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Abstract

In recent papers, Delves [2] and others [1],[3] described a Chebyshev series method for the numerical solution of integral equations with non-singular kernels or some particular singular kernels, for example Green's function kernel, logarithmic and Cauchy kernels and so on.

In this paper we describe a Fourier series expansion method for a class of singular integral equations with Hilbert kernel and constant coefficients. We give a number of numerical examples showing that Galerkin method works well in practice.

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INTRODUCTION

We consider here numerical solution of singular integral equation with Hilbert kernel of the form

$$a\varphi(x) + \frac{b}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\cot\frac{t-x}{2}\,dt$$

$$+\mu\int_{-\pi}^{\pi}k(t,x)\varphi(t)\,dt = f(x), \qquad (1)$$

$$-\pi \le x \le \pi,$$

where a, b and μ are real constants, with $b \neq 0$, $k(t,x)$ and $f(x)$ are real periodic functions of t and x with period 2π and are assumed known L_2-functions. $\varphi(x)$ is the unknown L_2-function with period 2π .

A theoretical consideration with existence and convergence theorems is described in [4] and [5].

As we know the set

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x,$$

$$\frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \ldots\}$$

is an orthonormal and complete basis for $L_2[-\pi,\pi]$. Therefore every square integrable function is completely determined (expect for its value at a finite number of points) by its Fourier series, whether or not this series converges. The Fourier series of a continuous, piecewise smooth function $f(x)$ (with period 2π) converges to $f(x)$ absolutely and uniformly [6], page 81 and 120.

The expansion method approximates φ by φ_N, where

$$\varphi(x) \simeq \varphi_N(x) = \frac{1}{2}a_0 \qquad (2)$$

$$+\sum_{i=1}^{N}(a_i\cos ix + b_i\sin ix),$$

and with above assumptions, $\varphi_N(x)$ converges to $\varphi(x)$ in the mean, i.e.,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}[\varphi(x)-\varphi_N(x)]^2\,dx = 0.$$

Obviously, κ the index of equation (1) is zero. Let

$$\theta \;=\; arg(a-ib),$$

being the characteristic number of (1). If $\kappa = 0$ then there are two cases, $[\theta]_\pi \neq \frac{\pi}{2}$ or $[\theta]_\pi = \frac{\pi}{2}$, where the notation $[x]_\pi$ denotes the number congruent to x in $[0,\pi)$ for the modulus π, [4].

Suppose $f(x)$ and $k(t,x)$ are continuously differentiable. If $[\theta]_\pi \neq \frac{\pi}{2}$ and μ is not an eigenvalue of $k(t,x)$ then (1) has a unique solution which can be obtained using Galerkin equations (otherwise (1) has an infinity of solutions). If $[\theta]_\pi = \frac{\pi}{2}$, when $a = 0$ and (1) is a first kind integral equations, under the constraint condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-\mu\int_{-\pi}^{\pi}k(t,x)\varphi(t)\,dt]\,dx = 0,$$

(1) has an infinity of solutions and if we impose the additional condition (unisolving condition)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\,dt = C,$$

where C is a given real constant, then (1) has a unique solution, [4].

The last condition imposes

$$a_0 \;=\; 2C,$$

to the Galerkin equations.

In next section, we use the following formulae:

$$\cos ix = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\frac{t-x}{2}\sin it\,dt,$$

$$\sin ix = \frac{-1}{2\pi}\int_{-\pi}^{\pi} \cot\frac{t-x}{2}\cos it\,dt,$$

in the sense of Cauchy principal integral, see [7], to obtain Galerkin equations.

THE AUGMENTED GALERKIN ALGORITHM

The Galerkin equations for the coefficients a_i and b_i in (2) are

$$\overline{\mathbf{A}}\mathbf{X} = \overline{\mathbf{F}}, \tag{3}$$

where, for $i = 0, 1, \ldots, N$,

$$\overline{\mathbf{A}}_{ij} = \eta_j[a\pi\delta_{ij}+$$
$$\mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k(t,x)\cos ix\cos jt\,dt\,dx],$$
$$j = 0, 1, \ldots, N,$$

and

$$\overline{\mathbf{A}}_{i(N+j)} = b\pi\delta_{ij}+$$
$$\mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k(t,x)\cos ix\sin jt\,dt\,dx,$$
$$j = 1, \ldots, N,$$

and for $i = 1, \ldots, N$,

$$\overline{\mathbf{A}}_{(N+i)j} = \eta_j[-b\pi\delta_{ij}+$$
$$\mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k(t,x)\sin ix\cos jt\,dt\,dx],$$
$$j = 0, 1, \ldots, N,$$

and

$$\overline{\mathbf{A}}_{(N+i)(N+j)} = a\pi\delta_{ij}+$$
$$\mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k(t,x)\sin ix\sin jt\,dt\,dx,$$
$$j = 1, \ldots, N.$$

For $i = 0, 1, \ldots, N$,

$$\overline{\mathbf{F}}_i = \int_{-\pi}^{\pi} f(x)\cos ix\,dx,$$

and for $i = 1, \ldots, N$

$$\overline{\mathbf{F}}_{(N+i)} = \int_{-\pi}^{\pi} f(x)\sin ix\,dx,$$

and

$$\mathbf{X} = [a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N]^t,$$

where

$$\eta_j = \begin{cases} \frac{1}{2} & j = 0, \\ 1 & j \neq 0, \end{cases}$$

and δ_{ij} is Kronecker delta. When $a = 0$ (in second case), we put only $a_0 = 2C$.

We use the augmented Galerkin scheme of [1] to find a solution of (3).

The assumption that (1) has an L_2-solution implies that the representation

$$\varphi(x) = \frac{1}{2}a_0 + \sum_{i=1}^{\infty}(a_i\cos ix + b_i\sin ix),$$

is convergent in L_2 space and therefore

$$\sum_{i=0}^{\infty} a_i^2 < \infty,$$

$$\sum_{i=1}^{\infty} b_i^2 < \infty.$$

Therefore we impose the following constraints

$$|a_i| \leq \delta_i = C_f\widehat{i}^{\,-r},\ i = 0, 1, \ldots$$

$$|b_i| \leq \delta_i = C_f\widehat{i}^{\,-r},\ i = 1, \ldots$$

with $C_f > 0$ and $r > \frac{1}{2}$ to (3) for having an L_2 solution, where $\widehat{i} = max(1, i)$, [1]. In other words, when the mentioned constraints are satisfied, we expect a_i's and b_i's belong to l_2 and hence the obtained solution belong to L_2. The constants C_f and r play the role of regularization parameters and some strategies are discussed in [1] to determine suitable values for them. We use here strategy one, in which

$$C_f = \lambda\|\overline{\mathbf{F}}\|_\infty/\|\overline{\mathbf{A}}\|_\infty,$$

where λ must be set heuristically, say $2 \leq \lambda \leq 10$, (it can be proved $\lambda \geq 1$).

COMPUTATIONAL DETAILS

To compute integrals in (3) we use m-panel Gauss-Kronrod Quadrature rule with t points. Hence,

$$\int_{-\pi}^{\pi} f(x)TR(ix)\,dx \simeq (-1)^i\frac{\pi}{m}$$

$$\sum_{p=1}^{m}\sum_{s=1}^{t} w_s f(\frac{\pi}{m}y_s^p-\pi)TR(\frac{i\pi}{m}y_s^p),$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k(z,x)TR(ix)TR'(jz)\,dz\,dx \simeq$$

$$(-1)^{i+j}\frac{\pi^2}{m^2}\sum_{p=1}^{m}\sum_{q=1}^{m}\sum_{s=1}^{t}\sum_{t=1}^{t} k(\frac{\pi}{m}y_s^p-\pi,$$

$$\frac{\pi}{m}y_t^q-\pi)TR(\frac{i\pi}{m}y_t^q)TR'(\frac{j\pi}{m}y_s^p),$$

where

$$y_j^i = x_j - 1 + 2i,$$

$TR(x)$ and $TR'(x)$ may be $sin(x)$ or $cos(x)$ and w_s and x_s are weights and nodes for t-point Gauss-Kronrod quadrature rule, respectively.

To solve (3), we consider the augmented Galerkin scheme of [1]

$$\begin{array}{ll} Minimize & \|\mathbf{AX}-\mathbf{F}\| \\ Subject\ to & \end{array} \qquad (4)$$

$$|a_i| \leq \delta_i = C_f\hat{i}^{\,-r},$$

$$|b_j| \leq \delta_j = C_f\hat{j}^{\,-r},$$

$$i = 0,1,\ldots,N,$$

$$j = 1,2,\ldots,N,$$

where $\mathbf{A}$ and $\mathbf{F}$ are numerical approximations to $\overline{\mathbf{A}}$ and $\overline{\mathbf{F}}$.

NUMERICAL EXAMPLES AND RESULTS

We consider a set of three examples. All computations were carried out on an IBM-PC using C language and long double precision. Computed errors are defined as follows:

$$\|E_N\|_2 = \sqrt{\frac{\sum_{i=1}^{99}\{\varphi(s_i)-\varphi_N(s_i)\}^2}{99}},$$

$$\|E_N\|_\infty = \max_{1\leq i\leq 99}|\varphi(s_i)-\varphi_N(s_i)|,$$

where $s_i = -\pi + i\pi/50$.

Example 1: For $-\pi < x < \pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\cot\frac{t-x}{2}\,dt$$

$$-\int_{-\pi}^{\pi}(\frac{t+x}{2})^2\varphi(t)\,dt = \pi - sin(x),$$

with solution $\varphi(x) = cos(x)$.

Here $f(x)$ and $k(x,y)$ are analytic functions and the Fourier series of them are known, and $\varphi(x)$ is in C^∞ and therefore the regularization parameter, r, can take any value.

Example 2: For $-\pi < x < \pi$,

$$\varphi(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\cot\frac{t-x}{2}\,dt$$

$$+\int_{-\pi}^{\pi}\frac{t+x}{2}\varphi(t)\,dt =$$

$$\frac{\pi^2}{3} - 4\sum_{i=1}^{\infty}(-1)^{i+1}\frac{cos(ix)-sin(ix)}{i^2},$$

with solution $\varphi(x) = x^2$. All functions have the same behavior as in example 1 and r can take any value.

Example 3: For $-\pi < x < \pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\cot\frac{t-x}{2}\,dt$$

$$+\int_{-\pi}^{\pi}sin(t)sin(x)\varphi(t)\,dt =$$

$$12(\pi sin(x) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{cos(ix)}{i^3}),$$

with solution $\varphi(x) = x(\pi - x)(\pi + x)$ and r can take any value.

Results for the above examples are presented in tables 1-3. Tables give the accuracy, $\|E_N\|_2$, obtained by the augmented Galerkin algorithm for $m = 1$ or $m = 2$ (number of panels in integration) and by different values for r and λ in examples with $t = 15$.

CONCLUSIONS

We conclude from the above results that the augmented Galerkin method allows the almost routine solution of Hilbert integral equations. In example **1**, the exact coefficients $a_0, a_2, a_3, \ldots$ and $b_1, b_2, \ldots$ are zero and our method for small value of N is very accurate. But for example **2**, $a_0, a_1, \ldots$ and for example **3**, $b_1, b_2, \ldots$ are non-zero and whence we must choose large value for N. Table 4 shows $\|E_N\|_2$ for exact Fourier coefficients of $\varphi(x)$ in examples **2** and **3**. Table 4 shows that our method work well in practice, these numbers are the lower bound of obtained $\|E_N\|_2$ in tables 2 and 3.

Table 4

N	Example 2	Example 3
2	0.370239	0.353477
3	0.221158	0.158695
4	0.148709	0.086192
5	0.107584	0.052678
6	0.081816	0.034878
7	0.064531	0.024464
8	0.052347	0.017922
9	0.043430	0.013581
10	0.036709	0.010575
12	0.027448	0.006824
14	0.021553	0.004685
16	0.017602	0.003364
18	0.014845	0.002497
20	0.012852	0.001899

Table 1 (Example 1)

$t = 15$

	$r = 5, \lambda = 4$					$r = 2, \lambda = 10$	
	$m = 1$		$m = 2$		$m = 2$		
N	$\|E_N\|_\infty$	$\|E_N\|_2$	$\|E_N\|_\infty$	$\|E_N\|_2$	N	$\|E_N\|_\infty$	$\|E_N\|_2$
2	3.68E-13	2.14E-13	3.75E-16	2.42E-16	2	3.75E-16	2.42E-16
3	3.50E-9	1.97E-9	1.23E-15	6.50E-16	3	1.23E-15	6.50E-16
4	1.50E-6	8.91E-7	3.33E-13	1.98E-13	4	3.33E-13	1.98E-13
5	1.15E-4	7.13E-5	3.33E-13	1.98E-13	5	3.33E-13	1.98E-13
6	2.60E-4	1.36E-4	3.05E-9	1.89E-9	6	3.05E-9	1.89E-9
7	3.31E-4	1.45E-4	3.05E-9	1.89E-9	7	3.05E-9	1.89E-9
8	3.66E-4	1.47E-4	1.37E-6	8.72E-7	8	1.37E-6	8.72E-7
9	3.59E-4	1.38E-4	1.37E-6	8.72E-7	9	1.37E-6	8.72E-7
10	1.51E-1	1.06E-1	1.42E-5	8.97E-6	10	1.04E-4	7.03E-5

Table 2 (Example 2)

$t = 15$

	$r = 1, \lambda = 10$					$r = 2, \lambda = 10$	
	$m = 1$		$m = 2$		$m = 1$		
N	$\|E_N\|_\infty$	$\|E_N\|_2$	$\|E_N\|_\infty$	$\|E_N\|_2$	N	$\|E_N\|_\infty$	$\|E_N\|_2$
2	1.20	3.70E-1	1.20	3.70E-1	2	1.20	3.70E-1
3	7.68E-1	2.21E-1	7.68E-1	2.21E-1	3	7.68E-1	2.21E-1
4	5.26E-1	1.49E-1	5.26E-1	1.49E-1	4	5.26E-1	1.49E-1
5	3.74E-1	1.08E-1	3.74E-1	1.08E-1	5	3.74E-1	1.08E-1
6	2.83E-1	8.22E-2	2.70E-1	8.18E-2	6	2.83E-1	8.22E-2
7	2.83E-1	1.63E-1	1.97E-1	6.45E-2	7	2.83E-1	1.63E-1
8	1.64	9.94E-1	1.59E-1	5.23E-2	8	5.10E-1	2.62E-1
9	3.47	1.68	1.45E-1	4.34E-2	9	7.37E-1	3.13E-1
10	4.38	1.86	1.33E-1	3.67E-2	10	7.47E-1	2.88E-1

Table 3 (Example 3)

$t = 15$

	$r=1, \lambda=10$					$r=2, \lambda=10$	
	$m=1$		$m=2$			$m=2$	
N	$\|E_N\|_\infty$	$\|E_N\|_2$	$\|E_N\|_\infty$	$\|E_N\|_2$	N	$\|E_N\|_\infty$	$\|E_N\|_2$
2	7.12E-1	3.53E-1	7.12E-1	3.53E-1	2	7.12E-1	3.53E-1
3	3.70E-1	1.59E-1	3.70E-1	1.59E-1	3	3.70E-1	1.59E-1
4	2.26E-1	8.62E-2	2.25E-1	8.62E-2	4	2.25E-1	8.62E-2
5	1.59E-1	5.39E-2	1.48E-1	5.27E-2	5	1.48E-1	5.27E-2
6	2.75E-1	1.61E-1	1.08E-1	3.49E-2	6	1.08E-1	3.49E-2
7	2.14	1.35	8.13E-2	2.45E-2	7	8.13E-2	2.45E-2
8	1.13E+1	6.59	6.15E-2	1.79E-2	8	6.15E-2	1.79E-2
9	3.57E+1	1.97E+1	4.73E-2	1.36E-2	9	4.73E-2	1.36E-2
10	3.75E+1	2.03E+1	4.02E-2	1.06E-2	10	4.02E-2	1.06E-2

It is important to note that the elements of matrix in (3) do not tend to zero as $N \to \infty$ and therefore direct methods or iterative methods can not be used to solve (3), but solving (4) is independent of N under some mild conditions that are valid here, [1].

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References

[1] Babolian,E. & Delves,L.M. "An Augmented Galerkin Method for First Kind Fredholm Equations ", *J. Inst. Maths. Applics.*, **24**, pp 157-174 (1979).

[2] Delves,L.M. "A Fast Method for the Solution of Fredholm Integral Equations ", *J. Inst. Math. Applics.*,**20**, pp 173-182 (1977).

[3] Delves,L.M. & Abd-Elal,L.F. & Hendry,J.A. "A Fast Galerkin Algorithm for Singular Integral Equations ", *J. Inst. Maths. Applics.*,**23**, pp 139-166 (1979).

[4] Du,Jin-Yuan "On the Numerical Solution for Singular Integral Equations with Hilbert Kernel", *Chinese J. Num. Math. & Appl.* **11:2**, pp 9-27 (1989).

[5] Ioakimidis,N.I. "A Natural Interpolation Formula for the Numerical Solution of Singular Integral Equations with Hilbert Kernel", *Bit*,**23**, pp 92-104 (1983).

[6] Tolstov,G.P. *Fourier Series,* DOVER publications, INC., New York (1976).

[7] Tricomi,F.G. *Integral Equations,* Interscience publishers, INC., New York, pp 167 (1965).