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INTERPOLATING SEQUENCES ON CURVES

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ABSTRACT. We establish a condition on boundary curves (ending at points) lying either in the unit disc or the upper half plane which implies that any consecutively separated sequence, in the hyperbolic distance, lying on one of these curves is an interpolating sequence for bounded holomorphic functions.

1. Introduction. In a previous paper by the authors and Charles Belna [1] certain geometric properties of a sequence $\{z_n\}$ were shown to be sufficient that $\{z_n\}$ be an interpolating sequence for the algebra of bounded holomorphic functions in the unit disc Δ or the upper half plane H in the complex plane. In this paper we are concerned with identifying a class of curves in either H or Δ such that any sequence on such a curve satisfying a minimal hyperbolic separation is an interpolating sequence. A sequence $\{z_n\}$ in Δ is an interpolating sequence for the algebra $H^\infty(\Delta)$ if, for each bounded sequence $\{w_n\}$, there exists a function $f \in H^\infty(\Delta)$ such that $f(z_n) = w_n$, for all n. For $\{z_n\}$ to be interpolating for $H^\infty(\Delta)$ it is necessary and sufficient that it be uniformly separated. That is

$$\inf_n \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \chi(z_n, z_k) > 0,$$

where $\chi(z, w)$ is the pseudo-hyperbolic distance in Δ. In case the domain is the upper half plane H we replace Δ by H in the above, retaining $\chi(z, w)$ as notation for the pseudo-hyperbolic distance in H. The necessity was established independently by L. Carleson [2], W.K. Hayman [4] and D.J. Newman [6]; the sufficiency was proved by Carleson [2]. A sequence $\{z_n\}$ in Δ or H is called separated if

$$\inf_{n\neq m} \chi(z_n, z_m) > 0. \tag{1.0}$$

Recently Gerber and Weiss [3] introduced a class $\mathscr{C}$ of subsets of Δ defined by the property that a sequence lying in $A \in \mathscr{C}$ is interpolating if and only if it is separated. They gave various characterizations of the class $\mathscr{C}$, one of which is that, for any set $A \in \mathscr{C}$, the closure of A in the maximal ideal space M contains only nontrivial homomorphisms. Other characterizations involve Carleson measures and hyperbolic distance. Our first result will be to give a geometric condition insuring that a curve is in $\mathscr{C}$. It is a simple application of Theorem 1 of [1]. The remainder of this paper is concerned with a slightly different problem. Hayman [4], Newman [6] and Kabaila [5] independently showed that if $\{z_n\}$ lies on a radius in Δ and satisfies

$$\inf_n \chi(z_n, z_{n+1}) > 0, \tag{1.1}$$

then $\{z_n\}$ is an interpolating sequence. A sequence $\{z_n\}$ in either Δ or H satisfying (1.1) is said to be consecutively separated. D.H. Wortman [8] showed that any consecutively separated sequence lying on a convex boundary curve in Δ ending at a point is interpolating. We show that the same type result holds for a class of curves containing all of the above type and which satisfies a fairly simple geometric property. This result depends on Corollary 1 of [1]. We also obtain several analytic descriptions of these curves. We recall a few definitions in order to restate the relevant results from [1].

A cone, $\mathscr{C}(0)$, with vertex angle α is a closed region of the form

$$\{re^{i\theta}: \theta_0 \leqq \theta \leqq \alpha + \theta_0\ r \geqq 0\}, \qquad \pi/2 < \alpha < \pi. \tag{1.2}$$

The rays $\theta = \theta_0$ and $\theta = \alpha + \theta_0$ are referred to as the right and left boundary rays respectively of $\mathscr{C}(0)$. The cone that is the image of $\mathscr{C}(0)$ under $T(z) = z + \zeta$ is denoted by $\mathscr{C}(\zeta)$, with the right and left boundary rays of $\mathscr{C}(\zeta)$ being the respective translated images of the right and left boundary rays of $\mathscr{C}(0)$. For simplicity we usually write $\mathscr{C}$ instead of $\mathscr{C}(0)$. We now give Theorem 1 and Corollary 1 of [1].

THEOREM 1. *Let $\{z_n\}$ be a sequence that lies in either Δ or $\mathscr{H}$, let α be a number strictly between $\pi/2$ and π, and let* N *be a nonnegative integer. If to each* n *there corresponds a cone $\mathscr{C}_n$ with vertex angle α such that $z_m \in \mathscr{C}_n(z_n)$ for all but at most* N *indices $m > n$, then $\{z_n\}$ is an interpolating sequence.*

COROLLARY 1. *Let $\{z_n\}$ be a consecutively separated sequence lying in either Δ or $\mathscr{H}$. If there exists a cone $\mathscr{C}$ such that*

$$z_{n+1} \in \mathscr{C}_n(z_n), \text{ for all } n, \tag{1.3}$$

then $\{z_n\}$ is an interpolating sequence.

We now apply directly Theorem 1 to boundary curves $z(t)$, $0 \leqq t < \infty$, in Δ or $\mathscr{H}$. By a boundary curve we mean a curve $z(t)$ such that, for any compact set A in the space, there is a t_0 such that, for $t \geqq t_0$, $z(t) \notin A$.

THEOREM 2. *Let $z(t)$, $0 \leqq t \leqq \infty$, be a boundary curve in Δ or $\mathscr{H}$ and* *let α be a number strictly between $\pi/2$ and π. If to each t there corresponds a cone $\mathscr{C}_t$ with vertex angle α such that*

$$z(s) \in \mathscr{C}_t(z(t)), \qquad s \geqq t,$$

then any separated sequence on $z(t)$ is interpolating.

The companion result which follows from Corollary 1 is equally obvious with the cones $\mathscr{C}_t$ replaced by the single cone $\mathscr{C}$, and the sequence required only to be consecutively separated. However we prefer to give a more general result which has a conformally invariant form as well as finding analytic conditions on $z(t)$ which imply that $z(s) \in \mathscr{C}(z(t))$, $s \geqq t$. To that end we limit ourselves now to either continuous piecewise differentiable or smooth curves in Δ or H. As above we choose the parameter interval to be $\mathbf{R}^+$, the non-negative real numbers, although the results are all independent of the parameter interval chosen. A piecewise differentiable curve $z(t)$ in the complex plane is a continuous curve such that $z'(t)$ exists and is nonzero except on a finite or countable set of isolated points $E_z \subseteq \mathbf{R}^+$, where $z'(t)$ either fails to exist or is zero. A smooth cure $z(t)$ has a continuous, nonzero, derivative for all $t \in \mathbf{R}^+$. In the sequel all curves are assumed to lie either in Δ or H unless so stated and to be at least piecewise differentiable even if not so specified. A boundary curve $z(t)$ ends at a point if $\lim_{t\to\infty} z(t) = \tau$, where τ is (necessarily) a boundary point of Δ or H(including ∞). A boundary curve $z(t)$ ending at a point τ is said to be moderately oscillating if there exists a cone $\mathscr{C}$ and a value t_0 such that if τ is finite,

$$z(s) \in \mathscr{C}(z(t)),\ s \geqq t, \text{ for all } t \geqq t_0; \tag{1.4}$$

or if $\tau = \infty$,

$$-\frac{1}{z(s)} \in \mathscr{C}\left(-\frac{1}{z(t)}\right), \qquad s \geqq t, \text{ all } t \geqq t_0. \tag{1.5}$$

It should be pointed out that any boundary curves $z(t)$ in Δ or H satisfying (1.4), (or, indeed, the hypothesis of Theorem 2) must end at a single boundary point. This was observed by Gerber and Weiss for curves in their class $\mathscr{C}$. One might ask whether one could define moderately oscillating for a curve $z(t)$ ending at ∞ by (1.4), rather than (1.5), i.e., similar to Corollary 1. The answer is that this definition is not conformally invariant (it is too restrictive for curves tending to ∞) whereas moderate oscillation, as defined, is conformally invariant. A proof of these remarks must await Theorem 3 of the next section, which gives an equivalent analytic formulation or fmoderate oscillation.

2. Analytic formulations equivalent to moderate oscillation. The equivalent analytic formulation of moderate oscillation is shown first and then several results are shown for boundary curves ending at finite points.

THEOREM 3. *Let $z(t)$ be a boundary curve ending at a point. Then $z(t)$ is moderately oscillating if and only if there exists real numbers θ_0, α, $\pi/2 < \alpha < \pi$, a determination of* $\arg z'(t)$, *and a value* $t_0 > 0$, *such that, for all* $t \geqq t_0$, $t \notin E_z$, *if $z(t)$ ends at finite boundary point,*

$$\theta_0 < \arg z'(t) < \theta_0 + \alpha; \tag{2.0}$$

or if $z(t)$ ends at ∞,

$$\theta_0 < \arg \frac{z'(t)}{z^2(t)} < \theta_0 + \alpha. \tag{2.1}$$

PROOF. We suppose always that $t \geqq t_0$ and consider first the case in which $z(t)$ ends at a finite boundary point and satisfies (2.0). We then show $z(t)$ satisfies (1.4). Naturally we take $\mathscr{C}(0)$ to be the cone $\{re^{i\theta} | \theta_0 \leqq \arg z \leqq \theta_0 + \alpha, r \geqq 0\}$. We begin by proving a modest version of (1.4). For any interval $(t', t'') \subset \mathbf{R}^+ - E_z$ we show that, for any $t' \leqq t_1 < t''$,

$$z(t) \in \mathscr{C}(z(t_1)), \qquad t_1 \leqq t \leqq t''. \tag{2.2}$$

Initially suppose $t < t_1 < t''$. By the differentiability of $z(t)$ we have that $\arg(z(t) - z(t_1))$ tends to $\arg z'(t_1)$ as $t \to t_1$. Thus there is some $\delta > 0$ such that

$$z(t) \in \mathscr{C}(z(t_1)), \qquad t_1 \leqq t \leqq t_1 + \delta. \tag{2.3}$$

Suppose $z(t)$ leaves $\mathscr{C}(z(t_1))$ for some $t < t''$; then there is a point t_2, $t_1 + \delta < t_2 < t''$, such that $z(t_2) \in \mathscr{C}(z(t_1))$ but for some $\varepsilon > 0$, $z(t) \notin \mathscr{C}(z(t_1))$, for all $t_2 < t < t_2 + \varepsilon$. But $z(t)$ is differentiable at t_2 and so (3.2) holds with t_2 in place of t_1. Thus (2.2) holds for $t_1 \leqq t < t''$ and by continuity holds at t'' as well. Suppose now that $t_1 = t'$. Let $\{t_n\}$ be any sequence in (t', t'') approaching t' monotonically. Because (2.2) is valid for each t_n, we have that $\mathscr{C}(z(t_n)) \subset \mathscr{C}(z(t_{n-1}))$, and the continuity of $z(t)$ implies $\mathscr{C}(z(t_n)) \to \mathscr{C}(z(t'))$, $n \to \infty$. Thus we have shown (2.2). To prove the full result (1.4) we need only repeat the argument that $z(t)$ cannot leave $\mathscr{C}(z(t_1))$, for any $t \geqq t_1$, except that we replace the differentiability argument by the more general (2.2), valid even if the point of departure, $z(t_2)$, has $t_2 \in E_z$.

If $z(t)$ ends at ∞ and satisfies (2.1) then (2.0) holds for $z_0(t) = -1/(z(t))$ and so $z_0(t)$ is moderately oscillating, which was to be shown.

Conversely suppose $z(t)$ is moderately oscillating and tends to a finite boundary point. Thus there are real numbers α, θ_0, $\pi/2 < \alpha < \pi$, and a value t_1 such that $\theta_0 \leqq \arg(z(s) - z(t)) \leqq \alpha + \theta_0$, $s > t$, $t \geqq t_1$. If $t \notin E_z$ we have $\arg(z(s) - z(t)) \to \arg z'(t)$ as $s \to t$ and (2.0) is satisfied. If $z(t)$ ends at ∞ the above argument applied to $-1/(z(t))$ gives (2.1).

With this equivalent formulation we are now able to show that moderate oscillation is a conformally invariant property either in Δ or H or between them. We begin with Δ so any 1–1 conformal map of Δ onto Δ has the form $L(z) = e^{i\theta}(z - z_1)/(1 - z\bar{z}_1)$, $z_1 \in \Delta$. If $z(t)$ ends at τ, then $\arg(L(z(t)))' = \arg(1 - |z_1|^2/(1 - z(t)\bar{z}_1)^2)) + \arg(z'(t))$t. As $t \to \infty$ the first argument on the right tends to $\arg(1 - |z_1|^2/((1 - \tau\,\bar{z}_1)^2))$ (any choice of the argument will do). Thus $\arg(L(z(t))'$ satisfies (2.0) if $\arg(z'(t))$ does, albeit with a different value for θ_0, and so is moderately oscillating. The 1–1 conformal maps of H onto H are of the form $L(z) = (az + b/(cz + d)$, a, b, c, d real numbers $ad - bc > 0$. Let $z(t)$ be a boundary curve in H ending at the point x which is moderately oscillating. If x is a finite point not equal to $-d/c$ the $L(z(t))$ is a boundary curve ending at a finite point also, and $\arg(L(z(t)))' = \arg(ad - bc)/((c\,z(t) + d)^2) + \arg\ z'(t)$. As $t \to \infty$ the first term on the right tends to $\arg(ad - bc)/((cx + d)^2)$ and so $L(z(t))$ is moderately oscillating. If $x = -d/c$, $c \neq 0$, then $L(z(t))$ ends at ∞, and $\arg(L(z(t)))'/(L(z(t)))^2 = \arg(ad - bc)/(a\,z(t) + b)^2 + \arg z'(t)$. The first term on the right tends to 2 arg c and so $L(z(t))$ is moderately oscillating. If $x = \infty$ and $L(z(t))$ ends at a finite point (that is $c \neq 0$), we have $\arg(L(z(t)))' = \arg(ad - bc/(c + d(z(t))^{-1})^2 + \arg(z'(t))/(z^2(t))$. The first term on the right tends to $\arg((ad - bc)/c^2)$ and so $L(z(t))$ is moderately oscillating. If $x = \infty$ and $L(z(t))$ ends at infinity, then $c = 0$, $a \neq 0$, $d \neq 0$. Thus $\arg(L(z(t)))'/(L(z(t)))^2 = \arg(ad)/(a + b\,z(t)^{-1})^2 + \arg(z'(t))/(z^2(t))$. The first term of the sum tends to $\arg(d/a)$ as t tends to infinity and so $L(z(t))$ is moderately oscillating. Finally, let $L(z)$ be any $1 - 1$ conformal map of Δ onto H, and $z(t)$ be a boundary curve in Δ ending at a point which is moderately oscillating. Because of the conformal invariances already shown we may suppose that $z(t)$ ends at 1 and that $L(z) = i(1 - z)/(1 + z))$. Then $\arg(L(z(t)))' = \arg(-2i)/(1 + z(t))^2 + \arg z'(t)$. As $t \to \infty$ the first term on the right tends to arg $(-i)$ and so $L(z(t))$ has moderate oscillation. Going from H to Δ we suppose $z(t)$ ends at 0 and $L(z) = (1 - z)/(1 + z)$. Then we proceed as above.

There does exist a boundary curve $z(t)$ in H ending at 0 which is moderately oscillating but such that the curve $-1/(z(t))$ does not satisfy (1.4) at ∞. Thus, in H, it is not enough to require that a curve $z(t)$ ending at ∞ have the property that $z(s) \in C(z(t))$, $s \geqq t$, for all $t \geqq t_0$, in order to have a conformally invariant property. Via geometry, it is not difficult to show that if $z(t)$ ends at ∞ and satisfies (1.4), then $-1/(z(t))$ satisfies (1.4). We first construct a consecutively separated sequence $\{z_n\}$ in H tending to 0 such that $\{z_n\}$, but not $\{-1/z_n\}$, satisfies (1.3). Joining the points $\{z_n\}$ with straight line segments will give a curve $z(t)$ with the prescribed properties. It also shows that Corollary 1 is too restrictive for contecutively separated sequences in H tending to ∞. We recast Corollary

1 later. To begin our construction we choose the cone $\mathscr{C}$: $\{re^{i\theta} | -\pi \leqq \theta \leqq -\pi/6,\ r \geqq 0\}$. We construct the sequence $\{z_j\}_0^\infty$ four elements at a time. Let $z_0 = i$. Then let z_1 be the point of intersection of the circle A, tangent to the real axis at 0 and passing through z_0, with the left boundary ray of $\mathscr{C}(z_0)$. (See Fig. 1 for this and subsequent constructions.) Let $z_2 = -\bar{z}_1$ and let z_3 be the intersection of the left boundary ray of $\mathscr{C}(z_2)$ and the imaginary axis. Repeat the construction with z_3 replacing z_0. Using elementary geometry and induction it is easy to show that $z_{3k} = i/2^k$, $z_{3k+1} = (1/2^k)(\sqrt{3} + i3)/4$, $z_{3k+2} = (1/2^k)(-\sqrt{3} + i3)/4$, $k = 0, 1, 2, \ldots$, and then to show that $\chi(z_j, z_{j+1}) \geqq 1/(2\sqrt{13})$, for all j. The sequence satisfies (1.3) and the rectilinear curve $z(t)$ through the z_j satisfies (1.4), both using the cone $\mathscr{C}$. However the points $-1/z_{3k}$, $-1/z_{3k+1}$, and $-1/z_{3k+2}$ all lie on the straight line $\operatorname{Im} z = 1/|z_{3k}|$ with $-1/z_{3k+1}$ lying to the left and $-1/z_{3k+2}$ lying to the right of $-1/z_{3k}$. If the sequence $\{-1/z_j\}$ and the curve $-1/z(t)$ satisfied (1.3) and (1.4) respectively, for some cone $\mathscr{C}^*$, the fact that $-1/z_{3k+1} \in \mathscr{C}^*(-1/z_{3k})$ and $-1/z_{3k+2} \in \mathscr{C}^*(-1/z_{3k})$ would imply that the vertex angle of $\mathscr{C}^*$ is at least π. This completes the example.

We now generalize Corollary 1.

Definition. Let $\{z_n\}$ be a sequence lying in Δ or H with $\lim_{n\to\infty} z_n = \tau$, a point on the boundary of Δ or H. The sequence is a conical sequence if there exists a cone $\mathscr{C}$ and an index N such that

(2.4) i) if $\tau \neq \infty$, $z_{n+1} \in \mathscr{C}(z_n)$, for all $n > N$;

(2.5) ii) if $\tau = \infty$, $-\dfrac{1}{z_{n+1}} \in \mathscr{C}\left(-\dfrac{1}{z_n}\right)$, for all $n > N$.

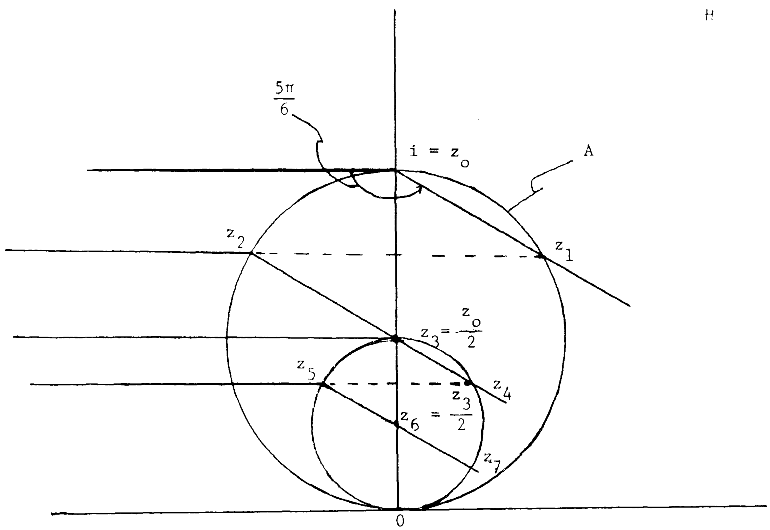

Figure 1.

THEOREM 4. *Any conical sequence $\{z_n\}$ in H or Δ which is consecutively separated is an interpolating sequence. Further, conical sequences are conformally invariant on Δ or H or between them.*

PROOF. First we show that $\{z_n\}$ is interpolating. If $\{z_n\}$ tends to a finite boundary point, then Corollary 1 applies to give interpolation. If $\{z_n\}$ tends to ∞ then $\{-1/z_n\}$ is consecutively separated and satisfies the hypothesis of Corollary 1. Hence $\{-1/z_n\}$, and thus $\{z_n\}$ is interpolating. To show that conical sequences are conformally invariant, all we need to find is a piecewise differentiable boundary curve $z(t)$ passing through the sequence points (with increasing parameter) ending at a point and which is moderately oscillating. Then, for any admissible conformal transformation $L(z)$, the curve $L(z(t))$ is moderately osciallating by the conformal invariance of this property. This implies that the sequence $\{L(z_n) = L(z(t_n))\}$, $t_n < t_{n+1}$, is conical. If $\{z_n\}$ tends to a finite boundary point, then we take $z(t)$ to be the curve obtained by successively joining z_n to z_{n+1} by a straight line segment. It certainly is piecewise differentiable, ends at a single point, and is moderately oscillating. If $\{z_n\}$ tends to ∞, we construct, as just described, the curve $z(t)$ passing through the sequence points $\{-1/z_n\}$. We then let $z_1(t) = -1/(z(t))$. Then $z_1(t)$ is also a piecewise differentiable curve ending at ∞ and passing through the points $\{z_n\}$. It is moderately oscillating because the sequence $\{-1/z_n\}$ satisfies (2.5) and so $z_1(t)$ satisfies (1.5).

For boundary curves in Δ ending at a point, there is an equivalent formulation for Theorem 3 in polar coordinates. Given such a curve $z(t) = r(t)e^{i\theta(t)}$, for $t \notin E_z$, let $\kappa(t)$ denote the angle from $z(t)$ to $z'(t)$, $-\pi < \kappa(t) \leqq \pi$. For future considerations we note that

$$\tan \kappa(t) = \frac{r(t)\,\theta'(t)}{r'(t)}. \tag{2.6}$$

We remark here that, for the remainder of the paper, when considering ratios of real number a/b, we assume that a and b are not both zero and if $b = 0$ we define $a/0 = +\infty$ if $a > 0$ and $a/0 = -\infty$ if $a < 0$.

THEOREM 5. *Let $z(t) = r(t)e^{i\theta(t)}$ be a boundary curve in Δ ending at $e^{i\varphi}$. Then $z(t)$ is moderately oscillating if and only if there exist real numbers θ_1, α', $\pi/2 < \alpha' < \pi$, and a value t_0 such that, for $t \geqq t_0$* and $t \notin E_z$,

$$\theta_1 < \kappa(t) < \theta_1 + \alpha'. \tag{2.7}$$

PROOF. Assume $t \geqq t_0$, and suppose $z(t)$ is moderately oscillating. We can assume without loss of generality that α and θ_0 are chosen so that

$$\varphi - \pi \leqq \theta_0 < \arg z'(t) < \theta_0 + \alpha < \varphi + \pi.$$

If $t \notin E_z$, $\arg z'(t) = \theta(t) + \kappa(t)$. For small $\varepsilon > 0$ there is a t_1 such that, for $t > t_1$ and $t \notin E_z$,

$$\theta_0 - \varphi - \varepsilon < \kappa(t) < \theta_0 - \varphi + \alpha + \varepsilon.$$

So for $t > t_1$, $\kappa(t)$ satisfies (2.7) with $\theta_1 = \theta_0 - \varphi - \varepsilon$ and $\alpha' = \alpha + 2\varepsilon$. Conversely, if $\kappa(t)$ satisfies (2.7), we may choose $\arg z'(t)$ so that $\arg z'(t) = \theta(t) + \kappa(t)$. So, for small ε, there is a t_2 such that, for $t > t_2$,

$$\theta_1 + \varphi - \varepsilon < \arg z'(t) < \theta_1 + \varphi + \alpha' + \varepsilon,$$

and so $z(t)$ is moderately oscillating.

We conclude this section with a condition on a smooth curve which is equivalent to moderate oscillation provided the boundary curve ends at a finite point.

THEOREM 6. *Let $z(t) = z(t) + iy(t)$, be a smooth boundary curve ending at a finite point. Then $z(t)$ has moderate oscillation if and only if there exists a $t_0 > 0$ such that the set of (extended) real numbers $\{(y'(t))/(x'(t)), t \geqq t_0\}$ omits some open interval.*

PROOF. For smooth curves, $\arg z'(t)$ has a continuous determination, and so the set $A(t_0) = \{\arg z'(t), t \geqq t_0\}$ is an interval on the real line. It contains an interval of length π if and only if the set $T(t_0) = \{\tan(\arg z'(t)) = (y'(t))/(x'(t)), t \geqq t_0\}$ omits at most one point. Thus $A(t_0)$ is contained in some interval of the form $[\theta_0, \alpha + \theta_0]$, $\pi/2 < \alpha < \pi$, if and only if $T(t_0)$ omits some open interval. Hence an application of Theorem 3 completes the proof. For the disc Δ there is a polar form of Theorem 6.

THEOREM 7. *Let $z(t) = r(t)e^{i\theta(t)}$ be a smooth boundary curve in Δ ending at a point. Then $z(t)$ has moderate oscillation if and only if there exists a $t_0 > 0$ such that the set $\{\theta'(t))/(r'(t)), t \geqq t_0\}$ omits some open interval.*

PROOF. The set $A(t_0) = \{\kappa(t), t \geqq t_0 > 0\}$ is an interval in the reals and so it contains an interval of length π if and only if $T(t_0) = \{\tan \kappa(t) = (r(t)\theta'(t))/(r'(t)), t \geqq t_0\}$ omits at most one point. Thus $A(t_0)$ is contained in an interval of the form $[\theta_1, \alpha' + \theta_1]$, $\pi/2 < \alpha' < \pi$, if and only if $T(t_0)$ omits an open interval. But the set $T^*(t_0) = \{(\theta'(t))/(r'(t)), t \geqq t_0\}$ is close to the set $T(t_0)$ for large t_0 and so there is a value t_0 such that $T^*(t_0)$ omits an interval if and only if $T(t_0)$ does. Involing Theorem 4 completes the argument.

In either Theorem 6 or 7 the ratios could be inverted giving the same result. In the next section we give more readily verifiable conditions necessary for a boundary curve ending at a finite point to be moderately oscillating.

3. Sufficient conditions for moderate oscillation. The theorems of this section apply only to boundary curves ending at a finite point. For curves tending to ∞, the best criterion for moderate oscillation remains Theorem 3.

THEOREM 8. *Let $z(t) = z(t) + iy(t)$ be a boundary curve ending at a finite point. Then $z(t)$ has moderate oscillation if condition (A) or (B) is satisfied.*

(A) *$x'(t)$, $t \notin E_z$, does not change sign and either*

$$\varliminf_{t\to\infty} \frac{y'(t)}{x'(t)} \text{ or } \varlimsup_{t\to\infty} \frac{y'(t)}{x'(t)}, \qquad t \notin E_z,$$

is finite;

(B) *$y'(t)$, $t \notin E_z$, does not change sign and either*

$$\varliminf_{t\to\infty} \frac{x'(t)}{y'(t)} \text{ or } \varlimsup_{t\to\infty} \frac{x'(t)}{y'(t)}, \qquad t \notin E_z,$$

is finite.

If $z(t)$ is assumed to be a smooth curve ending at a finite point, then it has moderate oscilltion if any one of the four above mentioned limits is finite.

PROOF. We begin with a piecewise differentiable curve which has $x'(t) \geqq 0$, $t \notin E_z$. It is easy to see that the continuity of $x(t)$ insures that $x(t)$ is non-decreasing for all $t \in \mathbf{R}^+$. Thus arg $z'(t)$ satisfies

$$(3.0) \qquad -\pi/2 \leqq \arg z'(t) \leqq \pi/2.$$

If $\varlimsup_{t\to\infty}((y'(t)/(x'(t)) = M < \infty$, $t \notin E_z$, then (3.0) gives, for large t and small $\varepsilon > 0$,

$$-\pi/2 \leqq \arg z'(t) \leqq \arctan M + \varepsilon,$$

and Theorem 3 implies $z(t)$ is moderately oscillating. The other three cases are entirely similar with arccot M replacing arctan M in case B. If $z(t)$ is a smooth curve, Theorem 6 is immediately applicable.

THEOREM 9. *Let $z(t) = r(t)e^{i\theta(t)}$ be a boundary curve in Δ ending at a point. Then $z(t)$ has moderate oscillation if condition (A)* or *(B) is satisfied.*

(A) *$'r(t) \geqq 0$, $t \notin E_z$, and either*

$$\varliminf_{t\to\infty} \frac{\theta'(t)}{r'(t)} \text{ or } \varlimsup_{t\to\infty} \frac{\theta'(t)}{r'(t)}, \qquad t \notin E_z$$

is finite;

(B) *$\theta'(t)$, $t \notin E_z$, does not change sign and either*

$$\varliminf_{t\to\infty} \frac{r'(t)}{\theta'(t)} \text{ or } \varlimsup_{t\to\infty} \frac{r'(t)}{\theta'(t)}, \qquad t \notin E_z,$$

is finite.

If $z(t)$ is a smooth boundary curve in Δ, then it is moderately oscillating if any one of the four above mentioned limits is finite.

PROOF. We begin with the piecewise difierentiable case and assume $r'(t) \geqq 0$, $t \notin E_z$. As in Theorem 8 this implies that $r(t)$ is increasing on $\mathbf{R}^+$. The differentiability of $z(t)$ together with $r(t)$ increasing makes it easy to show that

$$-\pi/2 \leqq \kappa(t) \leqq \pi/2. \tag{3.1}$$

Recalling that $\tan \kappa(t) = r(t)\,(\theta'(t)/(r'(t))$, if $\overline{\lim}_{t\to\infty}(\theta'(t)/(r'(t)) = M < \infty$, then because $r(t) \to 1$ as $t \to \infty$, there exist positive numbers ε and t_0 such that, for $t \geqq t_0$, $t \notin E_z$,

$$-\pi/2 - \varepsilon \leqq \kappa(t) \leqq \arctan M + \varepsilon. \tag{3.2}$$

Hence $z(t)$ has moderate oscillation. The proofs for the other cases are similar with case (B) using $\cot \kappa(t) = (r'(t)/(r(t)\,\theta'(t))$. If $z(t)$ is a smooth curve, then a direct application of Theorem 7 shows that $z(t)$ is moderately oscillating and the proof is complete.

If $z(t) = x(t) + iy(t) = r(t)e^{i\theta(t)}$ is a boundary curve ending at a finite point such that neither $x'(t)$ nor $y'(t)$ changes sign for $t \notin E_z$, neither does the quantity $(y'(t)/(x'(t))$, and so $z(t)$ has moderate oscillation. If $z(t)$ lies in Δ with neither $\theta'(t)$ nor $r'(t)$ changing signs, then $z(t)$ has moderate oscillation. It is possible to combine the various polar and rectangular conditions and we illustrate such a combination in the next theorem.

THEOREM 10. *Let $z(t) = x(t) + iy(t) = r(t)e^{i\theta(t)}$ be a boundary curve in Δ ending at 1. If $z(t)$ satisfies one of the following conditions for $t \notin E_z$, then $z(t)$ is moderately oscillating.*

(A) *$r'(t) \geqq 0$ and $y'(t)$ does not change sign;*

(B) *$x'(t) \geqq 0$ and $\theta'(t)$ does not change sign.*

PROOF. All the proofs are entirely similr and we show the proof in the case $r'(t) \geqq 0$ and $y'(t) \leqq 0$, $t \notin E_z$. The first inequality implies that $r(t)$ is non decreasing and so $-\pi/2 \leqq \kappa(t) \leqq \pi/2$; the second implies that $y(t)$ is non increasing and so $-\pi \leqq \arg z'(t) \leqq 0$. Because $\arg z'(t) = \theta(t) + \kappa(t)$, and $\theta(t) \to 0$ as $t \to \infty$, there is a $t_0 > 0$ such that $-(3\pi)/4 \leqq \arg z'(t) \leqq -\pi/6$, and, by Theorem 3, $z(t)$ had moderate oscillation. The other cases are shown in a similar manner. If $z(t)$ ends at $e^{i\varphi}$ other combinations imply that $z(t)$ is moderately oscillating.

The results of §3 on piecewise differentiable curves can be applied to sequences tending to a finite point to show they are conical sequences.

4. Conditions on sequences implying interpolation. If $\{z_n\}_0^\infty$ is a sequence of complex numbers, then the curve $z_1(t) = z_n + (t - n)\,(z_{n+1} - z_n)$,

$n \leqq t \leqq n+1$, $n = 0, 1, 2, \ldots$, is a continuous piecewise differentiable curve connecting the points z_n. If $z_n = x_n + iy_n$ and we set $z_1(t) = x(t) + iy(t)$, then $y'(t) = y_{n+1} - y_n$; $x'(t) = x_{n+1} - x_n$, $n < t < n + 1$, $n = 0, 1, 2, \ldots$.

For polar representation, setting $z_n = r_n e^{i\theta_n}$ define, for $n = 0, 1, 2, \ldots$,

$$z_2(t) = \begin{cases} (r_n + 2(r_{n+1} - r_n)(t-n))\, e^{i\theta_n}, & n \leqq t \leqq n + 1/2; \\ r_{n+1}\, e^{i(\theta_n + (\theta_{n+1} - \theta_n)(2t-2n-1))}, & n + 1/2 \leqq t \leqq n+1; \end{cases}$$

This curve is again a piecewise differentiable curve connecting the points z_n. If we set $z_2(t) = r(t)e^{i\theta(t)}$, then, for $n = 0, 1, 2, \ldots$,

$$r'(t) = \begin{cases} 2(r_{n+1} - r_n), & n < t < n + \frac{1}{2}; \\ 0, & n + \frac{1}{2} < t < n+1; \end{cases} \tag{4.0}$$

$$\theta'(t) = \begin{cases} 0, & n < t < n + \frac{1}{2}; \\ 2(\theta_{n+1} - \theta_n), & n + \frac{1}{2} < t < n+1. \end{cases} \tag{4.1}$$

COROLLARY 2. *Let $\{z_n = x_n + iy_n\}$ be a consecutively separated sequence in either Δ or H which tends to a finite boundary point. Then $\{z_n\}$ is an interpolating sequence if either condition* (A) *or* (B) *is satisfied.*

(A) *$\{x_n\}$ is a monotonic sequence and either*

$$\underline{\lim}_{n\to\infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \ \textit{or} \ \overline{\lim}_{n\to\infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n}$$

is finite;

(B) *$\{y_n\}$ is a monotonic sequence and either*

$$\underline{\lim}_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \ \textit{or} \ \overline{\lim}_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

is finite.

PROOF. The curve $z_1(t)$, defined by the sequence, satisfies the hypothesis of Theorem 8 and so is moderately oscillating; thus $\{z_n\}$ is a conical sequence and by Theorem 4 is an interpolating sequence.

The proofs of the next two corollaries are similar and based upon Theorem 9 and 10 respectively.

COROLLARY 3. *Let $\{z_n = r_n e^{i\theta_n}\}$ be a consecutively separated sequence in Δ tending to a boundary point. Then $\{z_n\}$ is an interpolating sequence if either conditions* (A) *or* (B) *is satisfied.*

(A) *$\{r_n\}$ is monotonically increasing and either*

$$\varliminf_{n\to\infty} \frac{\theta_{n+1} - \theta_n}{r_{n+1} - r_n} \text{ or } \varlimsup_{n\to\infty} \frac{\theta_{n+1} - \theta_n}{r_{n+1} - r_n}$$

is finite;

(B) $\{\theta_n\}$ *is a monotonic sequence and either*

$$\varliminf_{n\to\infty} \frac{r_{n+1} - r_n}{\theta_{n+1} - \theta_n} \text{ or } \varlimsup_{n\to\infty} \frac{r_{n+1} - r_n}{\theta_{n+1} - \theta_n}$$

is finite.

PROOF. After a rotation of Δ we may assume that $\{z_n\}$ tends to 1. We use the curve $z_2(t) = r(t)e^{i\theta(t)}$, defined by $\{z_n\}$. Recall that $\tan \kappa(t) = (r(t)\theta'(t))/(r'(t))$. Then (4.0) and (4.1) show that, as $t \to \infty$, $r'(t)$ and $\theta'(t)$ eventually are of one sign if $\{r_n\}$ and $\{\theta_n\}$, respectively, are monotonic, and $\tan \kappa(t)$ is asymptotic to $(\theta_{n+1} - \theta_n)/(r_{n+1} - r_n)$. Thus Theorem 9 shows that $z_2(t)$ is moderately oscillating if either A or B satisfied.

The proof for the next corollary follows along the same line so we omit it.

COROLLARY 4. *Let* $\{z_n = x_n + iy_n = r_n e^{i\theta_n}\}$ *be a consecutively separated sequence in* Δ *tending to* 1. *Then* $\{z_n\}$ *is an interpolating sequence if either condition* (A) *or* (B) *is satisfied.*

(A) $\{r_n\}$ *is a monotonically increasing sequence and* $\{y_n\}$ *is a monotonic sequence*;

(B) $\{x_n\}$ *is a monotonically increasing sequence and* $\{\theta_n\}$ *is a monotonic sequence.*

Theorem 8 contains the results on the M-sequences of Weiss [7] and Theorem 10 contains a results of Gerber and Weiss [3], while Theorem 2 contains Wortman's result [8] on convex curves.

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