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Abstract

In this paper, we will consider the interpolation of fuzzy data by a continuous fuzzy-valued function. We will use Lagrange polynomials, natural splines and complete splines.
Keywords: Fuzzy interpolation, Fuzzy spline, Fuzzy B-spline.

1. Introduction

The following problem was first posed by L.A. Zadeh: "Suppose we are given $n + 1$ (distinct) points x_0, \dots, x_n in \mathbb{R} , and for each of these points a 'fuzzy value' in \mathbb{R} , rather than a crisp one. Is it then possible to construct some function on \mathbb{R} with range also a collection of 'fuzzy values'; which coincide, on the given $n + 1$ points, with the given 'fuzzy values'; and which fulfills some natural 'smoothness' condition?"

Lowen, [6], with respect to Hausdorff metric gave a fuzzy interpolation polynomial of Lagrange type and proved its continuity, but there are not a numerical algorithm in [6]. Kaleva, [5], gave numerical algorithms based on fuzzy Lagrange polynomials and fuzzy cubic *Not-a-Knot* splines. Now let E denote the family of all normal, convex, upper semicontinuous and compactly supported fuzzy sets defined on \mathbb{R} . If $u \in E$ then $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}$, $0 < \alpha < 1$, denotes the α -level set of u . By $[u]^0$ we denote the support of u . It follows that the level sets of a $u \in E$ are closed, bounded intervals of \mathbb{R} .

It is well-know that the addition and multiplication by a real number can be extended to E . In based on level sets we have

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda u]^\alpha = \lambda[u]^\alpha,$$

for all $0 \leq \alpha \leq 1$, $u, v, \in E$ and $\lambda \in \mathbb{R}$.

In E we define the metric

$$h(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric in the space of nonempty compact, convex subsets of \mathbb{R} . By the corresponding properties of the Hausdorff metric, it is easily that for all $u, v, w \in E$ and $\lambda \in \mathbb{R}$

$$h(u + w, v + w) = h(u, v) \quad h(\lambda u, \lambda v) = \lambda h(u, v).$$

Furthermore, if $\lambda, \mu \in \mathbb{R}$ are of same sign then $(\lambda, \mu)u = \lambda u + \mu u$. It follows that the addition and scalar multiplication are continuous with respect to the metric h . Furthermore (E, h) is a complete metric space.

Let $\bar{y} = (y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ and suppose $P_{\bar{y}}(x)$ is the Lagrange interpolation polynomial which interpolates the data $(x_i, y_i); i = 0, 1, \dots, n$, i.e.,

$$P_{\bar{y}}(x_i) = y_i, \quad \forall i = 0, \dots, n,$$

$$P_{\bar{y}}(x) = \sum_{i=0}^n \prod_{j=0, j \neq i}^n \frac{(x - x_j)}{(x_i - x_j)}.$$

2. Fuzzy Interpolation Problem

Let $x_0 < x_1 < \dots < x_n$ be $n + 1$ points in \mathbb{R} and let $\{u_i\}_{i=0}^n$ be $n + 1$ fuzzy numbers in E . A formulation of Zadeh's problem now is the following.

Construct a function $F : \mathbb{R} \rightarrow E$, fulfilling the following properties:

- (i) For all $i = 0, 1, \dots, n : F(x_i) = u_i$,
- (ii) F is continuous,
- (iii) If for all $i = 0, 1, \dots, n$, $u_i = 1_{y_i}$ where $y_i \in \mathbb{R}$ and if f is the unique polynomial of degree $\leq n$ which passes through the points $\{(x_i, y_i)\}_{i=0}^n$, i.e., such that $f(x_i) = y_i$ for all $i = 0, 1, \dots, n$, then F reduces to the function $F(x) = 1_{f(x)}$ for all $x \in \mathbb{R}$.

We construct F in the following way. For each $\alpha \in I_0 = (0, 1]$ and $i = 0, 1, \dots, n$, let

$$J_i^\alpha = u_i^{-1}[\alpha, 1] = [u_i]^\alpha.$$

Note that each J_i^α is a compact nonempty interval. Finally for each $x \in \mathbb{R}$ we define the membership function of fuzzy number $F(x)$ by

$$F(x)(t) = \sup \{ \alpha \in I_0 \mid \exists \bar{y} \in \prod_{i=0}^n J_i^\alpha : P_{\bar{y}}(x) = t \}, \quad (1)$$

for all $t \in \mathbb{R}$.

Theorem 1. For each $x \in \mathbb{R}$ we have $F(x) \in E$, i.e., the assignment

$$F : \mathbb{R} \rightarrow E : x \rightarrow F(x)$$

is a well-defined function. Further we have:

- (i) For all $i = 0, 1, \dots, n$, $F(x_i) = u_i$,
- (ii) F is continuous,
- (iii) If for all $i = 0, 1, \dots, n$ we have $u_i = 1_{[a_i, b_i]}$ and for each $x \in \mathbb{R}$ we put

$$J(x) = \{ t \in \mathbb{R} \mid \exists \bar{y} \in \prod_{i=0}^n [a_i, b_i], P_{\bar{y}}(x) = t \},$$

then F reduces to the function $F(x) = 1_{J(x)}$.

To prove theorem 1, we require two lemmas.

Lemma 1. For any $(x, t) \in \mathbb{R} \times \mathbb{R}$,

$$F(x)(t) = \max \{ \alpha \in I_0 \mid \exists \bar{y} \in \prod_{i=0}^n J_i^\alpha, P_{\bar{y}}(x) = t \},$$

i.e., the supremum is attained.

Lemma 2. For any $\alpha \in I_0$,

$$\{ P_{\bar{y}} \mid \bar{y} \in \prod_{i=0}^n J_i^\alpha \}$$

is an equicontinuous family.

Kaleva, [5], gave a numerical algorithm for computing the fuzzy Lagrange interpolation. From (1), we have

$$\begin{aligned} [F(x)]^\alpha &= \{t \in \mathbb{R} | F(x)(t) \geq \alpha\} \\ &= \{t \in \mathbb{R} | \exists \bar{y} \in \prod_{i=0}^n J_i^\alpha : P_{\bar{y}}(x) = t\} \\ &= \sum_{i=0}^n L_i(x) J_i^\alpha, \end{aligned}$$

where $L_i(x) = \prod_{j=0, j \neq i}^n \frac{(x-x_j)}{(x_i-x_j)}$. Hence $F(x) = \sum_{i=0}^n L_i(x) u_i$.

Theorem 2. *The function F is continuous and if $x \in (x_i, x_{i+1})$ then for all $\alpha \in [0, 1]$*

$$\text{len}[F(x)]^\alpha \geq \min\{\text{len}[F(x_i)]^\alpha, \text{len}[F(x_{i+1})]^\alpha\},$$

where len denotes the length of an interval.

Theorem 3. $L_i(x)$, Lagrange polynomials, satisfies

- (a) L_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,
- (b) the sign of L_i does not change on $[x_j, x_{j+1}]$,
- (c) the sign of L_i changes at x_j for all $j \neq i$.

With respect to these theorems, if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $F(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number, where

$$\begin{aligned} m(x) &= \sum_{i=0}^n L_i(x) m_i, \\ l(x) &= \sum_{L_i(x) \geq 0} L_i(x) l_i - \sum_{L_i(x) < 0} L_i(x) r_i, \\ r(x) &= \sum_{L_i(x) \geq 0} L_i(x) r_i - \sum_{L_i(x) < 0} L_i(x) l_i. \end{aligned}$$

3. fuzzy splines

Kaleva, [5], gave a numerical algorithm for computing the not-a-knot fuzzy spline, also.

Definition 1. *Let $S_l(x_0, x_n)$ denote the family of splines of order l with knots $x_i, i = 0, 1, \dots, n$. In other words $s \in S_l(x_0, x_n)$ if*

- (i) $s \in C^{l-1}[x_0, x_n]$,
- (ii) s is a polynomial of order l in the interval $[x_i, x_{i+1}]$, for all $i = 0, 1, \dots, n-1$.

As in the case of interpolation polynomial we define an interpolating fuzzy spline Fs of order l as follows. If $Fs^\alpha(x)$ is the α -level set of $Fs(x)$ then

$$Fs^\alpha(x) = [Fs(x)]^\alpha = \{y \in \mathbb{R} | y = s_{\bar{y}}(x), \bar{y} = (y_0, y_1, \dots, y_n) \in \prod_{i=0}^n u_i^\alpha\},$$

where $s_{\bar{y}} \in S_l(x_0, x_n)$ is the spline, which interpolates the data $(x_i, y_i), i = 0, 1, \dots, n$.

If $s_i \in S_l(x_0, x_n)$ interpolates the data $(x_j, f_j), j = 0, 1, \dots, n$, where $f_j = \delta_{ij}$, Kronecker delta, then $s_{\bar{y}}(x) = \sum_{i=0}^n y_i s_i(x)$ and in the same way in section 2 we have

$$[Fs(x)]^\alpha = \sum_{i=0}^n s_i(x)[u_i]^\alpha, \quad Fs(x) = \sum_{i=0}^n s_i(x)u_i. \quad (2)$$

It is well-known that the interpolation conditions are not sufficient to uniquely determine the spline s_i . In addition we need two requirements. It appears that the sign of $s_i(x)$ depends on these requirements. If we adopt the not-a-knot condition, then the spline s_i behaves like L_i . The not-a-knot condition, means that the first and the second polynomial piece of $s \in S_l(x_0, x_n)$ are the same as well as the last and the last but one piece. In other words x_1 and x_{n-1} are not knots, i.e., the break points of polynomial pieces. It is obvious that if all basic cubic splines s_i satisfy the not-a-knot condition, then all cubic splines will satisfy it.

Theorem 4. *The function Fs is continuous and if $x \in (x_i, x_{i+1})$ then for all $\alpha \in [0, 1]$*

$$\text{len}[Fs(x)]^\alpha \geq \min\{\text{len}[Fs(x_i)]^\alpha, \text{len}[Fs(x_{i+1})]^\alpha\},$$

where len denotes the length of an interval.

Theorem 5. *Assume that $s_i \in S_3(x_0, x_n)$, interpolation spline of $\{(x_j, \delta_{ij})\}_{j=0}^n$ satisfies the not-a-knot condition. Then*

- (a) s_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,
- (b) the sign of s_i does not change on $[x_j, x_{j+1}]$,
- (c) the sign of s_i changes at x_j for all $j \neq i$.

Unfortunately, theorem 5 is not valid for any kind of splines, for example in $n = 5, x_i = i + 1, \bar{y} = (0, 0, 1, 0, 0, 0)$ with two additional conditions $s'(2) = s'(3) = 0$.

With respect to these theorems, if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $Fs(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number, where

$$\begin{aligned} m(x) &= \sum_{i=0}^n s_i(x)m_i, \\ l(x) &= \sum_{s_i(x) \geq 0} s_i(x)l_i - \sum_{s_i(x) < 0} s_i(x)r_i, \\ r(x) &= \sum_{s_i(x) \geq 0} s_i(x)r_i - \sum_{s_i(x) < 0} s_i(x)l_i. \end{aligned}$$

Author, [2], gave a numerical algorithm for computing the natural fuzzy spline of any odd order.

Definition 2. *A function $s : [x_0, x_n] \rightarrow \mathbb{R}$ is called a polynomial natural spline of odd degree $l = 2m - 1, m \geq 2$, provided that it possesses the following properties:*

- (a) $s \in C^{l-1}[x_0, x_n]$,
- (b) $s(x)$ is a polynomial of degree l for $x \in [x_i, x_{i+1}]; i = 0, 1, \dots, n - 1$,
- (c) $s^{(\nu)}(x_0) = s^{(\nu)}(x_n) = 0; \nu = m, \dots, 2m - 2$.

We denote the family of these splines by $S_l^N(x_0, x_n)$. If $s_i \in S_l^N(x_0, x_n)$ interpolates the data $(x_j, f_j), j = 0, 1, \dots, n$, where $f_j = \delta_{ij}$, Kronecker delta, then $s_{\overline{y}}(x) = \sum_{i=0}^n y_i s_i(x)$ and in the same way in (2), we have

$$[Fs^N(x)]^\alpha = \sum_{i=0}^n s_i(x)[u_i]^\alpha, \quad Fs^N(x) = \sum_{i=0}^n s_i(x)u_i,$$

where the interpolation function $Fs^N(x)$ is the natural fuzzy spline.

Theorem 6. *The function Fs^N is continuous and if $x \in (x_i, x_{i+1})$ then for all $\alpha \in [0, 1]$*

$$\text{len}[Fs^N(x)]^\alpha \geq \min\{\text{len}[Fs^N(x_i)]^\alpha, \text{len}[Fs^N(x_{i+1})]^\alpha\},$$

where len denotes the length of an interval.

Theorem 7. *Assume that $s_i \in S_l^N(x_0, x_n)$, interpolation spline of $\{(x_j, \delta_{ij})\}_{j=0}^n$. Then*

- (a) s_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,
- (b) the sign of s_i does not change on $[x_j, x_{j+1}]$,
- (c) the sign of s_i changes at x_j for all $j \neq i$.

With respect to these theorems, if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $Fs^N(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number as the same fuzzy spline with not-a-knot condition. Author, [1], gave a numerical algorithm for computing the complete fuzzy spline of any odd order.

Definition 3. *A function $s : [x_0, x_n] \rightarrow \mathbb{R}$ is called a polynomial complete spline of odd degree $l = 2m - 1, m \geq 2$, provided that it possesses the following properties:*

- (a) $s \in C^{l-1}[x_0, x_n]$,
- (b) $s(x)$ is a polynomial of degree l for $x \in [x_i, x_{i+1}]; i = 0, 1, \dots, n - 1$,
- (c) $s^{(\nu)}(x_0) = s^{(\nu)}(x_n) = 0; \nu = 1, \dots, m - 1$.

We denote the family of these splines by $S_l^C(x_0, x_n)$. If $s_i \in S_l^C(x_0, x_n)$ interpolates the data $(x_j, f_j), j = 0, 1, \dots, n$, where $f_j = \delta_{ij}$, Kronecker delta, then $s_{\overline{y}}(x) = \sum_{i=0}^n y_i s_i(x)$ and in the same way in (2), we have

$$[Fs^C(x)]^\alpha = \sum_{i=0}^n s_i(x)[u_i]^\alpha, \quad Fs^C(x) = \sum_{i=0}^n s_i(x)u_i,$$

where the interpolation function $Fs^C(x)$ is the complete fuzzy spline.

For this kind of spline, we have the theorems like as 6 and 7, and if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $Fs^C(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number.

The idea of fuzzy B-splines are introduced first by Anile et al. [3, 4]. Its power relies on the possibility of being used as approximating function both for fuzzy and crisp data. In the context of surface modeling, fuzzy B-splines are proposed as an integrated approach to uncertainty coding and data reduction. Fuzzy B-splines are suitable for representing and simplifying both crisp and imprecise surface data and support interrogation of the model at different presumption levels.

B-spline are parametric curves or surfaces that provide a flexible tool for the modeling and visualization of several kind of data. A B-spline of order h is a piecewise polynomial function of degree at most $h - 1$ such $f(x) : [x_0, x_n] \rightarrow \mathbb{R}$. The knot sequence of a B-spline is a non-decreasing sequence of real numbers $(x_0, x_1, \dots, x_m), m = k + 2(h - 1)$, where h is the order of

the B-spline. The knots in the sub-sequence $(x_{h-1}, \dots, x_{k+h-1})$, are the interior knots of the B-spline. The $f(x)$ is C^∞ in $[x_0, x_m]$ except in the knots: in a knot of multiplicity p , the $f(x)$ is only C^{h-p-1} . A B-spline $f(x)$ of order h over a sequence of $m = k + 2(h - 1)$ is a linear combination $f(x) = \sum_{i=0}^{k+h-1} c_i B_{i,h}(x)$, where c_i 's are the control coefficients, or control points and the $B_{i,h}(x)$'s are the B-spline basis function of order h . These functions could be very well defined according to a famous recursion

$$B_{i,1}(x) = \begin{cases} 1 & \text{if } x_i \leq x < x_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{i,h}(x) = \frac{x-x_i}{x_{i+h-1}-x_i} B_{i,h-1}(x) + \frac{x_{i+h}-x}{x_{i+h}-x_{i+1}} B_{i+1,h-1}(x), \text{ for } h > 1.$$

Definition 4. A fuzzy B-spline $Fb(x)$ relative to the crisp knot sequence (x_0, \dots, x_m) , $m = k + 2(h - 1)$ is a function $Fb : \mathbb{R} \rightarrow E$ as $Fb(x) = \sum_{i=0}^{k+h-1} F_i B_{i,h}(x)$, where F_i 's, the control coefficients, are fuzzy numbers and $B_{i,h}(x)$'s are the crisp B-spline basis functions of order h .

4. Conclusions

In this paper, some methods to modeling uncertain data has been presented. Unfortunately, for complexity in fuzzy multiplication or fuzzy division, the introducing of a full fuzzy interpolation is very difficult. Here, we present Zadeh's interpolation problem, which first solved by Lowen. Then the fuzzy Lagrange polynomials and fuzzy splines, natural and complete, and fuzzy B-splines are presented.

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