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Abstract

In this paper numerical algorithms for solving 'fuzzy ordinary differential equations' are considered. A scheme based on the 4th Runge-Kutta method in detail is discussed and this is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.

Keywords: Fuzzy differential equation,4th Runge-Kutta method, Fuzzy Cauchy problem.

AMS subject classification: 34A12,65L05.

1 Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [1] it was followed up by D.Dubois, H. Prade in [2], who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [5],[6] and by S. Seikkala in [7],.... The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [10] by the standard Euler method. paper organize as follows:
In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative ,which have been discussed by S. Seikkala in [10] are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this work. The numerically solving fuzzy differential equation by 4th Runge-kutta method is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

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2 Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) & = f(t, y(t)); \quad a \leq t \leq b, \\ y(a) & = \alpha, \end{cases} \tag{1}$$

The basis of all Runge-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = h \sum_{i=1}^{m} w_i k_i. \tag{2}$$

where for $i = 1, 2, \ldots m$, the w_i's are constants and

$$k_i = f(t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} k_j). \tag{3}$$

Equation (2) is to be exact for powers of h through h^m. Therefore, the truncation error T_m, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}),$$

The true magnitude of γ_m will generally be much less than the bound. Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1}$, as we expect it will be if h is small, then the bound on $\gamma_m h^{m+1}$ will usually be a bound on the error as a whole. The famous nonzero constants α_i , β_{ij} in 4th Runge-Kutta method are $m = 4$ and

$$\alpha_1 = 0 \ , \ \alpha_2 = \alpha_3 = 1/2 \ , \ \alpha_4 = 1 \ , \ \beta_{21} = 1/2 \ , \ \beta_{32} = 1/2 \ , \ \beta_{43} = 1,$$

and we have, see [9]

$$\begin{aligned} y_0 &= \alpha, \\ k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_1), \\ k_3 &= f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} k_2), \\ k_4 &= f(t_i + h, y_i + h k_3), \\ y_{i+1} &= y_i + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \tag{4}$$

where

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b \ and \ h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{5}$$

Theorem 1 *Let $f(t,y)$ belong to $C^4[a,b]$ and let it's partial derivatives are bounded and assume there exists, P, M, positive numbers, where*

$$|f(t,y)| < M \ , \ |\frac{\partial^{i+j} f}{\partial t^i \partial y^j}| < \frac{P^{i+j}}{M^{j-1}} \ , \ i+j \leq m$$

then in the 4th Runge-Kutta method $y(t_{i+1}) - y_{i+1} \approx \frac{73}{720} h^5 M P^4 + O(h^6)$.

Proof see [9].

A triangular fuzzy number v is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $v(x)$,the membership function of the fuzzy number v, is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$.
We specify v as $(a_1/a_2/a_3)$. We will write: $(1) v > 0$ if $a_1 > 0$; $(2) v \geq 0$ if $a_1 \geq 0$; $(3) v < 0$ if $a_3 < 0$; and $(4) v \leq 0$ if $a_3 \leq 0$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level intervals. It means that if $v \in E$ then the r-level set

$$[v]_r = \{s |\ v(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval which is denoted by

$$[v]_r = [v_1(r), v_2(r)].$$

Let I be a real interval. A mapping $x : I \to E$ is called a fuzzy process and its $r-$level set is denoted by

$$[x(t)]_r = [x_1(t;r), x_2(t;r)], \quad t \in I, \quad r \in (0,1].$$

Lemma 1 *Let $v, w \in E$ and s scalar, then for $r \in (0,1]$*

$$[v+w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[v-w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[v.w]_r = [\min\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\},$$
$$\max\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\}],$$

$$[sv]_r = s[v]_r.$$

3 A fuzzy cauchy problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) & = f(t, y(t)), \quad t \in I = [0, T], \\ y(0) & = y_0, \end{cases} \tag{6}$$

where f is a continuous mapping from $R_+ \times R$ into R and $y_0 \in E$ with r-level intervals

$$[y_0]_r = [y_1(0;r), y_2(0;r)], \quad r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t,y)$ when $y = y(t)$ is a fuzzy number

$$f(t,y)(s) = \sup\{y(\tau)|\ s = f(t,\tau)\}, \quad s \in R.$$

It follows that

$$[f(t,y)]_r = [f_1(t,y;r), f_2(t,y;r)], \quad r \in (0,1].$$

where

$$\begin{aligned} f_1(t,y;r) &= min\{f(t,u)|\ u \in [y_1(r), y_2(r)]\}, \\ f_2(t,y;r) &= max\{f(t,u)|\ u \in [y_1(r), y_2(r)]\}. \end{aligned} \tag{7}$$

Theorem 2 *Let f satisfy*

$$|f(t,v) - f(t,\overline{v})| \le g(t, |v - \overline{v}|), \quad t \ge 0, \quad v, \overline{v} \in R,$$

where $g : R_+ \times R_+$ is a continuous mapping such that $r \to g(t,r)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{8}$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.

Proof see [7].
In this paper we suppose (6) satisfies the hypothesis of theorem 2, also.

4 4th Runge-Kutta method

Let the exact solution $[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$ is approximated by some $[y(t)]_r = [y_1(t;r), y_2(t;r)]$. From (2),(3) we define

$$\begin{aligned} y_1(t_{n+1};r) - y_1(t_n;r) &= h \textstyle\sum_{i=1}^{4} w_i k_{i,1}(t_n, y(t_n;r)), \\ y_2(t_{n+1};r) - y_2(t_n;r) &= h \textstyle\sum_{i=1}^{4} w_i k_{i,2}(t_n, y(t_n;r)), \end{aligned} \tag{9}$$

where the w_i's are constants and

$$[k_i(t, y(t;r))]_r = [k_{i,1}(t, y(t;r)), k_{i,2}(t, y(t;r))], \quad i = 1,2,3,4$$

$$\begin{aligned} k_{i,1}(t_n, y(t_n;r)) &= f(t_n + \alpha_i h, y_1(t_n) + h \textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,1}(t_n, y(t_n;r))), \\ k_{i,2}(t_n, y(t_n;r)) &= f(t_n + \alpha_i h, y_2(t_n) + h \textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,2}(t_n, y(t_n;r))), \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
&k_{1,1}(t,y(t;r)) = min\{f(t,u)|u \in [y_1(t;r), y_2(t;r)]\},\\
&k_{1,2}(t,y(t;r)) = max\{f(t,u)|u \in [y_1(t;r), y_2(t;r)]\},\\
&k_{2,1}(t,y(t;r)) = min\{f(t+\tfrac{h}{2},u)|u \in [z_{1,1}(t,y(t;r)), z_{1,2}(t,y(t;r))]\},\\
&k_{2,2}(t,y(t;r)) = max\{f(t+\tfrac{h}{2},u)|u \in [z_{1,1}(t,y(t;r)), z_{1,2}(t,y(t;r))]\},\\
&k_{3,1}(t,y(t;r)) = min\{f(t+\tfrac{h}{2},u)|u \in [z_{2,1}(t,y(t;r)), z_{2,2}(t,y(t;r))]\},\\
&k_{3,2}(t,y(t;r)) = max\{f(t+\tfrac{h}{2},u)|u \in [z_{2,1}(t,y(t;r)), z_{2,2}(t,y(t;r))]\},\\
&k_{4,1}(t,y(t;r)) = min\{f(t+h,u)|u \in [z_{3,1}(t,y(t;r)), z_{3,2}(t,y(t;r))]\},\\
&k_{4,2}(t,y(t;r)) = max\{f(t+h,u)|u \in [z_{3,1}(t,y(t;r)), z_{3,2}(t,y(t;r))]\}.
\end{aligned}
\tag{11}
$$

Where in the 4th Runge-Kutta method,

$$
\begin{aligned}
&z_{1,1}(t,y(t;r)) = y_1(t;r) + \tfrac{h}{2}k_{1,1}(t,y(t;r)), \quad z_{1,2}(t,y(t;r)) = y_2(t;r) + \tfrac{h}{2}k_{1,2}(t,y(t;r)),\\
&z_{2,1}(t,y(t;r)) = y_1(t;r) + \tfrac{h}{2}k_{2,1}(t,y(t;r)), \quad z_{2,2}(t,y(t;r)) = y_2(t;r) + \tfrac{h}{2}k_{2,2}(t,y(t;r)),\\
&z_{3,1}(t,y(t;r)) = y_1(t;r) + hk_{3,1}(t,y(t;r)), \quad z_{3,2}(t,y(t;r)) = y_2(t;r) + hk_{3,2}(t,y(t;r)).
\end{aligned}
\tag{12}
$$

Define,

$$
\begin{aligned}
F[t,y(t;r)] = &\quad k_{1,1}(t,y(t;r)) + 2k_{2,1}(t,y(t;r)) + 2k_{3,1}(t,y(t;r)) + k_{4,1}(t,y(t;r)),\\
G[t,y(t;r)] = &\quad k_{1,2}(t,y(t;r)) + 2k_{2,2}(t,y(t;r)) + 2k_{3,2}(t,y(t;r)) + k_{4,2}(t,y(t;r)).
\end{aligned}
\tag{13}
$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$ respectively. The solution is calculated by grid points at (5). By (9),(13) we have

$$
\begin{aligned}
Y_1(t_{n+1};r) \approx &\quad Y_1(t_n;r) + \tfrac{h}{6}F[t_n, Y(t_n;r)],\\
Y_2(t_{n+1};r) \approx &\quad Y_2(t_n;r) + \tfrac{h}{6}G[t_n, Y(t_n;r)].
\end{aligned}
\tag{14}
$$

We define

$$
\begin{aligned}
y_1(t_{n+1};r) = &\quad y_1(t_n;r) + \tfrac{h}{6}F[t_n, y(t_n;r)],\\
y_2(t_{n+1};r) = &\quad y_2(t_n;r) + \tfrac{h}{6}G[t_n, y(t_n;r)].
\end{aligned}
\tag{15}
$$

The following lemmas will be applied to show convergence of these approximates i.e.,

$$
\begin{aligned}
\lim_{h\to 0} y_1(t;r) = &\quad Y_1(t;r),\\
\lim_{h\to 0} y_2(t;r) = &\quad Y_2(t;r).
\end{aligned}
$$

Lemma 2 *Let a sequence of numbers* $\{W_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N-1,$$

for some given positive constants A *and* B. *Then*

$$|W_n| \leq A^n|W_0| + B\frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.$$

Proof see [10].

Lemma 3 *Let a sequence of numbers* $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \leq |W_n| + A.max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A.max\{|W_n|, |V_n|\} + B,$$

for some given positive constants A *and* B, *and denote*

$$U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N.$$

Then

$$U_n \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, \quad 0 \leq n \leq N,$$

where $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$.

Proof see [10].

Let $F(t,u,v)$ and $G(t,u,v)$ are obtained by substituting $[y(t)]_r = [u,v]$ in (13),

$$\begin{aligned} F(t,u,v) = &\quad k_{1,1}(t,u,v) + 2k_{2,1}(t,u,v) + 2k_{3,1}(t,u,v) + k_{4,1}(t,u,v), \\ G(t,u,v) = &\quad k_{1,2}(t,u,v) + 2k_{2,2}(t,u,v) + 2k_{3,2}(t,u,v) + k_{4,2}(t,u,v). \end{aligned}$$

The domain where F and G are defined is therefore

$$K = \{(t,u,v) |\ 0 \leq t \leq T \ \ , \ \ -\infty < v < \infty \ \ , \ \ -\infty < u \leq v\}.$$

Theorem 3 *Let* $F(t,u,v)$ *and* $G(t,u,v)$ *belong to* $C^4(K)$ *and let the partial derivatives of* F *and* G *be bounded over* K. *Then, for arbitrary fixed* $r, 0 \leq r \leq 1$, *the approximately solutions (14) converge to the exact solutions* $Y_1(t;r)$ *and* $Y_2(t;r)$ *uniformly in* t.

Proof : It is sufficient to show

$$\begin{aligned} \lim_{h\to 0} y_1(t_N;r) = &\quad Y_1(t_N;r), \\ \lim_{h\to 0} y_2(t_N;r) = &\quad Y_2(t_N;r), \end{aligned}$$

where $t_N = T$. For n = 0,1, ...,N-1 , by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1};r) \approx \; & Y_1(t_n;r) + \tfrac{h}{6}F[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720}h^5MP^4 + O(h^6), \\ Y_2(t_{n+1};r) \approx \; & Y_2(t_n;r) + \tfrac{h}{6}G[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720}h^5MP^4 + O(h^6), \end{aligned} \tag{16}$$

denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$
$$V_n = Y_2(t_n;r) - y_2(t_n;r),$$

we have from (14) and (15)

$$W_{n+1} \approx W_n + \frac{h}{6}\{F[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{73}{720}h^5MP^4 + O(h^6),$$

$$V_{n+1} \approx V_n + \frac{h}{6}\{G[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{73}{720}h^5MP^4 + O(h^6).$$

Then

$$|W_{n+1}| \le |W_n| + \frac{1}{3}Lh.max\{|W_n|, |V_n|\} + \frac{73}{720}h^5MP^4 + O(h^6),$$
$$|V_{n+1}| \le |V_n| + \frac{1}{3}Lh.max\{|W_n|, |V_n|\} + \frac{73}{720}h^5MP^4 + O(h^6),$$

for $t \in [0, T]$ and $L > 0$ is a bound for the partial derivatives of F and G. Thus by lemma 3

$$|W_n| \le (1 + \frac{2}{3}Lh)^n|U_0| + (\frac{73}{360}h^5MP^4 + O(h^6))\frac{(1 + \frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

$$|V_n| \le (1 + \frac{2}{3}Lh)^n|U_0| + (\frac{73}{360}h^5MP^4 + O(h^6))\frac{(1 + \frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

where $|U_0| = |W_0| + |V_0|$. In particular

$$|W_N| \le (1 + \frac{2}{3}Lh)^N|U_0| + (\frac{73}{240}h^4MP^4 + O(h^5))\frac{(1 + \frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L},$$

$$|V_N| \le (1 + \frac{2}{3}Lh)^N|U_0| + (\frac{73}{240}h^4MP^4 + O(h^5))\frac{(1 + \frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L}.$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \le (\frac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT} - 1}{L}h^4 + O(h^5),$$

$$|V_N| \le (\frac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT} - 1}{L}h^4 + O(h^5),$$

and if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof.

5 Examples

Example 1 *Consider the fuzzy initial value problem, [10],*

$$\begin{cases} y'(t) &= y(t), \quad t \in I = [0,1], \\ y(0) &= (.75 + .25r\ ,\ 1.125 - .125r), \quad 0 < r \leq 1. \end{cases}$$

By using 4th Runge-Kutta method we have

$$y_1(t_{n+1}; r) = y_1(t_n; r)[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}],$$

$$y_2(t_{n+1}; r) = y_2(t_n; r)[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}].$$

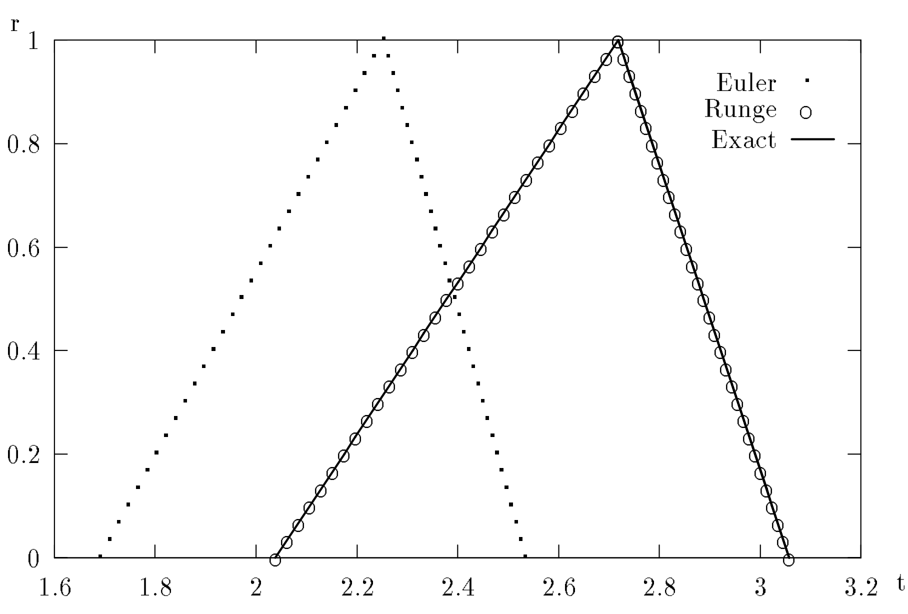
The exact solution is given by

$$Y_1(t; r) = y_1(0; r)e^t \quad , \quad Y_2(t; r) = y_2(0; r)e^t.$$

which at $t = 1$,

$$Y(1; r) = [(.75 + .25r)e\ ,\ (1.125 - .125r)e], \quad 0 < r \leq 1.$$

The exact and approximate solutions we obtained by Euler method [10], and 4th Runge-Kutta method are compared and plotted at $t = 1$ in Fig.(5.1).



$Fig.(5.1), (h = .5)$

Example 2 *Consider the fuzzy initial value problem, [10],*

$$\begin{cases} y'(t) &= ty(t), \quad [a,b] = [-1,1], \\ y(-1) &= (\sqrt{e} - .5(1-r), \sqrt{e} + .5(1-r)), \quad 0 < r \leq 1. \end{cases}$$

We separate between two steps.
$(a) t < 0$: The parametric form in this case is

$$y_1'(t;r) = ty_2(t;r), y_2'(t;r) = ty_1(t;r),$$

with the initial conditions given. The unique exact solution is

$$Y_2(t;r) = \frac{A+B}{2} y_2(0;r) + \frac{A-B}{2} y_1(0;r),$$

$$Y_1(t;r) = \frac{A-B}{2} y_2(0;r) + \frac{A+B}{2} y_1(0;r),$$

where $A = e^{\frac{(t^2-a^2)}{2}}, B = \frac{1}{A}$.
$(b) t \geq 0$: The parametric equations are

$$y_1'(t;r) = ty_1(t;r), y_2'(t;r) = ty_2(t;r),$$

with the initial conditions given. The unique exact solution at $t > 0$ is

$$Y_1(t;r) = y_1(0;r)e^{\frac{t^2}{2}}, Y_2(t;r) = y_2(0;r)e^{\frac{t^2}{2}}.$$

By using 4th Runge-Kutta method at $t_n, 0 \leq n \leq N$ we have

$$k_{1,1}(t_n;r) = min\{t.u | u \in [y_1(t_n;r), y_2(t_n;r)]\},$$

$$k_{1,2}(t_n;r) = max\{t.u | u \in [y_1(t_n;r), y_2(t_n;r)]\},$$

$$k_{2,1}(t_n;r) = min\{(t + \frac{h}{2}).u | u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},$$

$$k_{2,2}(t_n;r) = max\{(t + \frac{h}{2}).u | u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},$$

$$k_{3,1}(t_n;r) = min\{(t + \frac{h}{2}).u | u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},$$

$$k_{3,2}(t_n;r) = max\{(t + \frac{h}{2}).u | u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},$$

$$k_{4,1}(t_n;r) = min\{(t + h).u | u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\},$$

$$k_{4,2}(t_n;r) = max\{(t + h).u | u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\}.$$

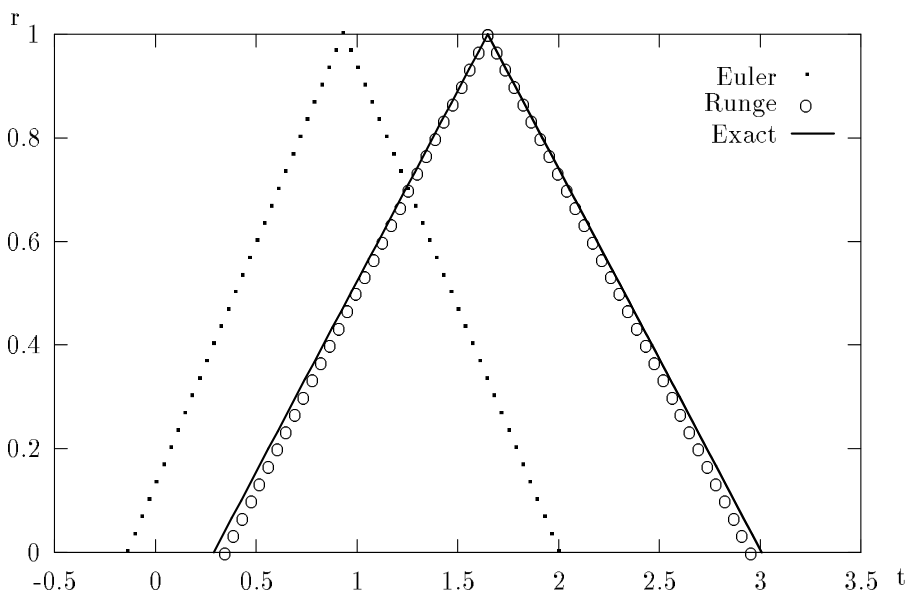
Where

$$z_{1,1}(t_n;r)=y_1(t_n;r)+\frac{h}{2}k_{1,1}(t_n;r),\quad z_{1,2}(t_n;r)=y_2(t_n;r)+\frac{h}{2}k_{1,2}(t_n;r),$$

$$z_{2,1}(t_n;r)=y_1(t_n;r)+\frac{h}{2}k_{2,1}(t_n;r),\quad z_{2,2}(t_n;r)=y_2(t_n;r)+\frac{h}{2}k_{2,2}(t_n;r),$$

$$z_{3,1}(t_n;r)=y_1(t_n;r)+hk_{3,1}(t_n;r),\quad z_{3,2}(t_n;r)=y_2(t_n;r)+hk_{3,2}(t_n;r).$$

By considering $t>0$ and $t<0$,the above minimizing and maximizing problems can be solving by GAMS software. The exact and approximate solutions are compared and plotted in Fig.(5.2).



$Fig.(5.2),(h=.4)$

Example 3 *Consider the fuzzy initial value problem*

$$y'(t)=c_1y^2(t)+c_2,\quad y(0)=0,$$

where $c_i>0$,*for* $i=1,2$ *are triangular fuzzy numbers, [11].*

The exact solution is given by

$$Y_1(t;r)=l_1(r)tan(w_1(r)t),$$

$$Y_2(t;r)=l_2(r)tan(w_2(r)t),$$

with

$$l_1(r)=\sqrt{c_{2,1}(r)/c_{1,1}(r)},l_2(r)=\sqrt{c_{2,2}(r)/c_{1,2}(r)},$$

$$w_1(r) = \sqrt{c_{1,1}(r).c_{2,1}(r)}, w_2(r) = \sqrt{c_{1,2}(r).c_{2,2}(r)},$$

where

$$[c_1]_r = [c_{1,1}(r), c_{1,2}(r)] \quad and \quad [c_2]_r = [c_{2,1}(r), c_{2,2}(r)].$$

The r-level sets of $y'(t)$ are

$$Y_1'(t;r) = c_{2,1}.(r)sec^2(w_1(r)t),$$

$$Y_2'(t;r) = c_{2,2}.(r)sec^2(w_2(r)t),$$

which defines a fuzzy number. We have

$$f_1(t,y;r) = \min\{c_1.u^2+c_2 | u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

$$f_2(t,y;r) = \max\{c_1.u^2+c_2 | u \in [y_1(t;r), y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

By using Rung-Kutta method at $t_n, 0 \leq n \leq N$

$$k_{1,1}(t_n;r) = (c_{1,1}(r).y_1^2(t_n;r) + c_{2,1}(r))$$

$$k_{1,2}(t_n;r) = (c_{1,2}(r).y_2^2(t_n;r) + c_{2,2}(r))$$

$$k_{2,1}(t_n;r) = (c_{1,1}(r).z_{1,1}^2(t_n;r) + c_{2,1}(r))$$

$$k_{2,2}(t_n;r) = (c_{1,2}(r).z_{1,2}^2(t_n;r) + c_{2,2}(r))$$

$$k_{3,1}(t_n;r) = (c_{1,1}(r).z_{2,1}^2(t_n;r) + c_{2,1}(r))$$

$$k_{3,2}(t_n;r) = (c_{1,2}(r).z_{2,2}^2(t_n;r) + c_{2,2}(r))$$

$$k_{4,1}(t_n;r) = (c_{1,1}(r).z_{3,1}^2(t_n;r) + c_{2,1}(r))$$

$$k_{4,2}(t_n;r) = (c_{1,2}(r).z_{3,2}^2(t_n;r) + c_{2,2}(r))$$

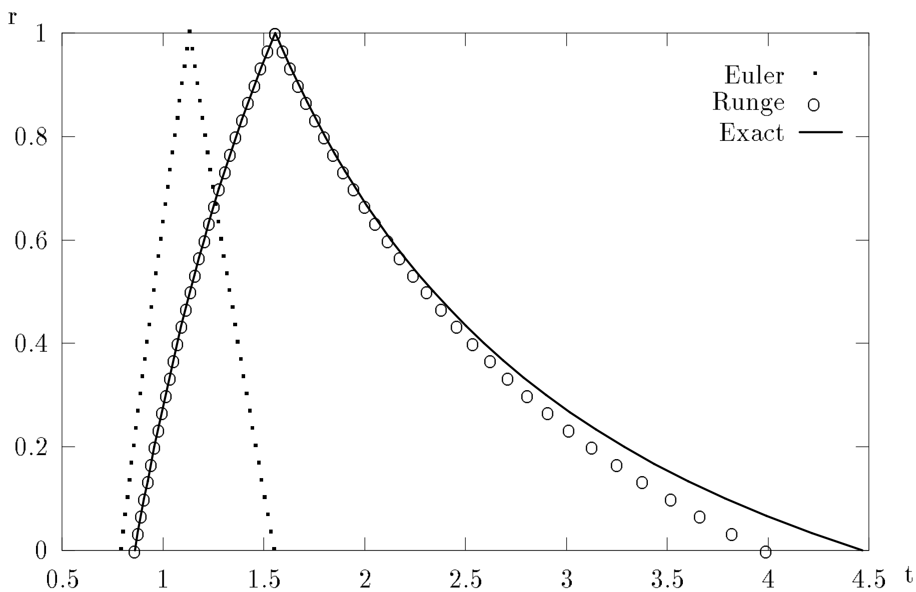
Where

$$z_{1,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{1,1}(t_n;r), \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{1,2}(t_n;r),$$

$$z_{2,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{2,1}(t_n;r), \quad z_{2,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{2,2}(t_n;r),$$

$$z_{3,1}(t_n;r) = y_1(t_n;r) + hk_{3,1}(t_n;r), \quad z_{3,2}(t_n;r) = y_2(t_n;r) + hk_{3,2}(t_n;r).$$

There are several nonlinear programings and can be solving by GAMS software. Thus the suggested 4th Rung-Kutta method in this paper can be used. The exact and approximate solutions are shown in Fig.(5.3) at $t = 1$.



$Fig.(5.3), (h = .5)$

6 Conclusion

We note that the convergence order of the Euler method in [10] is $O(h)$.It is shown that in proposed method, the convergence order is $O(h^4)$ and the comparison of solutions of example (1), (2) in this paper and [10] shows that this solutions are better for these examples.

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