2003

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Abstract

In this paper, we will consider the interpolation of fuzzy data by a continuous fuzzy-valued function. We will use Lagrange polynomials, natural splines and complete splines.

Keywords: Fuzzy interpolation, Fuzzy spline, Fuzzy B-spline.

1. Introduction

The following problem was first posed by L.A. Zadeh: "Suppose we are given $n+1$ (distinct) points $x_0, \dots, x_n$ in $\mathbb{R}$, and for each of these points a 'fuzzy value' in $\mathbb{R}$, rather than a crisp one. Is it then possible to construct some function on $\mathbb{R}$ with range also a collection of 'fuzzy values'; which coincide, on the given $n+1$ points, with the given 'fuzzy values'; and which fulfills some natural 'smoothness' condition?"

Lowen, [6], with respect to Hausdorff metric gave a fuzzy interpolation polynomial of Lagrange type and proved its continuity, but there are not a numerical algorithm in [6]. Kaleva, [5], gave numerical algorithms based on fuzzy Lagrange polynomials and fuzzy cubic *Not-a-Knot* splines.

Now let E denote the family of all normal, convex, upper semicontinuous and compactly supported fuzzy sets defined on $\mathbb{R}$. If $u \in E$ then $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}, 0 < \alpha < 1$, denotes the α-level set of u. By $[u]^0$ we denote the support of u. It follows that the level sets of a $u \in E$ are closed, bounded intervals of $\mathbb{R}$.

It is well-know that the addition and multiplication by a real number can be extended to E. In based on level sets we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [\lambda u]^\alpha = \lambda [u]^\alpha,$$

for all $0 \leq \alpha \leq 1, u, v, \in E$ and $\lambda \in \mathbb{R}$.

In E we define the metric

$$h(u,v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric in the space of nonempty compact, convex subsets of $\mathbb{R}$. By the corresponding properties of the Hausdorff metric, it is easily that for all $u, v, w \in E$ and $\lambda \in \mathbb{R}$

$$h(u+w, v+w) = h(u,v) \qquad h(\lambda u, \lambda v) = \lambda h(u,v).$$

Furthermore, if $\lambda, \mu \in \mathbb{R}$ are of same sign then $(\lambda, \mu)u = \lambda u + \mu u$. It follows that the addition and scalar multiplication are continuous with respect to the metric h. Furthermore (E, h) is a complete metric space.

Let $\overline{y} = (y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ and suppose $P_{\overline{y}}(x)$ is the Lagrange interpolation polynomial which interpolates the data $(x_i, y_i); i = 0, 1, \dots, n$, i.e.,

$$P_{\overline{y}}(x_i) = y_i, \quad \forall i = 0, \dots, n,$$

$$P_{\overline{y}}(x) = \sum_{i=0}^{n} \prod_{j=0, j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}.$$

2. Fuzzy Interpolation Problem

Let $x_0 < x_1 < \ldots < x_n$ be $n+1$ points in $\mathbb{R}$ and let $\{u_i\}_{i=0}^n$ be $n+1$ fuzzy numbers in E. A formulation of Zadeh's problem now is the following.
Construct a function $F : \mathbb{R} \to E$, fulfilling the following properties:

(i) *For all $i = 0, 1, \ldots, n : F(x_i) = u_i$,*

(ii) *F is continuous,*

(iii) *If for all $i = 0, 1, \ldots, n$, $u_i = 1_{y_i}$ where $y_i \in \mathbb{R}$ and if f is the unique polynomial of degree$\leq n$ which passes through the points $\{(x_i, y_i)\}_{i=0}^n$, i.e., such that $f(x_i) = y_i$ for all $i = 0, 1, \ldots, n$, then F reduces to the function $F(x) = 1_{f(x)}$ for all $x \in \mathbb{R}$.*

We construct F in the following way. For each $\alpha \in I_0 = (0, 1]$ and $i = 0, 1, \ldots, n$, let

$$J_i^\alpha = u_i^{-1}[\alpha, 1] = [u_i]^\alpha.$$

Note that each J_i^α is a compact nonempty interval. Finally for each $x \in \mathbb{R}$ we define the membership function of fuzzy number $F(x)$ by

$$F(x)(t) = \sup\{\alpha \in I_0 \mid \exists \overline{y} \in \prod_{i=0}^n J_i^\alpha : P_{\overline{y}}(x) = t\}, \tag{1}$$

for all $t \in \mathbb{R}$.
Theorem 1. *For each $x \in \mathbb{R}$ we have $F(x) \in E$, i.e., the assignment*

$$F : \mathbb{R} \to E : x \to F(x)$$

is a well-defined function. Further we have:

(i) *For all $i = 0, 1, \ldots, n, F(x_i) = u_i$,*

(ii) *F is continuous,*

(iii) *If for all $i = 0, 1, \ldots, n$ we have $u_i = 1_{[a_i, b_i]}$ and for each $x \in \mathbb{R}$ we put*

$$J(x) = \{t \in \mathbb{R} | \exists \overline{y} \in \prod_{i=0}^n [a_i, b_i], P_{\overline{y}}(x) = t\},$$

then F reduces to the function $F(x) = 1_{J(x)}$.
To prove theorem 1, we require two lemmas.
Lemma 1. *For any $(x, t) \in \mathbb{R} \times \mathbb{R}$,*

$$F(x)(t) = \max\{\alpha \in I_0 | \exists \overline{y} \in \prod_{i=0}^n J_i^\alpha, P_{\overline{y}}(x) = t\},$$

i.e., the supremum is attained.
Lemma 2. *For any $\alpha \in I_0$,*

$$\{P_{\overline{y}} | \overline{y} \in \prod_{i=0}^n J_i^\alpha\}$$

is an equicontinuous family.

Kaleva, [5], gave a numerical algorithm for computing the fuzzy Lagrange interpolation. From (1), we have

$$\begin{aligned} [F(x)]^\alpha &= \{t \in \mathbb{R} | F(x)(t) \geq \alpha\} \\ &= \{t \in \mathbb{R} | \exists \overline{y} \in \prod_{i=0}^{n} J_i^\alpha : P_{\overline{y}}(x) = t\} \\ &= \sum_{i=0}^{n} L_i(x) J_i^\alpha, \end{aligned}$$

where $L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}$. Hence $F(x) = \sum_{i=0}^{n} L_i(x) u_i$.

Theorem 2. *The function F is continuous and if $x \in (x_i, x_{i+1})$ then for all $\alpha \in [0,1]$*

$$len[F(x)]^\alpha \geq \min\{len[F(x_i)]^\alpha, len[F(x_{i+1})]^\alpha\},$$

where len denotes the length of an interval.

Theorem 3. *$L_i(x)$, Lagrange polynomials, satisfies*

(a) L_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,

(b) the sign of L_i does not change on $[x_j, x_{j+1}]$,

(c) the sign of L_i changes at x_j for all $j \neq i$.

With respect to these theorems, if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $F(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number, where

$$\begin{aligned} m(x) &= \sum_{i=0}^{n} L_i(x) m_i, \\ l(x) &= \sum_{L_i(x)\geq 0} L_i(x) l_i - \sum_{L_i(x)<0} L_i(x) r_i, \\ r(x) &= \sum_{L_i(x)\geq 0} L_i(x) r_i - \sum_{L_i(x)<0} L_i(x) l_i. \end{aligned}$$

3. fuzzy splines

Kaleva, [5], gave a numerical algorithm for computing the not-a-knot fuzzy spline, also.

Definition 1. *Let $S_l(x_0, x_n)$ denote the family of splines of order l with knots $x_i, i = 0, 1, \ldots, n$. In other words $s \in S_l(x_0, x_n)$ if*

(i) $s \in C^{l-1}[x_0, x_n]$,

(ii) s is a polynomial of order l in the interval $[x_i, x_{i+1}]$, for all $i = 0, 1, \ldots, n-1$.

As in the case of interpolation polynomial we define an interpolating fuzzy spline Fs of order l as follows. If $Fs^\alpha(x)$ is the α-level set of $Fs(x)$ then

$$Fs^\alpha(x) = [Fs(x)]^\alpha = \{y \in \mathbb{R} | y = s_{\overline{y}}(x), \overline{y} = (y_0, y_1, \ldots, y_n) \in \prod_{i=0}^{n} u_i^\alpha\},$$

where $s_{\overline{y}} \in S_l(x_0, x_n)$ is the spline, which interpolates the data $(x_i, y_i), i = 0, 1, \ldots, n$.

If $s_i \in S_l(x_0, x_n)$ interpolates the data $(x_j, f_j), j = 0, 1, \ldots, n$, where $f_j = \delta_{ij}$, Kronecker delta, then $s_{\overline{y}}(x) = \sum_{i=0}^{n} y_i s_i(x)$ and in the same way in section 2 we have

$$[Fs(x)]^\alpha = \sum_{i=0}^{n} s_i(x)[u_i]^\alpha, \qquad Fs(x) = \sum_{i=0}^{n} s_i(x) u_i. \tag{2}$$

It is well-known that the interpolation conditions are not sufficient to uniquely determine the spline s_i. In addition we need two requirements. It appears that the sign of $s_i(x)$ depends on these requirements. If we adopt the not-a-knot condition, then the spline s_i behaves like L_i. The not-a-knot condition, means that the first and the second polynomial piece of $s \in S_l(x_0, x_n)$ are the same as well as the last and the last but one piece. In other words x_1 and x_{n-1} are not knots, i.e., the break points of polynomial pieces. It is obvious that if all basic cubic splines s_i satisfy the not-a-knot condition, then all cubic splines will satisfy it.

Theorem 4. *The function Fs is continuous and if $x \in (x_i, x_{i+1})$ then for all $\alpha \in [0, 1]$*

$$len[Fs(x)]^\alpha \geq \min\{len[Fs(x_i)]^\alpha, len[Fs(x_{i+1})]^\alpha\},$$

where len denotes the length of an interval.

Theorem 5. *Assume that $s_i \in S_3(x_0, x_n)$, interpolation spline of $\{(x_j, \delta_{ij})\}_{j=0}^{n}$ satisfies the not-a-knot condition. Then*

(a) s_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,

(b) the sign of s_i does not change on $[x_j, x_{j+1}]$,

(c) the sign of s_i changes at x_j for all $j \neq i$.

Unfortunately, theorem 5 is not valid for any kind of splines, for example in $n = 5, x_i = i+1, \overline{y} = (0, 0, 1, 0, 0, 0)$ with two additional conditions $s'(2) = s'(3) = 0$.
With respect to these theorems, if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $Fs(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number, where

$$\begin{aligned} m(x) &= \sum_{i=0}^{n} s_i(x) m_i, \\ l(x) &= \sum_{s_i(x) \geq 0} s_i(x) l_i - \sum_{s_i(x) < 0} s_i(x) r_i, \\ r(x) &= \sum_{s_i(x) \geq 0} s_i(x) r_i - \sum_{s_i(x) < 0} s_i(x) l_i. \end{aligned}$$

Author, [2], gave a numerical algorithm for computing the natural fuzzy spline of any odd order.

Definition 2. *A function $s : [x_0, x_n] \to \mathbb{R}$ is called a polynomial natural spline of odd degree $l = 2m - 1$, $m \geq 2$, provided that it possesses the following properties:*

(a) $s \in C^{l-1}[x_0, x_n]$,

(b) $s(x)$ is a polynomial of degree l for $x \in [x_i, x_{i+1})$; $i = 0, 1, \ldots, n-1$,

(c) $s^{(\nu)}(x_0) = s^{(\nu)}(x_n) = 0$; $\nu = m, \ldots, 2m-2$.

We denote the family of these splines by $S_l^N(x_0, x_n)$. If $s_i \in S_l^N(x_0, x_n)$ interpolates the data $(x_j, f_j), j = 0, 1, \dots, n$, where $f_j = \delta_{ij}$, Kronecker delta, then $s_{\overline{y}}(x) = \sum_{i=0}^{n} y_i s_i(x)$ and in the same way in (2), we have

$$[Fs^N(x)]^\alpha = \sum_{i=0}^{n} s_i(x)[u_i]^\alpha, \qquad Fs^N(x) = \sum_{i=0}^{n} s_i(x)u_i,$$

where the interpolation function $Fs^N(x)$ is the natural fuzzy spline.

Theorem 6. *The function Fs^N is continuous and if $x \in (x_i, x_{i+1})$ then for all $\alpha \in [0,1]$*

$$len[Fs^N(x)]^\alpha \geq \min\{len[Fs^N(x_i)]^\alpha, len[Fs^N(x_{i+1})]^\alpha\},$$

where len denotes the length of an interval.

Theorem 7. *Assume that $s_i \in S_l^N(x_0, x_n)$, interpolation spline of $\{(x_j, \delta_{ij})\}_{j=0}^n$. Then*

(a) s_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,

(b) the sign of s_i does not change on $[x_j, x_{j+1}]$,

(c) the sign of s_i changes at x_j for all $j \neq i$.

With respect to these theorems, if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $Fs^N(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number as the same fuzzy spline with not-a-knot condition. Author, [1], gave a numerical algorithm for computing the complete fuzzy spline of any odd order.

Definition 3. *A function $s : [x_0, x_n] \to \mathbb{R}$ is called a polynomial complete spline of odd degree $l = 2m - 1$, $m \geq 2$, provided that it possesses the following properties:*

(a) $s \in C^{l-1}[x_0, x_n]$,

(b) $s(x)$ is a polynomial of degree l for $x \in [x_i, x_{i+1})$; $i = 0, 1, \dots, n-1$,

(c) $s^{(\nu)}(x_0) = s^{(\nu)}(x_n) = 0$; $\nu = 1, \dots, m-1$.

We denote the family of these splines by $S_l^C(x_0, x_n)$. If $s_i \in S_l^C(x_0, x_n)$ interpolates the data $(x_j, f_j), j = 0, 1, \dots, n$, where $f_j = \delta_{ij}$, Kronecker delta, then $s_{\overline{y}}(x) = \sum_{i=0}^{n} y_i s_i(x)$ and in the same way in (2), we have

$$[Fs^C(x)]^\alpha = \sum_{i=0}^{n} s_i(x)[u_i]^\alpha, \qquad Fs^C(x) = \sum_{i=0}^{n} s_i(x)u_i,$$

where the interpolation function $Fs^C(x)$ is the complete fuzzy spline.

For this kind of spline, we have the theorems like as 6 and 7, and if $u_i = (m_i, l_i, r_i)$ is a triangular fuzzy number then $Fs^C(x) = (m(x), l(x), r(x))$ is a triangular fuzzy number.

The idea of fuzzy B-splines are introduced first by Anile et al. [3, 4]. Its power relies on the possibility of being used as approximating function both for fuzzy and crisp data. In the context of surface modeling, fuzzy B-splines are proposed as an integrated approach to uncertainty coding and data reduction. Fuzzy B-splines are suitable for representing and simplifying both crisp and imprecise surface data and support interrogation of the model at different presumption levels.

B-spline are parametric curves or surfaces that provide a flexible tool for the modeling and visualization of several kind of data. A B-spline of order h is a piecewise polynomial function of degree at most $h - 1$ such $f(x) : [x_0, x_n] \to \mathbb{R}$. The knot sequence of a B-spline is a non-decreasing sequence of real numbers $(x_0, x_1, \dots, x_m), m = k + 2(h - 1)$, where h is the order of the B-spline. The knots in the sub-sequence $(x_{h-1}, \ldots, x_{k+h-1})$, are the interior knots of the B-spline. The $f(x)$ is C^∞ in $[x_0, x_m]$ except in the knots: in a knot of multiplicity p, the $f(x)$ is only C^{h-p-1}. A B-spline $f(x)$ of order h over a sequence of $m = k + 2(h-1)$ is a linear combination $f(x) = \sum_{i=0}^{k+h-1} c_i B_{i,h}(x)$, where c_i's are the control coefficients, or control points and the $B_{i,h}(x)$'s are the B-spline basis function of order h. These functions could be very well defined according to a famous recursion

$$\begin{aligned} B_{i,1}(x) &= \begin{cases} 1 & \text{if } x_i \leq x < x_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \\ B_{i,h}(x) &= \tfrac{x-x_i}{x_{i+h-1}-x_i} B_{i,h-1}(x) + \tfrac{x_{i+h}-x}{x_{i+h}-x_{i+1}} B_{i+1,h-1}(x), \text{ for } h > 1. \end{aligned}$$

Definition 4. *A fuzzy B-spline $Fb(x)$ relative to the crisp knot sequence $(x_0, \ldots, x_m), m = k + 2(h-1)$ is a function $Fb : \mathbb{R} \to E$ as $Fb(x) = \sum_{i=0}^{k+h-1} F_i B_{i,h}(x)$, where F_i's, the control coefficients, are fuzzy numbers and $B_{i,h}(x)$'s are the crisp B-spline basis functions of order h.*

4. Conclusions

In this paper, some methods to modeling uncertain data has been presented. Unfortunately, for complexity in fuzzy multiplication or fuzzy division, the introducing of a full fuzzy interpolation is very difficult. Here, we present Zadeh's interpolation problem, which first solved by Lowen. Then the fuzzy Lagrange polynomials and fuzzy splines, natural and complete, and fuzzy B-splines are presented.

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