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Abstract

In this paper numerical algorithms for solving 'fuzzy ordinary differential equations' are considered. A scheme based on the 2nd Rung-Kutta method in detail is discussed and this is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.

Keywords: Fuzzy differential equation, 2nd Rung-Kutta method, Fuzzy Cauchy problem.

AMS subject classification: 34A12, 65L05.

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1 Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [1] it was followed up by D.Dubois, H. Prade in [2], who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [5],[6] and by S. SeiKKala in [7],... The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [10] by the standard Euler method. paper organize as follows:

In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative ,which have been discussed by S. Seikkala in [10] are given. In section 3 we define the problem, this is a fuzzy Cuachy problem whose numerical solution is the main interest of this work. The numerically solving fuzzy differential equation by 2nd Rung-kutta method is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

2 Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); & a \leq t \leq b, \\ y(a) = \alpha, \end{cases} \quad (1)$$

The basis of all Rung-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = h \sum_{i=1}^m w_i k_i. \quad (2)$$

where for $i = 1, 2, \dots, m$, the w_i 's are constants and

$$k_i = f(t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} k_j). \quad (3)$$

Equation (2) is to be exact for powers of h through h^m . Therefore, the truncation error T_m , can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}),$$

The true magnitude of γ_m will generally be much less than the bound. Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1}$, as we expect it will be if h is small, then the bound on $\gamma_m h^{m+1}$ will usually be a bound on the error as a whole. The nonzero constants α_i , β_{ij} in 2nd Rung-Kutta method are $m = 2$ and for example

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \beta_{21} = 1.$$

Hence we have, see [9]

$$\begin{aligned} y_0 &= \alpha, \\ k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + h, y_i + h k_1), \\ y_{i+1} &= y_i + \frac{h}{2}(k_1 + k_2), \end{aligned} \quad (4)$$

where

$$a = t_0 \leq t_1 \leq \dots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \quad (5)$$

A triangular fuzzy number v is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $v(x)$, the membership function of the fuzzy number v , is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$.

We specify v as $(a_1/a_2/a_3)$. We will write: (1) $v > 0$ if $a_1 > 0$; (2) $v \geq 0$ if $a_1 \geq 0$; (3) $v < 0$ if $a_3 < 0$; and (4) $v \leq 0$ if $a_3 \leq 0$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r -level intervals. It means that if $v \in E$ then the r -level set

$$[v]_r = \{s \mid v(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval which is denoted by

$$[v]_r = [v_1(r), v_2(r)].$$

Let I be a real interval. A mapping $y : I \rightarrow E$ is called a fuzzy process and its r -level set is denoted by

$$[y(t)]_r = [y_1(t; r), y_2(t; r)], \quad t \in I, \quad r \in (0, 1].$$

Lemma 1 *Let $v, w \in E$ and s scalar, then for $r \in (0, 1]$*

$$[v + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[v - w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[v \cdot w]_r = [\min\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\},$$

$$\max\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}],$$

$$[sv]_r = s[v]_r.$$

Prrof see [2].

3 A fuzzy cuachy problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I = [0, T], \\ y(0) = y_0, \end{cases} \quad (6)$$

where f is a continuous mapping from $R_+ \times R$ into R and $y_0 \in E$ with r -level intervals

$$[y_0]_r = [y_1(0; r), y_2(0; r)], \quad r \in (0, 1].$$

The extension principle of Zadeh leads to the following definition of $f(t, y)$ when $y = y(t)$ is a fuzzy number

$$f(t, y)(s) = \sup\{y(\tau) \mid s = f(t, \tau)\}, \quad s \in R.$$

It follows that

$$[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)], \quad r \in (0, 1].$$

where

$$\begin{aligned} f_1(t, y; r) &= \min\{f(t, u) \mid u \in [y_1(r), y_2(r)]\}, \\ f_2(t, y; r) &= \max\{f(t, u) \mid u \in [y_1(r), y_2(r)]\}. \end{aligned} \quad (7)$$

Theorem 1 *Let f satisfy*

$$|f(t, v) - f(t, \bar{v})| \leq g(t, |v - \bar{v}|), \quad t \geq 0, \quad v, \bar{v} \in R,$$

where $g : R_+ \times R_+$ is a continuous mapping such that $r \rightarrow g(t, r)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \quad (8)$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.

Proof see [7].

In this paper we suppose (6) satisfies the hypothesis of theorem 1, also.

4 2nd Rung-Kutta method

Let the exact solution $[Y(t)]_r = [Y_1(t; r), Y_2(t; r)]$ is approximated by some $[y(t)]_r = [y_1(t; r), y_2(t; r)]$. From (2),(3) we define

$$\begin{aligned} y_1(t_{n+1}; r) - y_1(t_n; r) &= h \sum_{i=1}^2 w_i k_{i,1}(t_n, y(t_n; r)), \\ y_2(t_{n+1}; r) - y_2(t_n; r) &= h \sum_{i=1}^2 w_i k_{i,2}(t_n, y(t_n; r)), \end{aligned} \quad (9)$$

where the w_i 's are constants and

$$\begin{aligned} [k_i(t, y(t; r))]_r &= [k_{i,1}(t, y(t; r)), k_{i,2}(t, y(t; r))], \quad i = 1, 2 \\ k_{i,1}(t_n, y(t_n; r)) &= f(t_n + \alpha_i h, y_1(t_n) + h \sum_{j=1}^{i-1} \beta_{ij} k_{j,1}(t_n, y(t_n; r))), \\ k_{i,2}(t_n, y(t_n; r)) &= f(t_n + \alpha_i h, y_2(t_n) + h \sum_{j=1}^{i-1} \beta_{ij} k_{j,2}(t_n, y(t_n; r))), \end{aligned} \quad (10)$$

and

$$\begin{aligned} k_{1,1}(t, y(t; r)) &= \min\{f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}, \\ k_{1,2}(t, y(t; r)) &= \max\{f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}, \\ k_{2,1}(t, y(t; r)) &= \min\{f(t + h, u) | u \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))]\}, \\ k_{2,2}(t, y(t; r)) &= \max\{f(t + h, u) | u \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))]\}, \end{aligned} \quad (11)$$

Where in the 2nd Rung-Kutta method,

$$z_{1,1}(t, y(t; r)) = y_1(t; r) + h k_{1,1}(t, y(t; r)), \quad z_{1,2}(t, y(t; r)) = y_2(t; r) + h k_{1,2}(t, y(t; r)), \quad (12)$$

Define,

$$F[t, y(t; r)] = k_{1,1}(t, y(t; r)) + k_{2,1}(t, y(t; r)), \quad (13)$$

$$G[t, y(t; r)] = k_{1,2}(t, y(t; r)) + k_{2,2}(t, y(t; r)).$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n; r), Y_2(t_n; r)]$ and $[y(t_n)]_r = [y_1(t_n; r), y_2(t_n; r)]$ respectively. The solution is calculated by grid points which defined in (5). By (9),(13) we have

$$Y_1(t_{n+1}; r) \approx Y_1(t_n; r) + \frac{h}{2}F[t_n, Y(t_n; r)], \quad (14)$$

$$Y_2(t_{n+1}; r) \approx Y_2(t_n; r) + \frac{h}{2}G[t_n, Y(t_n; r)].$$

We define

$$y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{h}{2}F[t_n, y(t_n; r)], \quad (15)$$

$$y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{h}{2}G[t_n, y(t_n; r)].$$

The following lemmas will be applied to show convergence of these approximates i.e.,

$$\lim_{h \rightarrow 0} y_1(t; r) = Y_1(t; r),$$

$$\lim_{h \rightarrow 0} y_2(t; r) = Y_2(t; r).$$

Lemma 2 *Let a sequence of numbers $\{W_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N - 1,$$

for some given positive constants A and B . Then

$$|W_n| \leq A^n|W_0| + B\frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.$$

Proof see [10].

Lemma 3 *Let a sequence of numbers $\{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \leq |W_n| + A.\max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A \cdot \max\{|W_n|, |V_n|\} + B,$$

for some given positive constants A and B , and denote

$$U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N.$$

Then

$$U_n \leq \bar{A}^n U_0 + \bar{B} \frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \leq n \leq N,$$

where $\bar{A} = 1 + 2A$ and $\bar{B} = 2B$.

Proof see [10].

Let $F(t, u, v)$ and $G(t, u, v)$ are obtained by substituting $[y(t)]_r = [u, v]$ in (13)

$$F(t, u, v) = k_{1,1}(t, u, v) + k_{2,1}(t, u, v),$$

$$G(t, u, v) = k_{1,2}(t, u, v) + k_{2,2}(t, u, v).$$

The domain where F and G are defined is therefore

$$K = \{(t, u, v) \mid 0 \leq t \leq T, \quad -\infty < v < \infty, \quad -\infty < u \leq v\}.$$

Theorem 2 *Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^2(K)$ (the space of all continuous functions which their derivatives of order of one and two are exist.) and let the partial derivatives of F and G be bounded over K . Then, for arbitrary fixed $r, 0 \leq r \leq 1$, the approximately solutions (14) converge to the exact solutions $Y_1(t; r)$ and $Y_2(t; r)$ uniformly in t .*

Proof : It is sufficient to show

$$\lim_{h \rightarrow 0} y_1(t_N; r) = Y_1(t_N; r),$$

$$\lim_{h \rightarrow 0} y_2(t_N; r) = Y_2(t_N; r),$$

where $t_N = T$. For $n = 0, 1, \dots, N-1$, by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1}; r) &= Y_1(t_n; r) + \frac{h}{2}F[t_n, Y_1(t_n; r), Y_2(t_n; r)] + \frac{h^3}{4}Y^{(3)}(\xi_{n,1}), \\ Y_2(t_{n+1}; r) &= Y_2(t_n; r) + \frac{h}{2}G[t_n, Y_1(t_n; r), Y_2(t_n; r)] + \frac{h^3}{4}Y^{(3)}(\xi_{n,2}), \end{aligned} \quad (16)$$

where $\xi_{n,1}, \xi_{n,2} \in (t_n, t_{n+1})$. Denote

$$W_n = Y_1(t_n; r) - y_1(t_n; r),$$

$$V_n = Y_2(t_n; r) - y_2(t_n; r),$$

we have from (15) and (16)

$$\begin{aligned} W_{n+1} &= W_n + \frac{h}{2}\{F[t_n, Y_1(t_n; r), Y_2(t_n; r)] - F[t_n, y_1(t_n; r), y_2(t_n; r)]\} + \frac{h^3}{4}Y^{(3)}(\xi_{n,1}), \\ V_{n+1} &= V_n + \frac{h}{2}\{G[t_n, Y_1(t_n; r), Y_2(t_n; r)] - G[t_n, y_1(t_n; r), y_2(t_n; r)]\} + \frac{h^3}{4}Y^{(3)}(\xi_{n,2}). \end{aligned}$$

Then

$$\begin{aligned} |W_{n+1}| &\leq |W_n| + Lh \cdot \max\{|W_n|, |V_n|\} + \frac{h^3}{4}M_1, \\ |V_{n+1}| &\leq |V_n| + Lh \cdot \max\{|W_n|, |V_n|\} + \frac{h^3}{4}M_2, \end{aligned}$$

where

$$M_1 = \max |Y_1^{(3)}(t; r)|, \quad M_2 = \max |Y_2^{(3)}(t; r)|$$

for $t \in [0, T]$ and $L > 0$ is a bound for the partial derivatives of F and G . Thus by

lemma 3

$$\begin{aligned} |W_n| &\leq (1 + 2Lh)^n |U_0| + \frac{h^3}{2}M \frac{(1 + 2Lh)^n - 1}{2Lh}, \\ |V_n| &\leq (1 + 2Lh)^n |U_0| + \frac{h^3}{2}M \frac{(1 + 2Lh)^n - 1}{2Lh}, \end{aligned}$$

where $|U_0| = |W_0| + |V_0|$, $M = \max\{M_1, M_2\}$. In particular

$$|W_n| \leq (1 + 2Lh)^N |U_0| + h^2 M \frac{(1 + 2Lh)^{\frac{T}{h}} - 1}{4L},$$

$$|V_n| \leq (1 + 2Lh)^N |U_0| + h^2 M \frac{(1 + 2Lh)^{\frac{T}{h}} - 1}{4L},$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \leq h^2 M \frac{e^{2LT} - 1}{4L},$$

$$|V_N| \leq h^2 M \frac{e^{2LT} - 1}{4L},$$

and if $h \rightarrow 0$ we get $W_N \rightarrow 0$ and $V_N \rightarrow 0$ which completes the proof.

5 Examples

Example 1 Consider the fuzzy initial value problem, [10],

$$\begin{cases} y'(t) = y(t), & t \in I = [0, 1], \\ y(0) = (.75 + .25r, 1.125 - .125r), & 0 < r \leq 1. \end{cases}$$

By using 2nd Rung-Kutta method we have

$$y_1(t_{n+1}; r) = y_1(t_n; r) \left[1 + h + \frac{h^2}{2} \right],$$

$$y_2(t_{n+1}; r) = y_2(t_n; r) \left[1 + h + \frac{h^2}{2} \right].$$

The exact solution is given by

$$Y_1(t; r) = y_1(0; r)e^t, \quad Y_2(t; r) = y_2(0; r)e^t.$$

which at $t = 1$,

$$Y(1; r) = [(.75 + .25r)e, (1.125 - .125r)e], \quad 0 < r \leq 1.$$

The exact and approximate solutions are compared and plotted at $t = 1$ in Fig.(5.1).

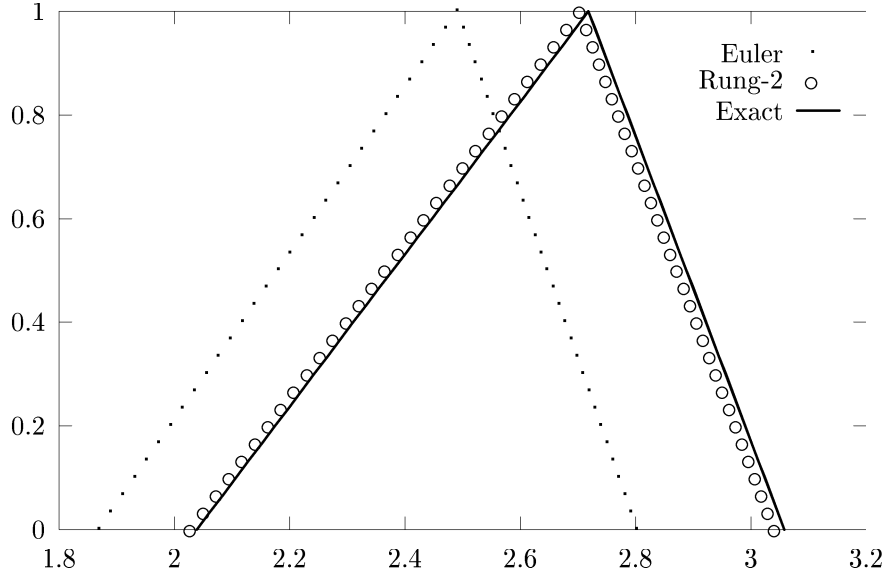


Fig.(5.1), ($h = .5$)

Example 2 Consider the fuzzy initial value problem, [10],

$$\begin{cases} y'(t) = ty(t), & [a, b] = [-1, 1], \\ y(-1) = (\sqrt{e} - .5(1 - r), \sqrt{e} + .5(1 - r)), & 0 < r \leq 1. \end{cases}$$

We separate between two steps.

(a) $t < 0$: The parametric form in this case is

$$y_1'(t; r) = ty_2(t; r), y_2'(t; r) = ty_1(t; r),$$

with the initial conditions given. The unique exact solution is

$$\begin{aligned} Y_2(t; r) &= \frac{A + B}{2}y_2(0; r) + \frac{A - B}{2}y_1(0; r), \\ Y_1(t; r) &= \frac{A - B}{2}y_2(0; r) + \frac{A + B}{2}y_1(0; r), \end{aligned}$$

where $A = e^{\frac{(t^2 - a^2)}{2}}$, $B = \frac{1}{A}$.

(b) $t \geq 0$: The parametric equations are

$$y_1'(t; r) = ty_1(t; r), y_2'(t; r) = ty_2(t; r),$$

with the initial conditions given. The unique exact solution at $t > 0$ is

$$Y_1(t; r) = y_1(0; r)e^{\frac{t^2}{2}}, Y_2(t; r) = y_2(0; r)e^{\frac{t^2}{2}}.$$

By using 2nd Rung-Kutta method at $t_n, 0 \leq n \leq N$ we have

$$k_{1,1}(t_n; r) = \min\{t.u|u \in [y_1(t_n; r), y_2(t_n; r)]\},$$

$$k_{1,2}(t_n; r) = \max\{t.u|u \in [y_1(t_n; r), y_2(t_n; r)]\},$$

$$k_{2,1}(t_n; r) = \min\{(t+h).u|u \in [z_{1,1}(t_n; r), z_{1,2}(t_n; r)]\},$$

$$k_{2,2}(t_n; r) = \max\{(t+h).u|u \in [z_{1,1}(t_n; r), z_{1,2}(t_n; r)]\},$$

Where

$$z_{1,1}(t_n; r) = y_1(t_n; r) + hk_{1,1}(t_n; r), \quad z_{1,2}(t_n; r) = y_2(t_n; r) + hk_{1,2}(t_n; r),$$

By considering $t > 0$ and $t < 0$, the above minimizing and maximizing problems can be solving by GAMS software. The exact and approximate solutions are compared and plotted in Fig.(5.2).

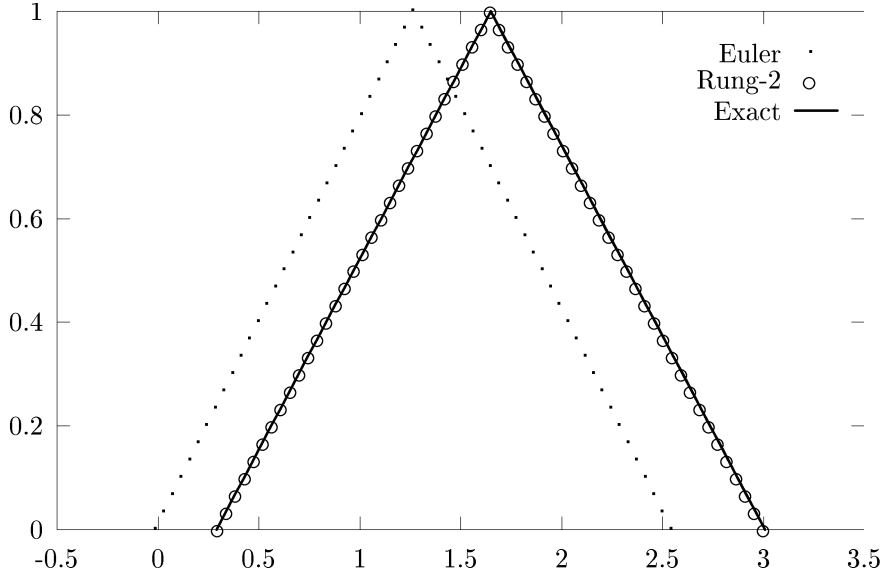


Fig.(5.2), ($h = .4$)

Example 3 Consider the fuzzy initial value problem

$$y'(t) = c_1 y^2(t) + c_2, \quad y(0) = 0,$$

where $c_i > 0$, for $i = 1, 2$ are triangular fuzzy numbers, [11].

The exact solution is given by

$$Y_1(t; r) = l_1(r) \tan(w_1(r)t),$$

$$Y_2(t; r) = l_2(r) \tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)}, \quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)},$$

$$w_1(r) = \sqrt{c_{1,1}(r) \cdot c_{2,1}(r)}, \quad w_2(r) = \sqrt{c_{1,2}(r) \cdot c_{2,2}(r)},$$

where

$$[c_1]_r = [c_{1,1}(r), c_{1,2}(r)] \quad \text{and} \quad [c_2]_r = [c_{2,1}(r), c_{2,2}(r)].$$

The r -level sets of $y'(t)$ are

$$Y_1'(t; r) = c_{2,1}(r) \sec^2(w_1(r)t),$$

$$Y_2'(t; r) = c_{2,2}(r) \sec^2(w_2(r)t),$$

which defines a fuzzy number. We have

$$f_1(t, y; r) = \min\{c_1.u^2 + c_2|u \in [y_1(t; r), y_2(t; r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

$$f_2(t, y; r) = \max\{c_1.u^2 + c_2|u \in [y_1(t; r), y_2(t; r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

By using Rung-Kutta method at $t_n, 0 \leq n \leq N$

$$k_{1,1}(t_n; r) = (c_{1,1}(r).y_1^2(t_n; r) + c_{2,1}(r))$$

$$k_{1,2}(t_n; r) = (c_{1,2}(r).y_2^2(t_n; r) + c_{2,2}(r))$$

$$k_{2,1}(t_n; r) = (c_{1,1}(r).z_{1,1}^2(t_n; r) + c_{2,1}(r))$$

$$k_{2,2}(t_n; r) = (c_{1,2}(r).z_{1,2}^2(t_n; r) + c_{2,2}(r))$$

Where

$$z_{1,1}(t_n; r) = y_1(t_n; r) + hk_{1,1}(t_n; r), \quad z_{1,2}(t_n; r) = y_2(t_n; r) + hk_{1,2}(t_n; r),$$

There are several nonlinear programings and can be solving by GAMS software. Thus the suggested 2nd Rung-Kutta method in this paper can be used. The exact and approximate solutions are shown in Fig.(5.3) at $t = 1$.

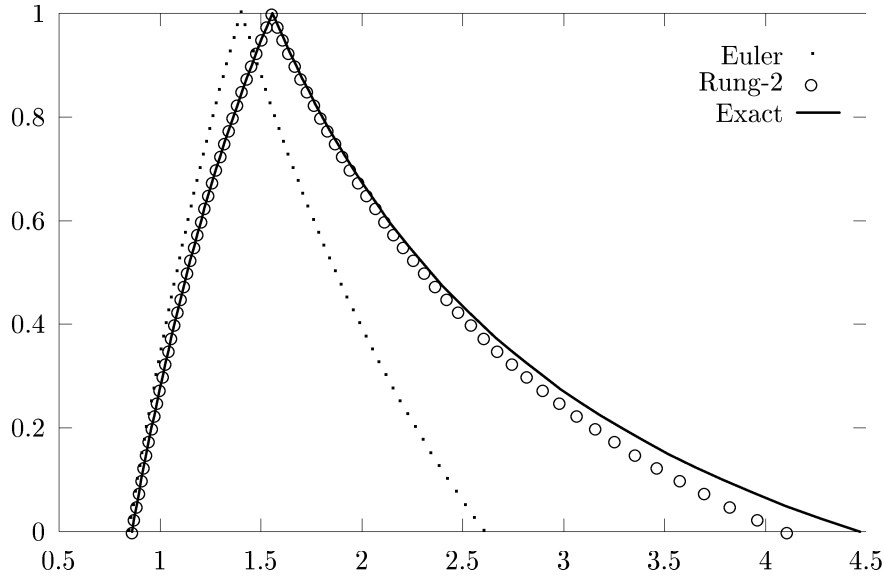


Fig.(5.3), ($h = .5$)

6 Conclusion

We note that the convergence order of the Euler method in [10] is $O(h)$. It is shown that in proposed method, the convergence order is $O(h^2)$ and the comparison of solutions of example (1), (2) in this paper and [10] shows that this solutions are better for these examples.

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