in the lower bounding approach in Proposition 6.5.

Proposition 6.5. *Let* $w = \left(w(\xi)\right)_{\xi\in\Xi}$ *where* $w(\xi) \in \mathbb{R}^{n_1}$ *satisfy* $\sum_{\xi(n_t)\in\Xi(n_t)} p_\xi w(\xi) = 0$ *for each* $n_t \in N_t$. *Let*

$$D_\xi\left(w(\xi)\right) := \min\left\{\begin{array}{l}\sum_{t=1,\ldots,T}\theta_t + \sum_{t=1}^{T-1} w_t^T(\xi)\theta_t : \\ \beta_t + M\cdot\theta_t \geq c_t(\xi_t)^T x_t, \forall t=1,\ldots,T \\ x_t(\xi)\in X_\xi, p_\xi x_t(\xi) - p_\xi \hat{x}\left(n(\xi_t)\right) = 0, \forall \xi\in\Xi, \forall t=1,\ldots,T\end{array}\right\} \tag{18}$$

Then $D(w) := \sum_{\xi\in\Xi} p_\xi D_\xi\left(w(\xi)\right) \leq z_{EP}^*$.

6.6 Lower bounds on risk-averse stochastic mixed-integer problem with E-EE

Similar to Proposition 6.5, we can easily derive the lower bounding approach for multistage risk-averse stochastic mixed-integer programs with $\mathbb{E}$-EE in Proposition 6.6.

Proposition 6.6. *Let* $w = \left(w(\xi)\right)_{\xi\in\Xi}$ *where* $w(\xi)\in\mathbb{R}^{n_1}$ *satisfy* $\sum_{\xi(n_t)\in\Xi(n_t)} p_\xi w(\xi) = 0$ *for each* $n_t \in N_t$. *Let*

$$D_\xi\left(w(\xi)\right) := \min\left\{\begin{array}{l}\sum_{t=1,\ldots,T} e_t + \sum_{t=1}^{T-1} w_t^T(\xi) e_t : \\ \gamma_t + M\cdot e_t \geq c_t(\xi_t)^T x_t, \forall t=1,\ldots,T \\ x_t(\xi)\in X_\xi, p_\xi x_t(\xi) - p_\xi \hat{x}\left(n(\xi_t)\right) = 0, \forall \xi\in\Xi, \forall t=1,\ldots,T\end{array}\right\} \tag{19}$$

Then $D(w) := \sum_{\xi\in\Xi} p_\xi D_\xi\left(w(\xi)\right) \leq z_{EE}^*$.

7. Numerical results

In this section, we study the performance of the lower bounding approach for risk-averse stochastic mixed-integer test instances with $\mathbb{E}$-CVaR. We investigate the interaction between the strategies for choosing the PH ρ parameter and the quality of PH lower bounds as well as the scenario bundling strategies on a financial portfolio optimization instance. We consider summary results of the performance of the lower bounding approach on a number of lot sizing instances. We further examine the lower bounding approach on a risk-averse large-scale power generation expansion planning instance whose extensive form is too large to solve.

We use PySP (Watson et al. 2012), an open-source software package for modeling and solving stochastic programs, to implement PH algorithm and a plugin called phboundextension to implement the lower bounding approach for PH algorithm. CPLEX is used to solve the

deterministic mixed-integer linear programs. All the experiments are conducted on a Linux server with 31 GB and 8 processors with 4 cores per processor.

7.1. Portfolio optimization problem

The application of multistage stochastic programming has gained popularity in the financial industry to address the stochastic nature of financial problems. The multistage portfolio optimization (MPO) problem, or multistage financial asset allocation problem, finds the optimal decisions to rebalance the portfolio over time to maximize the expected value of the portfolio by the end of the planning horizon. We modify the portfolio optimization formulation from Dantzig and Infanger (1993). At the initial time period, a certain amount of wealth is available to a decision maker in asset $i = 1,\ldots n$ and in cash which we index as asset $n+1$ with $x_i^0, i = 1,\ldots,n+1$ to be the dollar value of initially available assets. At each time period $t = 1,\ldots T$, an investor can sell off an amount of asset i worth y_i^t for cash or buy an amount of asset i worth z_i^t from trades in previous periods, and his resulting amount of asset i at period t is denoted as x_i^t . Buying and selling causes transaction costs proportional to the dollar value of the asset traded. Buying one unit of asset i requires $1+v_i$ units of cash and selling one unit of asset i results in $1-\mu_i$ units of cash. At time period t, the return rate r_i^t of asset i from period t to period $t+1$ is not known to the decision maker until after the decision is made on rebalancing the portfolio for period t. Only the return rate on cash, r_{n+1}^t and the return rate on asset i from initial period, r_i^0 are assumed known. In addition to Dantzig and Infanger's formulation, it is required that the amount of assets sold or bought must be either zero or a positive value between its lower and upper bounds. A multistage stochastic mixed-integer programming formulation of MPO problem is:

$$\max \sum_{\xi\in\Xi} p_\xi \sum_{i=1}^{n+1} r_i^T(\xi)\, x_i^T(\xi) \tag{20a}$$

$$r_i^{t-1}(\xi)\, x_i^{t-1}(\xi) + z_i^t(\xi) - y_i^t(\xi) = x_i^t(\xi), \forall i = 1,\ldots n, t = 1,\ldots T, \xi \in \Xi \tag{20b}$$

$$r_{n+1}^{t-1}(\xi)\, x_{n+1}^{t-1}(\xi) - \sum_{i=1}^{n}(1+v_i)\, z_{n+1}^t(\xi) + \sum_{i=1}^{n}(1-\mu_i)\, y_{n+1}^t(\xi) = x_{n+1}^t(\xi), \forall t = 1,\ldots T, \xi \in \Xi \tag{20c}$$

$$m_i^t(\xi) l_i^y \le y_i^t(\xi) \le m_i^t(\xi) u_i^y, n_i^t(\xi) l_i^z \le z_i^t(\xi) \le n_i^t(\xi) u_i^z, \forall i = 1,\ldots n, t = 1,\ldots T, \xi \in \Xi \tag{20d}$$

$$x_i^t(\xi), y_i^t(\xi), z_i^t(\xi) \ge 0, m_i^t(\xi), n_i^t(\xi) \in \{0,1\}, \forall i = 1,\ldots n+1, t = 1,\ldots T, \xi \in \Xi \tag{20e}$$

We generate a test instance with 5 assets, 3 stages and 10 branches emanating from each scenario tree node. The 10 branches from each scenario tree node are sampled from normal distributions of stochastic parameters r_i^t with identical probabilities for each asset for each period. The means

of normal distributions are displayed in Table 1 and the standard deviations are 0.5. The fixed input parameters are displayed in Table 2.

Table 1. Mean values of normal distributions of return rates of assets for MPO test instance

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Period 1	1	1.1	1.2	1.3	1.4
Period 2	1	0.9	0.8	0.7	0.6
Period 3	0.8	0.9	1	1.1	1.2

Table 2. Input parameters for MPO test instance

$x_i^0, i = 1, \ldots, n+1$	100
$v_i, i = 1, \ldots, n$	0
$\mu_i, i = 1, \ldots, n$	0.5%
r_{n+1}^t	1.02
l_i^y, l_i^z	30
u_i^y, u_i^z	300

Here, we performed computational studies on the $\mathbb{E}$-CVaR problem of this MPO instance given the upper and lower bounds for $\mathbb{E}$-CVaR variables. The preselected probability is set to be $\alpha_t = 0.8$ for each stage t such that we are only concerned with the 20% worst scenarios at each stage. We perform multiple runs of the PH algorithm on this instance, varying the values of the penalty parameter ρ. Specifically, we consider fixed $\rho \in \left\{FX\left(10^{-2}\right), FX\left(10^{-3}\right), FX\left(10^{-4}\right)\right\}$ and record the time-series of the lower bound $D\left(w, w', w''\right)$ obtained at each PH iteration during each run. The lower bound results are shown in Figure 1 (a), which additionally displays the optimal solution value obtained from solving its extensive form. We also consider the PH lower bounds when bundling scenarios. Specifically, we vary the number of scenarios in each bundle considered by PH, while holding ρ constant. Each scenario bundle is formed by some scenarios emanating from the same scenario tree node. An illustrative example is shown in Figure 1 (b), with $\rho = FX\left(10^{-3}\right)$.

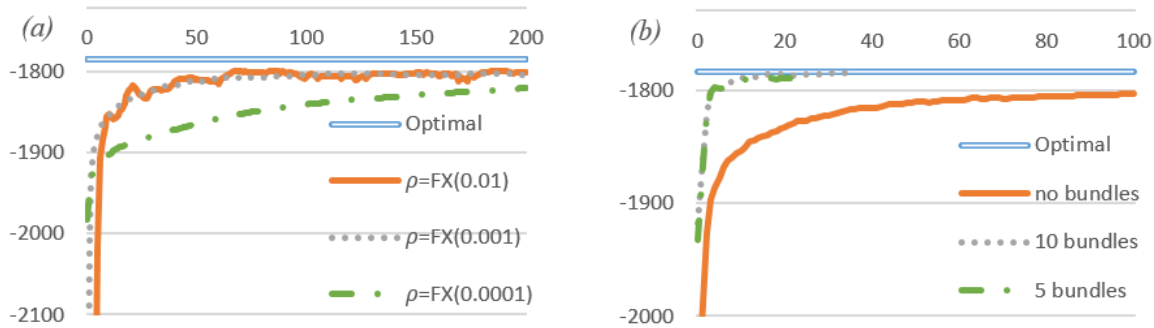

Figure 1. Lower bounds from PH and optimal value from solving extensive from for MPO instance (a) different penalty parameter values; (b) different scenario bundling strategies with $\rho = FX\left(10^{-3}\right)$

As displayed in the PH lower bounding results in Figure 1 (a), larger ρ values can lead to oscillations in the convergence of lower bounds. In contrast, lower ρ values smoothen the convergence of lower bounds but can also slow down their convergence. Figure 1 (b) shows the advantage of scenario bundling for improving the quality of lower bound convergence but the disadvantage is that each PH iteration takes longer.

Table 3 further demonstrates that scenario bundling may reduce the number of PH iterations to converge as well as the total PH computational time. The computation time consumed per iteration, however, increases with the number of scenarios per bundle.

Table 3. Computation time for MPO test instance with different scenario bundles

Number of bundles	100	10	5
Number of scenarios per bundle	1	10	20
Number of PH iterations to converge	200	36	25
Average PH execution time per iteration (seconds)	4.5	11.8	16.9
Total PH execution time (seconds)	909	425	423

7.2. Lot sizing problem

The multistage lot-sizing problem (MLS) has been widely used as a test case for multistage stochastic integer programming algorithms (Burhaneddin and Özaltın 2014). It seeks to determine a minimum cost production and inventory holding schedule for a product to satisfy its stochastic demand over a finite discrete planning horizon. A multistage stochastic mixed-integer programming formulation of the MLS problem is:

$$\min \sum_{\xi \in \Xi} p_{\xi} \sum_{t \in T} \left(\alpha_t x_t(\xi) + \beta_t y_t(\xi) + h_t s_t(\xi) \right) \tag{21a}$$

$$s_{t-1}(\xi) + x_t(\xi) = d_t(\xi) + s_t(\xi), \forall t = 1, \ldots T, \xi \in \Xi \tag{21b}$$

$$x_t(\xi) \le My_t(\xi), \forall t = 1,\ldots T, \xi \in \Xi \tag{21c}$$

$$s_0(\xi) = 0, \forall \xi \in \Xi \tag{21d}$$

$$y_t(\xi) \in \{0,1\}; x_t(\xi), s_t(\xi) \ge 0, \forall \xi \in \Xi \tag{21e}$$

where the decision variables x_t, s_t, and y_t denote production level, inventory level, and setup indicator at period $t = 1,\ldots T$, the parameters $\alpha_t, \beta_t, h_t, d_t$ denote production cost, setup cost, inventory cost, and demand at period $t \in T$, the parameter M denotes production capacity, and the parameter p_ξ denotes the probability for each scenario $\xi \in \Xi$. Objective (21a) minimizes the total expected production, setup and inventory costs. Constraints (21b) enforce inventory balance conditions, (21c) enforce the production capacity limits, (21d) enforces no initial inventory, and (21e) enforce variable restrictions.

We populate data for MLS instances as in Guan et al. (2006). We generate a test instance with 4 stages and 5 branches emanating from each scenario tree node. The 5 branches from each scenario tree node are sampled from uniform distributions of stochastic parameters $d_t \sim U[0,100]$ with identical probabilities for each time period. The fixed input parameters are displayed in Table 4. The capacity is assigned to be 200.

Table 4. Input parameters for MLS test instance

t	1	2	3	4
h_t	3	8	6	5
α_t	18	22	17	20
β_t	99	91	102	108

Here, we performed computational studies on $\mathbb{E}$-CVaR problems of (3-stage, 5-branch), (3-stage, 10-branch), (4-stage, 5-branch), and (4-stage, 10-branch) MLS instances with the upper and lower bounds for $\mathbb{E}$-CVaR variables and preselected probability set to be $\alpha_t = 0.2$ for each stage t. We perform multiple runs of the PH algorithm on this instance, varying the values of the penalty parameter ρ. Specifically, we consider fixed $\rho \in \{FX(10^{-3}), CP(10^{-3}), CP(10^{-4})\}$ and record the time-series of the lower bound $D(w, w', w'')$ obtained at each PH iteration during each run. The lower bound results for various lot sizing instances are shown in Figure 2, which additionally displays the optimal solution value obtained from solving its extensive form.

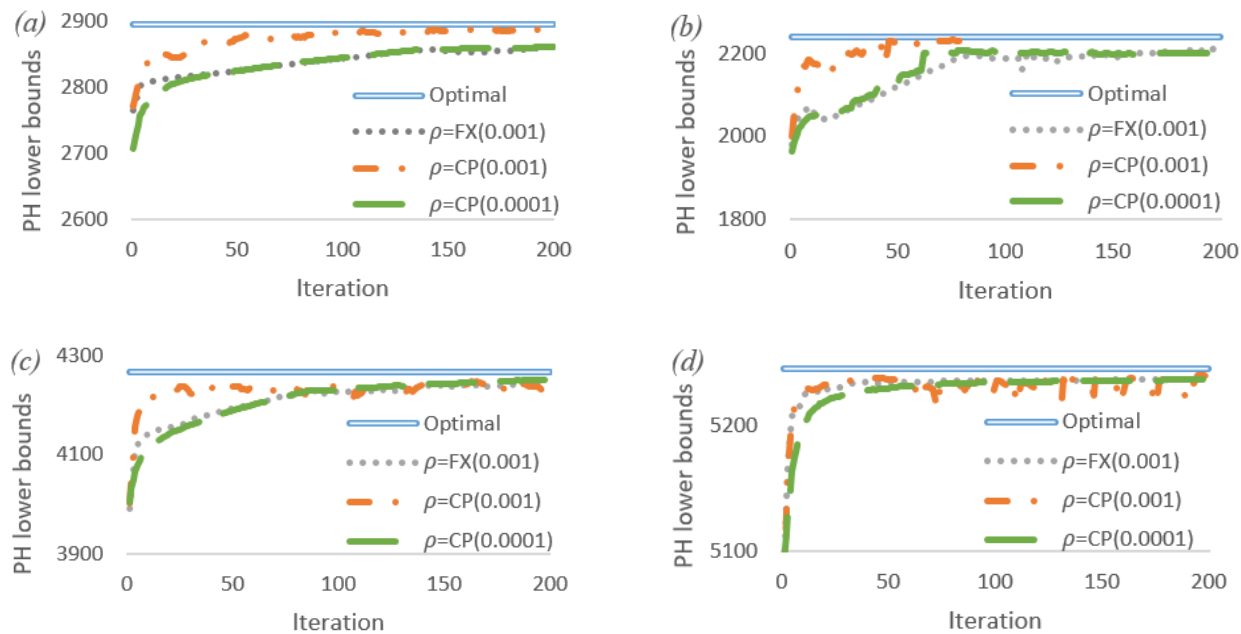

Figure 2. Lower bounds from PH and optimal value from solving extensive form for MLS instances

(a) 3-stage, 5-branch; (b) 3-stage, 10-branch; (c) 4-stage, 5-branch; (d) 4-stage, 10-branch

7.3. Generation expansion planning problem

In a power generation expansion planning (GEP) problem, one seeks to determine a long-term construction and generation plan for different types of generators, taking into account the uncertainties in future demand and fuel prices. Suppose there are T time stages and n types of expansion technologies available. Let x_{it} represent the numbers of generators to be built for generator type i in stage t, and y_{it} represent the amount of electricity produced by generator type i in stage t. The parameters a_{it}, b_{it} denote investment and generation cost for generator type i in stage t. The parameters r_i, u_i, d_t denote the capacity rating of generator type i, the construction limits on generator type i, and the electricity demand at stage t. A multistage stochastic mixed-integer programming formulation of GEP problem is:

$$\min \sum_{\xi \in \Xi} p_\xi \sum_{t=1}^{T} \sum_{i=1}^{n} \left(a_{it} x_{it}(\xi) + b_{it}(\xi) y_{it}(\xi) \right) \tag{22a}$$

$$y_{it}(\xi) \le r_i \sum_{s=1}^{t} x_{is}(\xi), \forall i = 1, \ldots n, t = 1, \ldots T, \xi \in \Xi \tag{22b}$$

$$\sum_{t=1}^{T} x_{it}(\xi) \le u_i, \forall i = 1, \ldots n, \xi \in \Xi \tag{22c}$$

$$\sum_{i=1}^{n} y_{it}(\xi) \geq d_t(\xi), \forall t = 1, \ldots T, \xi \in \Xi \tag{22d}$$

$$x_{it} \in \mathbb{Z}_+, y_{it} \in \mathbb{R}_+, \forall i = 1, \ldots n, t = 1, \ldots T, \xi \in \Xi \tag{22e}$$

In the above formulation, objective (22a) minimizes the total expected investment cost and generation cost. Constraints (22b) enforce generation capacity, (22c) enforce the limitation on total number of generators, (22d) enforce demand satisfaction, and (22e) enforce variable restrictions.

We consider an instance of the GEP problem with a 10-year planning horizon where each year is considered as one period. There are 6 types of generators available for capacity expansion, namely Coal, Combined Cycle (CC), Combined Turbine (CT), Nuclear, Wind, and Integrated Gasification Combined Cycle (IGCC). Among these 6 types of generators, both CC and CT power generators are fueled by natural gas. All the input parameters are deterministic except demand and natural gas price. We populate data for GEP problem as in Jin et al. (2011).

While Jin et al. (2011) consider a 10-period GEP instance as a two-stage problem, we consider a 10-period GEP instance as a 8-stage problem according to the division of the planning horizon and scenario tree generation in Feng et al. (2013). In our instance, each of the first six stages represent one period and each of the last two stages represent two periods. The stochastic parameters of demand and natural gas price are generated from two correlated geometric Brownian motions as in Jin et al. (2011). From each scenario tree node, 3 realizations of the pair of uncertain parameters and their probabilities are computed using moment matching method (Feng and Ryan 2013), thus leading to a large-scale mixed-integer problem with 2,187 scenarios. The data of fixed input parameters are obtained from GEP instance in Jin et al. (2011). We formulate its risk-averse $\mathbb{E}$-CVaR formulation with preselected probability set to be $\alpha_t = 0.2$ for each stage t, which has 244,944 variables and 205,578 constraints in total.

Due to the large number of scenarios and variables in this $\mathbb{E}$-CVaR problem, its extensive form failed to compute an optimal objective with 48-hour time limit. To deal with this issue, the lower bounding approach from PH for risk-averse problems is employed here to compute a feasible solution with a reasonable optimality gap. In this instance, the objective cost coefficients of decision variables are in the unit of millions while the cost coefficients of the $\mathbb{E}$-CVaR related variables η_t and v_t are no greater than one. The unbalanced cost coefficients in objective prevent the PH algorithm from obtaining good variable-specific penalty parameters ρ, which significantly slows down its progress. Thus, additional variables $\lambda_t = 10^{-6}\eta_t$ and $u_t = 10^{-6}v_t$ are substituted for η_t and v_t in the $\mathbb{E}$-CVaR optimization problem. In addition, the variable fixing and slamming strategies from Watson and Woodruff's PH extensions (2011) are adopted here to accelerate PH convergence by forcing early agreement of variables at the expense of sub-optimal solutions. We fix decision variables once their value has stabilized to a fixed value over the past 3 iterations. After 10 iterations, we enforced slamming where a decision variable is fixed to its maximum solution across all scenarios for every 2 subsequent iterations.

Table 5 shows that while the extensive form failed to solve within 48 hours, the Progressive Hedging algorithm is able to provide a feasible solution with 3.2% optimality gap within 5 hours. The lower bounding approach allows us to assess the quality of feasible solutions generated by the algorithm by an upper bound on its optimality gap as the difference between the upper and lower bounds.

Table 5. PH run-time and optimality gap on 8-stage GEP instance with 48-hour time limit

	PH iterations	Run-time (hours)	Optimal objective (cost in thousand million dollars)	Lower bound	Upper bound	Optimality gap
Extensive Form		48	N/A	N/A	N/A	N/A
Progressive Hedging	12	6	N/A	3.41	3.45	1.2%
	26	14	N/A	3.41	3.42	0.3%

As the preselected probability α varies, the optimal solutions to the risk-averse programs change corresponding to optimize the expected values of costs in the $(1-\alpha)\cdot 100\%$ worst scenarios. Table 6 reports the best feasible solutions to first-stage decision variables for the E-CVaR problem of 8-stage GEP instance with different values of α.

Table 6. First-stage variable solutions for different values of α

Number of generators to build by type	α		
	0	0.2	0.8
Baseload	0	0	4
CC	0	0	0
CT	0	0	0
Nuclear	1	1	1
Wind	22	21	40
IGCC	0	0	4

Table 6 indicates that the optimal solutions may vary significantly according to the preselected probability and the optimal solutions to risk-neutral models do not necessarily guarantee best performance for risk-averse models.

Conclusions

We have proposed the scenario decomposition reformulations of multistage risk-averse stochastic programs with a variety of ECRMs. Based on the scenario reformulation, we presented a lower bounding approach from the PH algorithm. We discussed strategies for choosing the PH ρ parameter and applied the scenario bundling strategy to help improve the

quality of the PH lower bounds. Computing lower bounds for the PH algorithm allows us to assess the quality of the solutions generated by PH algorithm and also integrate with exact algorithms that rely on lower bounds. The integration of this lower bounding approach for risk-averse models with other exact algorithms remains as a promising area for potential future research. We also provided a remedy for the issue of unbounded optimization in the lower bounding problems introduced by $\mathbb{E}$-CVaR. Numerical results indicate that this lower bounding approach obtains convergent and tight lower bounds and displays its advantage in solving near-optimal solutions within reasonable run-time for large-scale stochastic problems whose extensive form fails to solve.

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