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Abstract

In recent papers, Delves [2] and others [1],[3] described a Chebyshev series method for the numerical solution of integral equations with non-singular kernels or some particular singular kernels, for example Green's function kernel, logarithmic and Cauchy kernels and so on.

In this paper we describe a Fourier series expansion method for a class of singular integral equations with Hilbert kernel and constant coefficients. We give a number of numerical examples showing that Galerkin method works well in practice.

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INTRODUCTION

We consider here numerical solution of singular integral equation with Hilbert kernel of the form

$$a\varphi(x) + \frac{b}{2\pi} \int_{-\pi}^{\pi} \varphi(t) \cot \frac{t-x}{2} dt + \mu \int_{-\pi}^{\pi} k(t,x)\varphi(t) dt = f(x), \quad (1)$$

$$-\pi \leq x \leq \pi,$$

where a, b and μ are real constants, with $b \neq 0$, $k(t, x)$ and $f(x)$ are real periodic functions of t and x with period 2π and are assumed known L_2 -functions. $\varphi(x)$ is the unknown L_2 -function with period 2π .

A theoretical consideration with existence and convergence theorems is described in [4] and [5].

As we know the set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos x, \right. \\ \left. \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 2x, \dots \right\}$$

is an orthonormal and complete basis for $L_2[-\pi, \pi]$. Therefore every square integrable function is completely determined (except for its value at a finite number of points) by its Fourier series, whether or not this series converges. The Fourier series of a continuous, piecewise smooth function $f(x)$ (with period 2π) converges to $f(x)$ absolutely and uniformly [6], page 81 and 120.

The expansion method approximates φ by φ_N , where

$$\varphi(x) \simeq \varphi_N(x) = \frac{1}{2}a_0 \quad (2)$$

$$+ \sum_{i=1}^N (a_i \cos ix + b_i \sin ix),$$

and with above assumptions, $\varphi_N(x)$ converges to $\varphi(x)$ in the mean, i.e.,

$$\lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} [\varphi(x) - \varphi_N(x)]^2 dx = 0.$$

Obviously, κ the index of equation (1) is zero. Let

$$\theta = \arg(a - ib),$$

being the characteristic number of (1). If $\kappa = 0$ then there are two cases, $[\theta]_{\pi} \neq \frac{\pi}{2}$ or $[\theta]_{\pi} = \frac{\pi}{2}$, where the notation $[x]_{\pi}$ denotes the number congruent to x in $[0, \pi)$ for the modulus π , [4].

Suppose $f(x)$ and $k(t, x)$ are continuously differentiable. If $[\theta]_{\pi} \neq \frac{\pi}{2}$ and μ is not an eigenvalue of $k(t, x)$ then (1) has a unique solution which can be obtained using Galerkin equations (otherwise (1) has an infinity of solutions). If $[\theta]_{\pi} = \frac{\pi}{2}$, when $a = 0$ and (1) is a first kind integral equations, under the constraint condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x) - \mu \int_{-\pi}^{\pi} k(t, x)\varphi(t) dt] dx = 0,$$

(1) has an infinity of solutions and if we impose the additional condition (unsolving condition)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) dt = C,$$

where C is a given real constant, then (1) has a unique solution, [4].

The last condition imposes

$$a_0 = 2C,$$

to the Galerkin equations.

In next section, we use the following formulae:

$$\begin{aligned}\cos ix &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot \frac{t-x}{2} \sin it \, dt, \\ \sin ix &= \frac{-1}{2\pi} \int_{-\pi}^{\pi} \cot \frac{t-x}{2} \cos it \, dt,\end{aligned}$$

in the sense of Cauchy principal integral, see [7], to obtain Galerkin equations.

THE AUGMENTED GALERKIN ALGORITHM

The Galerkin equations for the coefficients a_i and b_i in (2) are

$$\overline{\mathbf{A}}\mathbf{X} = \overline{\mathbf{F}}, \quad (3)$$

where, for $i = 0, 1, \dots, N$,

$$\begin{aligned}\overline{\mathbf{A}}_{ij} &= \eta_j [a\pi\delta_{ij} + \\ &\mu \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} k(t, x) \cos ix \cos jt \, dt \, dx], \\ j &= 0, 1, \dots, N,\end{aligned}$$

and

$$\begin{aligned}\overline{\mathbf{A}}_{i(N+j)} &= b\pi\delta_{ij} + \\ &\mu \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} k(t, x) \cos ix \sin jt \, dt \, dx, \\ j &= 1, \dots, N,\end{aligned}$$

and for $i = 1, \dots, N$,

$$\begin{aligned}\overline{\mathbf{A}}_{(N+i)j} &= \eta_j [-b\pi\delta_{ij} + \\ &\mu \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} k(t, x) \sin ix \cos jt \, dt \, dx], \\ j &= 0, 1, \dots, N,\end{aligned}$$

and

$$\overline{\mathbf{A}}_{(N+i)(N+j)} = a\pi\delta_{ij} +$$

$$\begin{aligned}\mu \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} k(t, x) \sin ix \sin jt \, dt \, dx, \\ j = 1, \dots, N.\end{aligned}$$

For $i = 0, 1, \dots, N$,

$$\overline{\mathbf{F}}_i = \int_{-\pi}^{\pi} f(x) \cos ix \, dx,$$

and for $i = 1, \dots, N$

$$\overline{\mathbf{F}}_{(N+i)} = \int_{-\pi}^{\pi} f(x) \sin ix \, dx,$$

and

$$\mathbf{X} = [a_0, a_1, \dots, a_N, b_1, b_2, \dots, b_N]^t,$$

where

$$\eta_j = \begin{cases} \frac{1}{2} & j = 0, \\ 1 & j \neq 0, \end{cases}$$

and δ_{ij} is Kronecker delta. When

$a = 0$ (in second case), we put only $a_0 = 2C$.

We use the augmented Galerkin scheme of [1] to find a solution of (3).

The assumption that (1) has an L_2 -solution implies that the representation

$$\varphi(x) = \frac{1}{2}a_0 + \sum_{i=1}^{\infty} (a_i \cos ix + b_i \sin ix),$$

is convergent in L_2 space and therefore

$$\sum_{i=0}^{\infty} a_i^2 < \infty,$$

$$\sum_{i=1}^{\infty} b_i^2 < \infty.$$

Therefore we impose the following constraints

$$|a_i| \leq \delta_i = C_f \hat{i}^{-r}, \quad i = 0, 1, \dots$$

$$|b_i| \leq \delta_i = C_f \hat{i}^{-r}, \quad i = 1, \dots$$

with $C_f > 0$ and $r > \frac{1}{2}$ to (3) for having an L_2 solution, where $\hat{i} = \max(1, i)$, [1]. In

other words, when the mentioned constraints are satisfied, we expect a_i 's and b_i 's belong to l_2 and hence the obtained solution belong to L_2 . The constants C_f and r play the role of regularization parameters and some strategies are discussed in [1] to determine suitable values for them. We use here strategy one, in which

$$C_f = \lambda \|\overline{\mathbf{F}}\|_\infty / \|\overline{\mathbf{A}}\|_\infty,$$

where λ must be set heuristically, say $2 \leq \lambda \leq 10$, (it can be proved $\lambda \geq 1$).

COMPUTATIONAL DETAILS

To compute integrals in (3) we use m -panel Gauss-Kronrod Quadrature rule with t points. Hence,

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) TR(ix) dx &\simeq (-1)^i \frac{\pi}{m} \\ &\sum_{p=1}^m \sum_{s=1}^t w_s f\left(\frac{\pi}{m} y_s^p - \pi\right) TR\left(\frac{i\pi}{m} y_s^p\right), \\ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} k(z, x) TR(ix) TR'(jz) dz dx &\simeq \\ (-1)^{i+j} \frac{\pi^2}{m^2} \sum_{p=1}^m \sum_{q=1}^m \sum_{s=1}^t \sum_{t=1}^t k\left(\frac{\pi}{m} y_s^p - \pi, \right. \\ &\left. \frac{\pi}{m} y_t^q - \pi\right) TR\left(\frac{i\pi}{m} y_t^q\right) TR'\left(\frac{j\pi}{m} y_s^p\right), \end{aligned}$$

where

$$y_j^i = x_j - 1 + 2i,$$

$TR(x)$ and $TR'(x)$ may be $\sin(x)$ or $\cos(x)$ and w_s and x_s are weights and nodes for t -point Gauss-Kronrod quadrature rule, respectively.

To solve (3), we consider the augmented Galerkin scheme of [1]

$$\begin{aligned} &Minimize \quad \|\mathbf{A}\mathbf{X} - \mathbf{F}\| \quad (4) \\ &Subject to \end{aligned}$$

$$|a_i| \leq \delta_i = C_f \hat{i}^{-r},$$

$$|b_j| \leq \delta_j = C_f \hat{j}^{-r},$$

$$i = 0, 1, \dots, N,$$

$$j = 1, 2, \dots, N,$$

where \mathbf{A} and \mathbf{F} are numerical approximations to $\overline{\mathbf{A}}$ and $\overline{\mathbf{F}}$.

NUMERICAL EXAMPLES AND RESULTS

We consider a set of three examples. All computations were carried out on an IBM-PC using C language and long double precision. Computed errors are defined as follows:

$$\|E_N\|_2 = \sqrt{\frac{\sum_{i=1}^{99} \{\varphi(s_i) - \varphi_N(s_i)\}^2}{99}},$$

$$\|E_N\|_\infty = \max_{1 \leq i \leq 99} |\varphi(s_i) - \varphi_N(s_i)|,$$

where $s_i = -\pi + i\pi/50$.

Example 1: For $-\pi < x < \pi$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x) \cot \frac{t-x}{2} dt$$

$$- \int_{-\pi}^{\pi} \left(\frac{t+x}{2}\right)^2 \varphi(t) dt = \pi - \sin(x),$$

with solution $\varphi(x) = \cos(x)$.

Here $f(x)$ and $k(x, y)$ are analytic functions and the Fourier series of them are known,

and $\varphi(x)$ is in C^∞ and therefore the regularization parameter, r , can take any value.

Example 2: For $-\pi < x < \pi$,

$$\begin{aligned} \varphi(x) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x) \cot \frac{t-x}{2} dt \\ + \int_{-\pi}^{\pi} \frac{t+x}{2} \varphi(t) dt = \\ \frac{\pi^2}{3} - 4 \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\cos(ix) - \sin(ix)}{i^2}, \end{aligned}$$

with solution $\varphi(x) = x^2$. All functions have the same behavior as in example 1 and r can take any value.

Example 3: For $-\pi < x < \pi$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x) \cot \frac{t-x}{2} dt \\ + \int_{-\pi}^{\pi} \sin(t) \sin(x) \varphi(t) dt = \\ 12(\pi \sin(x) + \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\cos(ix)}{i^3}), \end{aligned}$$

with solution $\varphi(x) = x(\pi - x)(\pi + x)$ and r can take any value.

Results for the above examples are presented in tables 1-3. Tables give the accuracy, $\|E_N\|_2$, obtained by the augmented Galerkin algorithm for $m = 1$ or $m = 2$ (number of panels in integration) and by different values for r and λ in examples with $t = 15$.

CONCLUSIONS

We conclude from the above results that the augmented Galerkin method allows the almost routine solution of Hilbert integral equations. In example 1, the exact coefficients a_0, a_2, a_3, \dots and b_1, b_2, \dots are zero and our method for small value of N is very accurate. But for example 2, a_0, a_1, \dots and for example 3, b_1, b_2, \dots are non-zero and whence we must choose large value for N . Table 4 shows $\|E_N\|_2$ for exact Fourier coefficients of $\varphi(x)$ in examples 2 and 3. Table 4 shows that our method work well in practice, these numbers are the lower bound of obtained $\|E_N\|_2$ in tables 2 and 3.

Table 4

N	Example 2	Example 3
2	0.370239	0.353477
3	0.221158	0.158695
4	0.148709	0.086192
5	0.107584	0.052678
6	0.081816	0.034878
7	0.064531	0.024464
8	0.052347	0.017922
9	0.043430	0.013581
10	0.036709	0.010575
12	0.027448	0.006824
14	0.021553	0.004685
16	0.017602	0.003364
18	0.014845	0.002497
20	0.012852	0.001899

Table 1 (Example 1) $t = 15$

$r = 5, \lambda = 4$					$r = 2, \lambda = 10$		
$m = 1$			$m = 2$		$m = 2$		
N	$\ E_N\ _\infty$	$\ E_N\ _2$	$\ E_N\ _\infty$	$\ E_N\ _2$	N	$\ E_N\ _\infty$	$\ E_N\ _2$
2	3.68E-13	2.14E-13	3.75E-16	2.42E-16	2	3.75E-16	2.42E-16
3	3.50E-9	1.97E-9	1.23E-15	6.50E-16	3	1.23E-15	6.50E-16
4	1.50E-6	8.91E-7	3.33E-13	1.98E-13	4	3.33E-13	1.98E-13
5	1.15E-4	7.13E-5	3.33E-13	1.98E-13	5	3.33E-13	1.98E-13
6	2.60E-4	1.36E-4	3.05E-9	1.89E-9	6	3.05E-9	1.89E-9
7	3.31E-4	1.45E-4	3.05E-9	1.89E-9	7	3.05E-9	1.89E-9
8	3.66E-4	1.47E-4	1.37E-6	8.72E-7	8	1.37E-6	8.72E-7
9	3.59E-4	1.38E-4	1.37E-6	8.72E-7	9	1.37E-6	8.72E-7
10	1.51E-1	1.06E-1	1.42E-5	8.97E-6	10	1.04E-4	7.03E-5

Table 2 (Example 2) $t = 15$

$r = 1, \lambda = 10$					$r = 2, \lambda = 10$		
$m = 1$			$m = 2$		$m = 1$		
N	$\ E_N\ _\infty$	$\ E_N\ _2$	$\ E_N\ _\infty$	$\ E_N\ _2$	N	$\ E_N\ _\infty$	$\ E_N\ _2$
2	1.20	3.70E-1	1.20	3.70E-1	2	1.20	3.70E-1
3	7.68E-1	2.21E-1	7.68E-1	2.21E-1	3	7.68E-1	2.21E-1
4	5.26E-1	1.49E-1	5.26E-1	1.49E-1	4	5.26E-1	1.49E-1
5	3.74E-1	1.08E-1	3.74E-1	1.08E-1	5	3.74E-1	1.08E-1
6	2.83E-1	8.22E-2	2.70E-1	8.18E-2	6	2.83E-1	8.22E-2
7	2.83E-1	1.63E-1	1.97E-1	6.45E-2	7	2.83E-1	1.63E-1
8	1.64	9.94E-1	1.59E-1	5.23E-2	8	5.10E-1	2.62E-1
9	3.47	1.68	1.45E-1	4.34E-2	9	7.37E-1	3.13E-1
10	4.38	1.86	1.33E-1	3.67E-2	10	7.47E-1	2.88E-1

Table 3 (Example 3) $t = 15$

$r = 1, \lambda = 10$					$r = 2, \lambda = 10$		
$m = 1$			$m = 2$		$m = 2$		
N	$\ E_N\ _\infty$	$\ E_N\ _2$	$\ E_N\ _\infty$	$\ E_N\ _2$	N	$\ E_N\ _\infty$	$\ E_N\ _2$
2	7.12E-1	3.53E-1	7.12E-1	3.53E-1	2	7.12E-1	3.53E-1
3	3.70E-1	1.59E-1	3.70E-1	1.59E-1	3	3.70E-1	1.59E-1
4	2.26E-1	8.62E-2	2.25E-1	8.62E-2	4	2.25E-1	8.62E-2
5	1.59E-1	5.39E-2	1.48E-1	5.27E-2	5	1.48E-1	5.27E-2
6	2.75E-1	1.61E-1	1.08E-1	3.49E-2	6	1.08E-1	3.49E-2
7	2.14	1.35	8.13E-2	2.45E-2	7	8.13E-2	2.45E-2
8	1.13E+1	6.59	6.15E-2	1.79E-2	8	6.15E-2	1.79E-2
9	3.57E+1	1.97E+1	4.73E-2	1.36E-2	9	4.73E-2	1.36E-2
10	3.75E+1	2.03E+1	4.02E-2	1.06E-2	10	4.02E-2	1.06E-2

It is important to note that the elements of matrix in (3) do not tend to zero as $N \rightarrow \infty$ and therefore direct methods or iterative methods can not be used to solve (3), but solving (4) is independent of N under some mild conditions that are valid here, [1].

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