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THE SHARPNESS OF SOME CLUSTER SET RESULTS

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Abstract. We show that a recent cluster set theorem of Rung is sharp in a certain sense. This is accomplished through the construction of an interpolating sequence whose limit set is closed, totally disconnected and porous. The results also generalize some of Dolzenko's cluster set theorems.

Key words. Cluster sets, interpolating sequences, porous sets. 1980 Mathematics Subject Classification. Primary 30D40, Secondary 30E05.

1. INTERPOLATING SEQUENCES. We begin by considering a closed totally disconnected set P on the boundary $\partial\Delta$ of the unit disc Δ in the complex plane. Thus $\tilde{P}=\partial\Delta-P$ is the union of countably many disjoint arcs. Our first objective is to construct interpolating sequences on certain curves in the unit disc Δ whose limit points are all the points of P. In a special case Dolzenko [1] used this construction apparently not realizing that he was dealing with interpolating sequences. We wish to define an approach to a point $\tau \in \partial\Delta$ inside a reasonably nice subdomain of Δ. Let $h(t)$ be a real-valued function defined for $-1 \le t \le 1$. We require that

(i) h be continuous.
(ii) $h(t)=h(-t)$.
(iii) $h(0)=0,\ h(1)=1,\ h(t_1) \le h(t_2),\ 0 \le t_1 \le t_2 \le 1.$ (1.1)
(iv) $h(t) \le t$.
(v) $h''(t) > 0,\ t \neq 0$.

Such an h is said to be a convex approach function. This function h determines a convex boundary domain $\Omega(\theta,h)$ at $\tau=e^{i\theta}$ as follows (See Fig. 1):

$$\Omega(\theta,h) = \{re^{it}:0 \le r \le 1-h(t-\theta);\ |t-\theta| \le 1\} \quad (1.2)$$

For example $h(t)=t$ defines the usual nontangential approach; $h(t)=t^2$ defines the horocyclic approach and so on. The boundary of the domain Ω

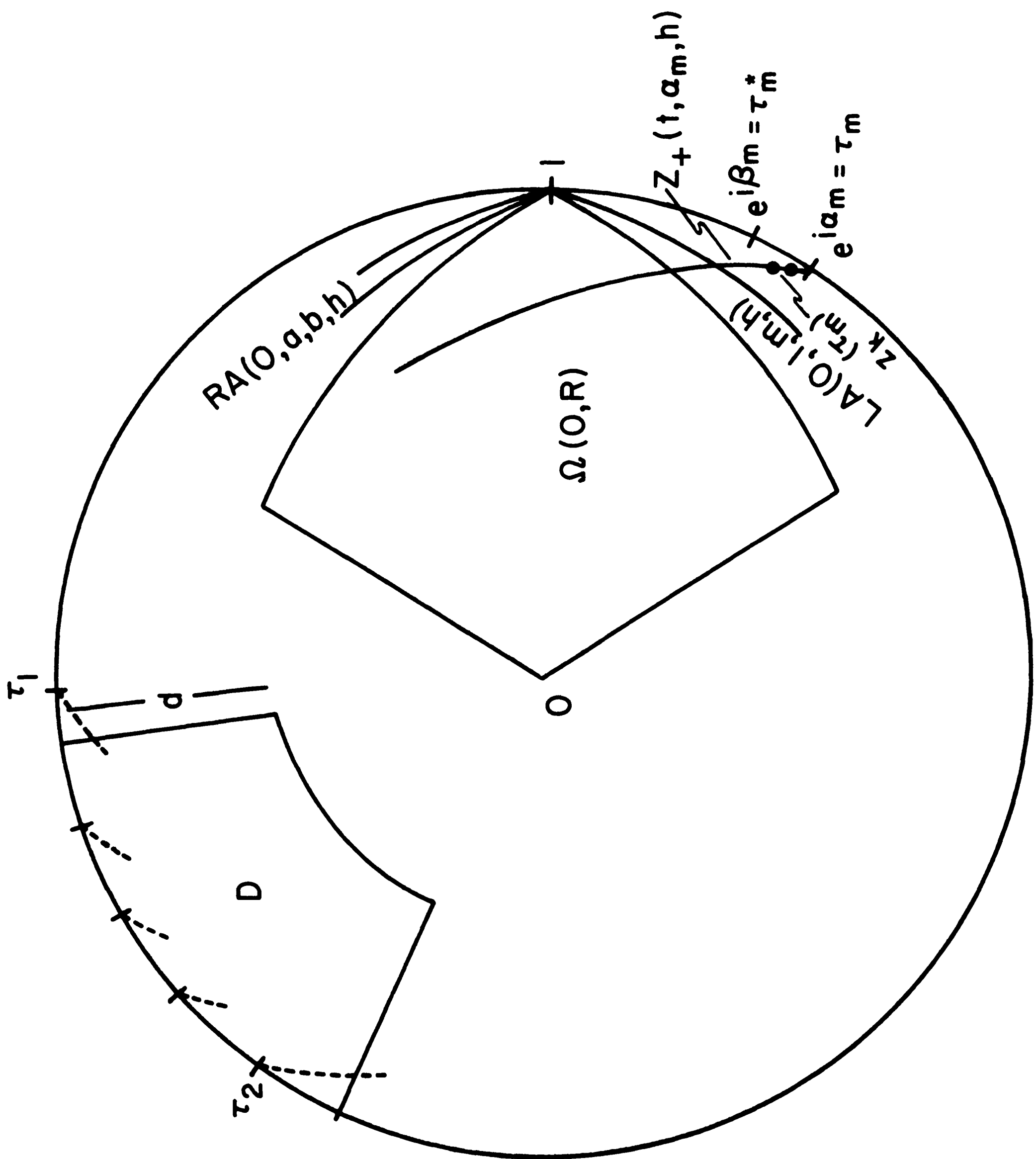

Fig. 1 The graph of h-angle domains, the sequence $\{z_k\}$, and the set D.

consists of two h-curves Z_+ and Z_- which meet at $\tau=e^{i\theta}$ and are defined by

$$Z_+(t,\theta,h) = [1-h(t-\theta)]e^{it}, \ 0 \le t-\theta \le 1,$$
$$Z_-(t,\theta,h) = [1-h(t-\theta)]e^{it}, \ 0 \le \theta-t \le 1. \tag{1.3}$$

The first curve in (1.3) is called the right h-curve at τ and the second curve in (1.3) is called the left h-curve at τ (See Fig. 1). Clearly these curves are rotations of the corresponding curves at 1.

We construct an interpolating sequence which has P as its limit set. Recall that a sequence $\{z_n\}$ is an interpolating sequence if, for each bounded sequence of complex numbers $\{w_n\}$, there exists a function f in H^∞ such that $f(z_n)=w_n$ for every n. We shall use the characterization of

Garnett [2] for interpolating sequences. For $a,b \in \Delta$, set

$$\chi(a,b) = \left|\frac{a-b}{1-a\bar{b}}\right| ,$$

the pseudohyperbolic distance on Δ The sequence $\{z_n\}$ is interpolating if and only if

$$\begin{aligned} &\text{(i)} \quad \varliminf_{n\neq m} \chi(z_n,z_m) > 0; \\ &\text{(ii)} \quad \text{There exists a constant } A \text{ such that for any domain} \end{aligned} \tag{1.4}$$

$D = \{re^{i\theta}: 1-d \leq r \leq 1,\ |\theta-\theta_0| \leq d\}$,

$$\sum_{z\in D} (1-|z_n|) \leq Ad.$$

Such a domain D is shown in Fig. 1.

Recall that $\tilde{P}=\partial\Delta-P$ is the countable union of disjoint open arcs which we denote by (τ_n,τ_n^*), oriented in the usual counterclockwise sense. We denote the length of this arc by $|(\tau_n,\tau_n^*)|$. Fix such an arc (τ_n,τ_n^*). We now construct a sequence $\{z_k\}_0^\infty$ on the right h-curve Z_+ ending at τ_n (We could equally well define these sequences on the left h-curve or both. For simplicity we put them only on the right h-curves). Because any Z_+ intersects each circle $|z|=r$, $0 < r < 1$, in at most one point it is enough to define $\{z_k\}$ by specifying $|z_k|$. Thus let

$$\begin{aligned} |z_0| &= 1 - h(|(\tau_n,\tau_n^*)|/16) = 1-K \\ 1-|z_k| &= (1-|z_0|)/2^k = K/2^k , \quad k \geq 1. \end{aligned} \tag{1.5}$$

Consequently, if $z_k = |z_k|\, e^{i\theta_k}$ and $\tau_n=e^{i\theta_n}$, then from (1.3) we see that z_k lies "over" the interval (τ_n,τ_n^*) and is of the form

$$z_k = [1-h(\theta_k-\theta_n)]e^{i\theta_k}, \quad 0 < \theta_k-\theta_n < \pi/8, \quad |z_k| > 1/2. \tag{1.6}$$

We note that

$$\chi(z_k,z_{k+1}) \geq \chi(|z_k|,|z_{k+1}|) = \left|\frac{\frac{1}{2^k} - \frac{1}{2^{k+1}}}{1-\left(1-\frac{K}{2^k}\right)}\right| = \frac{1}{3-K/2^k} ,$$

and thus $\chi(z_n,z_m) > 1/3$, $n\neq m$. To show (1.4)(ii), for a given domain D let n be the least index such that $z_n \in D$. Note that

$$\sum_{z_k\in D}(1-|z_k|) \leq \sum_{k=n}^{\infty} (1-|z_k|) = \sum_{k=n}^{\infty} (1-|z_n|)/2^k = 2(1-|z_n|) < 2d,$$

hence the sequence $\{z_k\}$ is interpolating.

We want to construct a larger interpolating sequence by taking the union of all sequences at the points τ_n. For each arc (τ_n,τ_n^*) in $\tilde{P}$ construct a sequence on Z_- in exactly the same manner as above. We claim that the (countable) union, S, of these sequences is still an interpolating sequence (Note that any rearrangement of an interpolating sequence is still interpolating). We first show that (1.4) is satisfied, beginning with (1.4)(i).

We will show that if $a,b \in S$, then

$$\chi(a,b) > \frac{1}{4\pi} . \tag{1.7}$$

The proof of the above inequality can be done in two steps. First, if a and b lie on the same h-curve then we have shown that $\chi(a,b) > 1/3$. Second, if a,b lie on two different h-curves then we use the following inequality (See [3], p.474) which is valid for any $a,b \in \Delta$

$$\frac{\chi(a,b)}{1+|a|\chi(a,b)} \le \frac{|a-b|}{1-|a|^2} \le \frac{\chi(a,b)}{1-|a|\chi(a,b)} . \tag{1.8}$$

The right inequality implies that if

$$\frac{|a-b|}{1-|a|} > \frac{1}{\pi} \tag{1.9}$$

then

$$\chi(a,b) > \frac{1}{4\pi} . \tag{1.10}$$

Thus we show that (1.9) is valid. Let $z_m^{\tau_n}$ be an element of the sequence on the h-curve ending at τ_n and let $z_k^{\tau_j}$ be an element of the sequence on the h-curve ending at τ_j as shown in Fig. 1. Set $\arg z_m^{\tau_n} = \theta_m$ and $\arg \tau_n = \theta_n$. We may assume $\arg \tau_n > \arg \tau_j$. Then it is clear that

$$|z_m^{\tau_n} - z_k^{\tau_j}| > |z_m^{\tau_n}| \sin(\theta_m - \theta_n) \tag{1.11}$$

and

$$\sin(\theta_m - \theta_n) > (2/\pi)(\theta_m - \theta_n) \tag{1.12}$$

Thus

$$\frac{|z_m^{\tau_n} - z_k^{\tau_j}|}{1-|z_m^{\tau_n}|} > \frac{|z_m^{\tau_n}| \sin(\theta_m - \theta_n)}{h(\theta_m - \theta_n)} \ge \frac{1}{\pi} ,$$

where we used (1.1), (1.6), (1.11) and (1.12). This proves that the sequence S satisfies (1.4)(i).

For (1.4)(ii), let D be the domain specified there. We claim that

$$\sum_{a \in S \cap D} (1-|a|) \le 5d. \tag{1.13}$$

The points of S that lie in D belong to curves that end at the boundary of D except for at most two curves which might end outside D (See Fig. 1). Partition the points of $S \cap D$ into two sets A and B as follows: $z_n^{\tau_m} \in A$ if and only if $(\tau_m, \tau_m^*) \subseteq \partial D \cap \partial\Delta$, otherwise put $z_n^{\tau_m} \in B$. Thus from (1.1) and (1.5) we have

$$\sum_{z_n^{\tau_m}\in A} (1-|z_n^{\tau_m}|) = \sum_m \sum_n (1-|z_n^{\tau_m}|) \le \sum_m \sum_{n=0}^{\infty} (1-|z_n^{\tau_m}|) = 2\sum_m (1-|z_0^{\tau_m}|) \le \frac{1}{8} \sum_m |(\tau_m,\tau_m^*)| \le \frac{d}{8}.$$

If $\{z_n^{\tau_m}\} \subseteq B$ then τ_m has at most two values, say τ_1 and τ_2 (See Fig. 1). Let $z_{n_i}^{\tau_i}$ denote the first term of each sequence lying in D then

$$\sum_{i=1}^{2} \sum_{z_n^{\tau_i}\in B} (1-|z_n^{\tau_i}|) \le \sum_{i=1}^{2} (1-|z_{n_i}^{\tau_i}|) \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= 2 \sum_{i=1}^{2} (1-|z_{n_i}^{\tau_i}|) \le 4d.$$

where we used (1.5). Thus

$$\sum_{a\in S\cap D} (1-|a|) \le 5d,$$

which implies that S is an interpolating sequence.

2. <u>POROSITY AND RIGHT h-ANGLES</u>. In this section we add another restriction on the set $P \subseteq \partial\Delta$. We assume that P is porous. The notion of porosity was introduced in 1967 by Dolzenko [1] and later used by Rung [4] and Yoshida [5] to generalize some of the cluster theory results. We note that in 1976 Zajicek [6] generalized the definition of porosity and proved a variety of interesting properties of porous sets.

Let $P \subseteq \partial\Delta$. For each $e^{i\theta} \in \partial\Delta$, let $\eta(\theta,\varepsilon,P)$ be the length of the largest subarc of the arc $(e^{i(\theta-\varepsilon)}, e^{i(\theta+\varepsilon)})$ which does not meet P. If no such arc exists define $\eta(\theta,\varepsilon,P)=0$. According to Dolzenko [1], P is porous at $e^{i\theta}$ if

$$\overline{\lim_{\varepsilon\to 0}} \frac{\eta(\theta,\varepsilon,P)}{\varepsilon} > 0. \tag{2.1}$$

A set $P \subseteq \partial\Delta$ is porous if it is porous at each $p \in P$; P is σ-porous if it is the finite or countable union of porous sets. A porous set is nowhere dense and thus a σ-porous set is of the first Baire category.

We now define a right h-angle in Δ at $\tau=e^{i\theta} \in \partial\Delta$. For any positive constant c, set $h^c(x)=h\left(\frac{x}{c}\right)$, $-c \le x \le c$; Then h^c is also a convex approach function. For any constants $0 < a < b$, define

$$RA(\theta,a,b,h) = \{re^{i\phi}: 1-h^a(\phi-\theta) < r < 1-h^b(\phi-\theta),\ 0 \le \phi-\theta \le a\}. \tag{2.2}$$

The boundary curve of $RA(\theta,a,b,h)$ defined by the left inequality will be called the lower boundary curve of the right h-angle domain and the other boundary curve is called the upper boundary curve (See Fig. 1). The left h-angle domain at τ, $LA(\theta,a,b,h)$ is defined by replacing $\phi-\theta$ by $\theta-\phi$ in (2.2) with upper and lower boundary curves defined by the same inequality. If $h(t)=t$ then this represents a typical Stolz angle domain.

If $E \subseteq \Delta$ and if $E \cap \partial\Delta \ne \emptyset$ then the cluster set of a function f along E will be denoted by $C(f,E)$. Our final objective is to investigate the sharpness of the following theorem of Rung [4].

Theorem R. Let f be defined in Δ taking values in the extended complex plane, and h a given approach function. Then for all $e^{i\theta} \in \partial\Delta$, except for a σ-porous set, for any choice of $0 < a < b$ and $c > 0$,

$$C(f, RA(\theta, a, b, h)) = C(f, LA(\theta, a, b, h)) = C(f, \Omega(\theta, h^c))$$

We start by introducing two well-known results due to Garnett (See Koosis [7], p.281-282) and Kerr-Lawson [8].

Lemma 1 (Garnett) If there is an $\eta > 0$ such that $\chi(a_n, a_m) \geq \eta$, for $n \neq m$, and if $\sum_{a_n \in D} (1-|a_n|^2) \leq Ad$, where D is given by (1.4) and A is a constant then

$$\inf_m \prod_{\substack{n=1 \\ m\neq n}}^{\infty} \chi(a_n, a_m) \geq \delta,$$

where δ depends only on η and A.

Lemma 2 (Kerr-Lawson). Let $B(z)$ be the Blaschke function whose zeros are given by the sequence $\{a_n\}$. Suppose that

$$\inf_m \prod_{\substack{n=1 \\ m\neq n}}^{\infty} \chi(a_n, a_m) \geq \delta > 0.$$

Then given a number $\delta_0 > 0$ there exists a number $\varepsilon > 0$ which depends only on δ and δ_0 such that the set $\{z : |B(z)| < \varepsilon\}$ is contained in the union of disjoint pseudohyperbolic discs $N(a_n, \delta_0)$ with χ-center a_n and χ-radius δ_0.

Theorem 1. Let P be a closed totally disconnected porous subset of $\partial\Delta$ and let h be a convex approach function. Then there exists a Blaschke function $B(z)$ with the following properties:

(i) $B(z)$ is defined and analytic in $\bar{\Delta}-P$.

(ii) There exists an $\varepsilon > 0$ such that for each $e^{i\beta} \in \partial\Delta$, $\underline{\lim}\, |B(z)| \geq \varepsilon$ as $z \to e^{i\beta}$, with $z \in RA(\beta, \frac{1}{3}, \frac{1}{2}, h)$;

(iii) For each $\tau = e^{i\beta} \in P$, there exists either a $RA(\beta, a, b, R)$ or a $LA(\beta, a, b, h)$ which contain infinitely many zeros of $B(z)$. (The choice of a and b vary with $e^{i\beta}$.)

Proof. Let $B(z)$ be the infinite Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left(\frac{a_n - z}{1-\bar{a}_n z}\right), \qquad z \in \Delta,$$

where $\{a_n\}$ is any arrangement of the interpolating sequence S defined in section 1 relative to the set P. Since $B(z)$ is a Blaschke product then (i) follows.

We now prove (ii). If $\tau=e^{i\beta} \in \partial\Delta-P$ then (ii) is obvious. If $\tau \in P$ then we first show that (ii) holds for the case when $\tau=e^{i\beta}$ is an isolated point of P. In this circumstance τ is the initial endpoint of an arc (τ,τ^*) contained in $\partial\Delta-P$. Using the results of Rung [4, p.204] and Satyanarayana and Weiss [9, p.65] we find that the pseudohyperbolic distance near τ between the boundaries of the right h-angle domain (2.2) is at least $|b-a|/2|b+a|$. Choosing $a=1/2$, $b=3/2$, the above distance equals to 1/4. Thus for large n, each $N(z_n,\frac{1}{8}) \subseteq RA(\beta,\frac{1}{2},\frac{3}{2},h)$.

Lemmas 1 and 2 imply that there exists a positive number ε such that $|B(z)| \geq \varepsilon$ for $z \notin \bigcup_{n=1}^{\infty} N(z_n^{\tau},1/8)$. Thus we have $\underline{\lim}_{z\to\tau} |B(z)| \geq \varepsilon$, for $z \in RA(\beta,\frac{1}{3},\frac{1}{2},h)$. Recall that the points of $\{a_n\}$ lie on right h-curves ending at isolated points of P and lying over the corresponding interval of $\partial\Delta-P$.

Consequently if τ is a limit point of P then it is easy to see that $\bigcup_{n=1}^{\infty} N(a_n,1/8)$ still does not meet this $RA(\beta,\frac{1}{3},\frac{1}{2},h)$ so $\underline{\lim}_{z\to\tau} |B(z)| \geq \varepsilon$, when $z \in RA(\beta,\frac{1}{3},\frac{1}{2},h)$.

Finally we prove (iii). If $\tau=e^{i\beta}$ is an isolated point then clearly $RA(\beta,\frac{1}{2},\frac{3}{2},h)$ contains infinitely many zeros of $B(z)$. Suppose $\tau=e^{i\beta}$ is a limit point of isolated points $\tau_n \in P$. We shall show that there exists an integer $m \geq 2$ such that the right h-angle domain $RA(\beta,1,m,h)$ contains infinitely many points of S. Let $\tilde{P}$ be the union of the arcs

$$\tilde{P} = \bigcup_{n=1}^{\infty} (\tau_n,\tau_n^*)$$

and without loss of generality we take $\beta=0$. By (2.1) there is a sequence of arcs $(e^{i\theta_k},e^{i\theta_k})$ and subarcs $(e^{i\alpha_k},e^{i\beta_k}) \subseteq \partial\Delta-P$ with

$$\lim_{k\to\infty} \frac{\beta_k-\alpha_k}{\theta_k} > 0. \tag{2.3}$$

We consider two cases according as to whether the subsequence $(e^{i\alpha_k},e^{i\beta_k})$ approach 1 from above or below. If both, then select a subsequence approaching from one side of 1. The case of $(e^{i\alpha_k},e^{i\beta_k})$ approaching 1 from below is slightly more complicated and so we prove this case. We suppose to the contrary that none of the left h angle domains $LA(0,1,m,h)$, $m=2,3,4,\ldots$ contain infinitely many points of S. For each m, there must be an infinite subsequence of the arcs $(e^{i\alpha_k},e^{i\beta_k})$ such that the corresponding Blaschke sequence $\{z_m^{(k)}\}$ on $Z_+(t,\alpha_k,h)$ has its first term, $z_0^{(k)}$, between the upper boundary curve $Z_-(t,0,h)$ of $LA(0,1,m,h)$ (defined in (2.2)) and $\partial\Delta$ (See Fig. 1). To see this we first may assume that $|\alpha_k| \leq 1$. Then note that $Z_+(t,\alpha_k,h)$ and $Z_-(t,\beta_k,h)$ meet at a point on the radius to the midpoint of the arc $(e^{i\alpha_k},e^{i\beta_k})$. Because $\arg z_0^{(k)} = \frac{\beta_k-\alpha_k}{16} < \frac{\beta_k-\alpha_k}{2}$, then $z_0^{(k)}$ lies between $Z_-(t,\beta_k,h)$ and $\partial\Delta$ and certainly between $Z_-(t,0,h)$, the lower boundary curve of $LA(0,1,m,h)$, and $\partial\Delta$. But then $z_0^{(k)}$ must lie between the upper boundary curve of $LA(0,1,m,h)$, $Z\_(t,0,h^m)$, and $\partial\Delta$ else there would be infinitely many points of S inside $LA(0,1,m,h)$. For each m choose $(e^{i\alpha_{k_m}}, e^{i\beta_{k_m}})$ satisfying $\alpha_{k_m} \to 0$, $m \to \infty$. According to (1.5) the first Blaschke sequence term $z_0^{(k_m)}$ associated with $e^{i\alpha_{k_m}}$ has

$$1 - |z_0^{(k_m)}| = h(|(e^{i\alpha_{k_m}}, e^{i\beta_{k_m}})|/16) = h\left(\frac{\beta_{k_m}-\alpha_{k_m}}{16}\right) .$$

Thus the point $w_m = |w_m| e^{i\psi_m}$ on $Z\_(t,0,h^m)$ which lies on the same radius as $z_0^{(k_m)}$ satisfies

$$h(\psi_m/m) = 1 - |w_m| > 1 - |z_0^{(k_m)}| = h\left(\frac{\beta_{k_m}-\alpha_{k_m}}{16}\right) .$$

Now $|\psi_m| < \theta_{k_m}$ and so the properties of h^{-1} together with the above two inequalities imply that

$$\frac{\beta_{k_m}-\alpha_{k_m}}{\theta_{k_m}} \le \frac{(\beta_{k_m}-\alpha_{k_m})}{|\psi_m|} \le \frac{16}{m}$$

and this last expression tends to 0 as $m \to \infty$. This contradicts (2.3). When the intervals approach 1 from above the left angle at 1 is replaced by the corresponding right angle at 1 and the proof proceeds along the same lines as before. This completes the proof of the theorem.

Remark 1. Note that the constant ε appearing in property (ii) of Theorem 1 depends only on the three constants $\delta_0=1/8$, $\eta=1/4\pi$, $A=5$ (which appear in (1.7), (1.10) and (1.13) respectively) so that any Blaschke product whose zeros are an interpolating sequence with these three constants satisfies $|B(z)| \ge \varepsilon$ for a single constant ε.

A slight modification of a result of Dolzenko [1, p.8] gives the following lemma.

Lemma 3. *A set* $P \subseteq \partial\Delta$ *given by*

$$P = \bigcup_{n=1}^{\infty} P_n,$$

where P_n *are closed, totally disconnected and porous sets can be written in the form*

$$P = \bigcup_{k=1}^{\infty} F_k$$

where F_k *are disjoint, closed, and porous. Moreover if* $p \neq q$ *then each of the sets* F_p *and* F_q *lies entirely on an arc complementary to the other with respect to* $\partial\Delta$.

Theorem 2. *Given a σ-porous set $P \subseteq \partial\Delta$ which can be written as $P = \bigcup_{n=1}^{\infty} P_n$, where P_n are closed and porous. There exists a bounded holomorphic function $f(z)$ in Δ with the following properties:*

(i) *$f(z)$ is continuous from within Δ at each point of $\partial\Delta - P$;*

(ii) *For each point $\tau \in P$, there exist two h angle domains s at τ such that the cluster sets of f along these angles are different.*

Proof. Consider the set $P = \bigcup_{n=1}^{\infty} P_n$. Lemma 3 implies that there exist mutually disjoint, closed, and porous sets F_k such that $P = \bigcup_{k=1}^{\infty} F_k$. Furthermore F_k lies entirely in an arc complementary to F_j for $k \neq j$. Corresponding to each set F_k we construct a function $B_k(z)$ $(B_k(z) = B(z),\ P = F_k)$ as we have done in Theorem 1. Following Dolzenko we define $f(z)$ as the infinite series

$$f(z) = \sum_{k=1}^{\infty} \varepsilon^{2k} B_k(z), \quad z \in \Delta, \tag{2.4}$$

where ε is the fixed constant appearing in Theorem 2. (Recall that this value ε obtained in property (ii) of Theorem 1 is independent of the particular $B_k(z)$; see Remark 1 after Theorem 1. There is no loss of generality in assuming $0 < \frac{\varepsilon}{1-\varepsilon^2} < 1/2$.) The series (2.4) is clearly uniformly and absolutely convergent on compact subsets of Δ and so $f(z)$ is analytic and bounded on Δ. It is also clear that $f(z)$ is continuous from within Δ at each point $\tau \in C-P$. This proves (i) of the theorem.

We now proceed to prove (ii). Consider a fixed F_{k_0} and let $\tau = e^{i\beta} \in F_{k_0}$. Then for $k \neq k_0$ the functions $B_k(z)$ are continuous at the point τ. Moreover we have for $z \in \Delta$

$$\left| \sum_{k=k_0+1}^{\infty} \varepsilon^{2k} B_k(z) \right| \le \varepsilon^{2(k_0+1)}/(1-\varepsilon^2). \tag{2.5}$$

We now use Theorem 1, specifically property (ii) of $B_{k_0}(z)$, property (2.5), and the continuity of $B_k(z)$ at τ for $k < k_0$ to show the following limit. Set $a = \sum_{k=1}^{k_0-1} \varepsilon^{2k} B_k(\tau)$. Thus for $z \in RA(\beta, \frac{1}{3}, \frac{1}{2}, h)$ we find

$$\begin{aligned}
&\underline{\lim}_{z\to\tau} |f(z) - a| \\
&= \underline{\lim}_{z\to\tau} \left| \varepsilon^{2k_0} B_{k_0}(z) + \sum_{k=1}^{k_0-1} \varepsilon^{2k}[B_k(z) - B_k(\tau)] + \sum_{k_0+1}^{\infty} \varepsilon^{2k} B_k(z) \right| \\
&\ge \varepsilon^{2k_0} \underline{\lim}_{z\to\tau} |B_{k_0}(z)| - \overline{\lim}_{z\to\tau} \left| \sum_{k=0}^{k_0-1} \varepsilon^{2k}[B_k(z) - B_k(\tau)] \right| \\
&\quad - \overline{\lim}_{z\to\tau} \left| \sum_{k_0+1}^{\infty} \varepsilon^{2k} B_k(z) \right|
\end{aligned} \tag{2.6}$$

$$\geq \varepsilon^{2k_0-1} - \varepsilon^{2(k_0+1)}/(1-\varepsilon^2)$$

$$= \varepsilon^{2k_0+1}\left[1 - \frac{\varepsilon}{1-\varepsilon^2}\right] = r_0 > 0.$$

Note that (2.6) implies that there are no points of $C(f, RA(\beta, \frac{1}{3}, \frac{1}{2}, h))$ within the disc $|f(z) \ a| < r_0$. On the other hand property (iii) of Theorem 1 shows that there exists a sequence $\{z_n\}$ of zeros of $B_{k_0}(z)$ contained in an h angle domain at τ such that $z_n \to \tau$. Consequently using (2.5) and the continuity of the finite sum we have

$$\overline{\lim_{z_n\to\tau}} |f(z_n)-a| = \overline{\lim_{z_n\to\tau}} \left| \sum_{k_0+1}^{\infty} \varepsilon^{2k} B_k(z_n) + \sum_{k=1}^{k_0-1} \varepsilon^{2k}[B_k(z_n)-B_k(z)]\right| \tag{2.7}$$
$$\leq \varepsilon^{2(k_0-1)}/(1-\varepsilon^2) = r_1.$$

Now $r_1 < r_0$ because

$$r_0 - r_1 = \varepsilon^{2k_0+1}\left[1 - \left[\frac{2\varepsilon}{1-\varepsilon^2}\right]\right] > 0. \tag{2.8}$$

Expressions (2.7) and (2.8) imply states that the cluster set of $f(z)$ along the h angle domain containing the zeros of $B_{k_0}(z)$ contains points in the circle $|f(z)\text{-}a| < r_0$. This completes the proof of (ii). If the sets P_n are not assumed to be porous then the zeros of $B_{k_0}(z)$ only accumulate at τ and so the best that can be said is that the total cluster set of f at τ is different from the cluster set of f along $RA(\beta, \frac{1}{3}, \frac{1}{2}, h)$.

Thus we have generalized Dolzenko's results and have shown the sharpness of Theorem R when the exceptional σ-porous set is the union of closed porous sets.

REFERENCES

1. DOLZENKO, E.P. Boundary Properties of Arbitrary Functions, Izv. Acad. Nauk SSSR 31, 3-14, (1967) (Russian); English translation; Math. USSR-Izv. 1, 1-12, (1967).
2. GARNETT, J. Interpolating Sequences for Bounded Harmonic Functions, Indiana Univ. Math. J. 21(1971), 187-192.
3. BELNA, C., OBAID, S.A. and RUNG, D.C. Geometric Conditions for Interpolation, Proc. Amer. Math. Soc. 88(1983), 469-475.
4. RUNG, D.C. Meier Type Theorems for General Boundary Approach and σ-Porous Exceptional Sets, Pacific J. Math. 76(1978), 201-213.
5. YOSHIDA, H. On Some Generalizations of Meier's Theorems, Pacific J. Math. 46 (1973), 609-621.
6. ZAJICEK, L. Sets of σ-Porosity and Sets of σ-Porosity (q), Casopis Pest. Math. 101(1976), 350-359.
7. KOOSIS, P. Introduction to H^p Spaces, London Math. Soc. Lecture Note Ser(2), Vol. 40, Cambridge Univ. Press, Cambridge, 1980.
8. KERR-LAWSON, A. Some Lemmas on Interpolating Blaschke Products and a Correction, Canad. J. Math. 17(1969), 531-534.
9. STAYAMARAYANA, U.V. and WEISS, M.L. The Geometry of Convex Curves Tending to 1 in the Unit Disc, Proc. Amer. Math. Soc. 41(1973), 159-166.