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Effective calculation of multiple solutions of mixed convection in a porous medium

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ABSTRACT

This paper considers an important model of boundary value problem with a condition at infinity namely combined free and forced convection over a plane of arbitrary shape embedded in a fluid-saturated porous medium; this model admits dual solutions, and uses a technique, which is to some extent modification of homotopy analysis method (HAM), in order to obtain dual solutions analytically with high accuracy.

KEY WORDS:
Homotopy analysis method, rule of multiplicity of solutions, prescribed parameter, convergence-controller parameter.

1 INTRODUCTION

The homotopy analysis method (HAM) has been developed by Liao [1] to obtain series solutions of controllable convergence to various nonlinear problems. This technique has successfully been applied in the latter decade to several nonlinear problems such as the viscous flows of non-Newtonian fluids [2], the KdV-type equations [3, 4], nonlinear heat transfer [5], finance problems [6], projectile motion [7], Glauert-jet flow [8], nonlinear water waves [9], groundwater flows [10], time-dependent Emden–Fowler type equations [11], differential difference equation [12], Laplace equation with Dirichlet and Neumann boundary conditions [13], and so on. As shown by Liao [1], the HAM logically contains some other non-perturbation techniques, such as Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method. On the other hand, the calculation of the all solutions or equivalently not losing any solutions of nonlinear boundary value problems is so important in engineering and physical sciences. In this regard, the present paper introduces a procedure for calculating the multiple solutions analytically at the same time. The idea behind this method is to reconstruct the homotopy analysis method [1] by adding rule of multiplicity of solutions and the so-called prescribed parameter in order to achieve this important goal by obtaining all branches of solutions analytically using one auxiliary linear operator, one auxiliary function and particularly the same initial approximation guess.

The aforesaid idea, for the first time, has been successfully tested on some nonlinear models such as model of diffusion and reaction in porous catalysts [14, 15]. In the present paper, we demonstrate the whole theory behind this technique and apply it on a model treated by A Nakayama and H Koyama [16] namely a kind of the problem of combined free and forced convection over a plane or axisymmetric body of arbitrary shape which is embedded in a fluid-saturated porous medium. This model accepts multiple (dual) solutions that is why it has been chosen for examining the proposed method.

2 THE PROPOSED METHOD

Consider the nonlinear differential equation:

$$\mathrm{N}\,[u(r)] = 0, \quad r \in \Omega \tag{1}$$

with boundary conditions

$$\mathrm{B}\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{2}$$

where N represents a general nonlinear operator, B is a boundary operator, and Γ is the boundary of the domain Ω. The critical step of the technique is that the boundary value problem (1) and (2) should be converted to an equivalent problem so that conditions (2) containing an unknown parameter, the so-called prescribed parameter δ and are split to

$$\mathrm{B}'\left(u, \delta, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \quad u(\alpha) = \beta \tag{3}$$

where $u(\alpha) = \beta$ is the forcing condition. The Homotopy analysis method can now applied on the problem as follows:

$$\mathrm{N}\,[u(r)] = 0, \quad r \in \Omega \tag{4}$$

$$\mathrm{B}'\left(u, \delta, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma. \tag{5}$$

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2.1 The zero-order deformation equation

We suppose that all the solutions $u = u(r)$ of the problem (4) can be expressed by a set of base functions

$$\{w_i(r), i = 0,1,2,...\}$$

in the form

$$u = u(r) = \sum_{n=0}^{+\infty} a_n w_n(r) \qquad (6)$$

where a_n are the coefficients to be determined. Let $u_0(r,\delta)$, with δ as a prescribed parameter, denote an initial approximation guess of the exact solution $u(r)$ which satisfies automatically the boundary conditions (5). By denoting $\hbar \neq 0$ as the convergence-controller parameter, $H(r) \neq 0$ an auxiliary function, and Λ an auxiliary linear operator and using $p \in [0,1]$ as an embedding parameter, we construct the general zero-order deformation equation and the corresponding boundary conditions as follow:

$$(1-p)\Lambda[\phi(r,\delta;p) - u_0(r,\delta)] = p\hbar \mathrm{H}(r)\mathrm{N}\,[\phi(r,\delta;p)], \qquad (7)$$

$$\mathrm{B}'\left(\phi(r,\delta;p), \delta, \frac{\partial \phi(r,\delta;p)}{\partial n}\right) = 0, \quad r \in \Gamma, \qquad (8)$$

where $\phi(r,\delta;p)$ is an unknown function to be determined. When $p=0$, the zero order deformation equation (7) becomes

$$\Lambda[\phi(r,\delta;0) - u_0(r,\delta)] = 0 \qquad (9)$$

which gives $\phi(r,\delta;0) = u_0(r,\delta)$. When $p=1$, the Eq. (7) leads to

$$\mathrm{N}\,[\phi(r,\delta;1)] = 0 \qquad (10)$$

which is exactly the same as the original Eq. (1) provided that $\phi(r,\delta;1) = u(r,\delta)$. We now expand the function $\phi(r,\delta;p)$ in a Taylor series to the embedding parameter p. This Taylor expansion can be written in the form

$$\phi(r,\delta;p) = u_0(r,\delta) + \sum_{m=1}^{+\infty} u_m(r,\delta)p^m, \qquad (11)$$

where

$$u_m(r,\delta) = \frac{1}{m!}\frac{\partial^m \phi(r,\delta;p)}{\partial p^m}\Big|_{p=0}, \; m = 0,1,2,... \qquad (12)$$

It is a fact that when the linear operator Λ, the initial approximation $u_0(r,\delta)$, the auxiliary parameter $\hbar \neq 0$, and the auxiliary function $H(r) \neq 0$ are chosen properly, the series (11) converges for $p=1$, and thus

$$u(r,\delta) = u_0(r,\delta) + \sum_{m=1}^{+\infty} u_m(r,\delta) = \sum_{n=0}^{+\infty} a_n w_n(r). \qquad (13)$$

2.2 High-order deformation equation

Assume that the linear operator Λ, the initial approximation $u_0(r,\delta)$, and the auxiliary function $H(r) \neq 0$ are chosen properly (it is worth mentioning here that the so-called convergence-controller parameter, $\hbar \neq 0$, will be determined later), the unknown functions $u_m(r,\delta)$ in Eq. (13) can be determined with the aid of the high-order deformation equations as follows. At first we define the vector $\overleftarrow{u}_n = \{u_0(r), u_1(r), , u_n(r)\}$ then, differentiating the zero-order deformation equation (7) m times with respect to the embedding parameter p, dividing it by $m!$, setting subsequently $p=0$ and taking into account the boundary conditions (8), one obtains the mth-order deformation equation

$$\Lambda[u_m(r,\delta) - \chi_m u_{m-1}(r,\delta)] = \hbar \mathrm{H}(r) R_m(\vec{u}_{m-1}, r, \delta), \qquad (14)$$

subject to the boundary conditions

$$\frac{\partial^m\left\{\mathrm{B}'\left(\phi(r,\delta;p), \delta, \frac{\partial \phi(r,\delta;p)}{\partial n}\right)\right\}}{\partial p^m}\Big|_{p=0} = 0, \quad r \in \Gamma, \qquad (15)$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \qquad (16)$$

and

$$R_m(\vec{u}_{m-1}, r, \delta) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}\,[\phi(r,\delta;p)]}{\partial p^{m-1}}\Big|_{p=0}$$

$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}[\sum_{n=0}^{+\infty} u_n(r,\delta)p^n]}{\partial p^{m-1}}\Big|_{p=0}. \quad (7) \qquad (17)$$

The high-order deformation equation (14) obviously, is the ordinary differential equation with boundary condition (15) and, it can be easily solved by using some software programs such as Mathematica or Maple. In this way, starting by $u_0(r,\delta)$, we obtain the functions $u_m(r,\delta)$ for $m = 1,2,3,...$ from Eq. (14) and (15) successively. Accordingly, the Mth order approximate solution of the problem (4) and (5) becomes

$$u(r,\delta) \approx U_M(r,\delta,\hbar) = u_0(r,\delta) + \sum_{m=1}^{M} u_m(r,\delta) = \sum_{n=0}^{M'} a_n w_n(r). \qquad (18)$$

2.3 Prediction of the multiplicity of solutions

It should be noted that up to this stage, the linear operator Λ, the initial approximation guess $u_0(r,\delta)$, and the auxiliary function $H(r) \neq 0$ have been chosen properly so that the series solutions (18) would be convergence. However, there are still two unknown parameters in the series (18) namely δ and $\hbar$, prescribed parameter and convergence-controller parameter, should be determined. It is essential that existence of unique or multiple solutions in terms of the basic functions (6) for the original boundary value problem (1) depends on the fact whether the forcing condition (3) ($u(\alpha) = \beta$), admits unique or multiple values for the formally introduced parameter δ in the boundary conditions (3). This stage is called rule of multiplicity of solutions that is a criterion in order to know how many solutions the boundary value problem (1) admits. The so-called rule of multiplicity of solutions is applied as follows:

Consider the Mth order approximate solution (18) and set $u(\alpha) = \beta$ in the forcing condition (3) to derive

$$u(\alpha) \approx U_M(\alpha,\delta,\hbar) = \beta. \qquad (19)$$

The above equation has two unknown parameters namely δ (prescribed parameter), and $\hbar$ which controls the convergence of the HAM series (18). It is a basic feature of HAM that the series solution (18) converges at $r=\alpha$ only in that range of $\hbar$, where the parameter δ does not change with the variation of $\hbar$. This means that in the plot of δ as function of $\hbar$ according to Eq. (19) implicitly, in the convergence range of the series $u(\alpha)$ some plateau occurs. The number of such horizontal plateaus where $\delta(\hbar)$ becomes constant, gives the multiplicity of the solutions of problems (1) and (2).

3 APPLICATIONS

Consider a plane of arbitrary shape embedded in a fluid-saturated porous medium. The geometry and wall temperature of the heated body are specified by the functions $r(x)$ and $T_w(x)$, where x is the coordinate measured along the surface of the body from its lower stagnation point. Also, the tangential component of the acceleration due to gravity g_x is also function of the wall geometry $r(x)$. Under the boundary layer and Darcy-Boussinesq approximations the governing equations of the steady mixed convection flow past a plane of arbitrary shape were reduced to the following form [16, 17]:

$$f' = 1+\left(\frac{Ra_x}{Pe_x}\right)\theta \tag{20}$$

$$\theta'' + \left(\frac{1}{2}-nI\right)f\theta' - nIf'\theta = xI\left(f'\frac{\partial\theta}{\partial x}-\theta'\frac{\partial f}{\partial x}\right) \tag{21}$$

along with the boundary conditions

$$f=0 \text{ and } \theta=1 \quad \text{at } \eta=0$$
$$f' \to 1 \text{ and } \theta\to 0 \quad \text{as } \theta\to+\infty \tag{22}$$

where primes denote partial differentiation with respect to η, f and θ are dimensionless stream function and temperature profiles, respectively. The product nI is called the lumped parameter which has the form

$$nI = \frac{\lambda}{1+3\lambda}, \tag{23}$$

after assuming that

$$\frac{Ra_x}{Pe_x} = \text{const.} \equiv b. \tag{24}$$

Now, Eqs. (20) and (21) represents ordinary differential equations. In addition, by assuming $\lambda=-1$ final steady mixed convection flow is given by

$$2f''' + f' - \left(f'\right)^2 = 0 \tag{25}$$

with the boundary conditions

$$f(0)=0,\ f'(0)=1+b,\ f'(+\infty)=1. \tag{26}$$

3.1 Exact dual solutions

As it has been shown in [17], Eqs. (25)-(26) admit dual solutions for $f'(\eta)$, for any given value of the parameter $b\in\left(-\frac{3}{2},0\right)$ which are

$$f'(\eta) = -\frac{1}{2}+\frac{3}{2}\tanh^2\left[\frac{\eta}{2\sqrt{2}}\pm\frac{1}{2}\ln\left(\frac{\sqrt{3}+\sqrt{3+2b}}{\sqrt{3}-\sqrt{3+2b}}\right)\right] \tag{27}$$

therefore, the corresponding dual temperature profiles can be expressed by

$$\theta(\eta) = \frac{f'(\eta)-1}{b} = -\frac{3}{2b}\cosh^{-2}\left[\frac{\eta}{2\sqrt{2}}\pm\frac{1}{2}\ln\left(\frac{\sqrt{3}+\sqrt{3+2b}}{\sqrt{3}-\sqrt{3+2b}}\right)\right] \tag{28}$$

so that the physical interest wall skin friction and wall heat flux, respectively are obtained as follows:

$$f''(0) = \pm b\sqrt{\frac{2b+3}{6}}, \tag{29}$$

$$\theta'(0) = \frac{f''(0)}{b} = \pm\sqrt{\frac{2b+3}{6}}. \tag{30}$$

The wall skin friction versus b is plotted in Figure 1. As we see, any given $b\in\left(-\frac{3}{2},0\right)$ corresponds to two $f''(0)$, also dual velocity and dual temperature profiles with respect to similarity variable η are shown in Figures 2 and 3. It is seen that both dual solutions satisfy the boundary conditions.

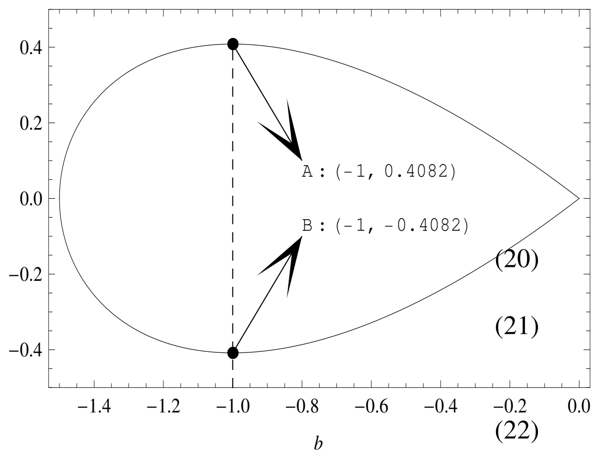


Fig. 1: The wall skin friction $f''(0)$ via constant b.

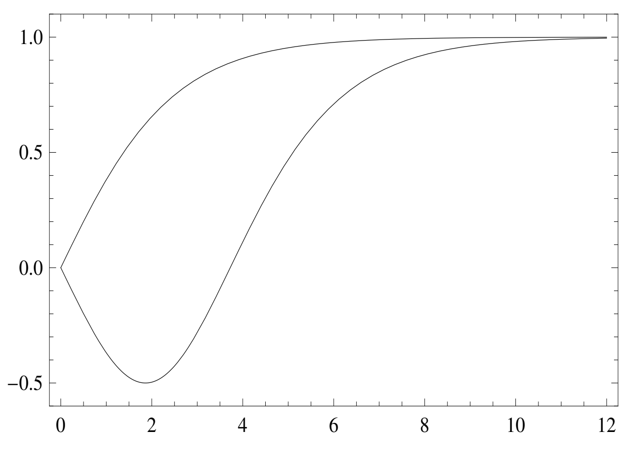


Fig. 2: Dual non-dimensional velocity profiles $f'(\eta)$ via similarity variable η.

3.2 Prediction of dual solutions by the rule of multiplicity of solutions

The aim of this subsection is to show how one can discover existence of dual solutions for Eqs. (25)-(26); to be more specific assume that $b=-1$ (Points A and B in Figure 1). Supposing that $u=f'$ and $u'(0)=f''(0)=\alpha$, we obtain

$$2u''+u-u^2=0 \tag{31}$$

$$u(0)=0,\quad u'(0)=\alpha \tag{32}$$

with additional forcing condition

$$u(+\infty)=1. \tag{33}$$

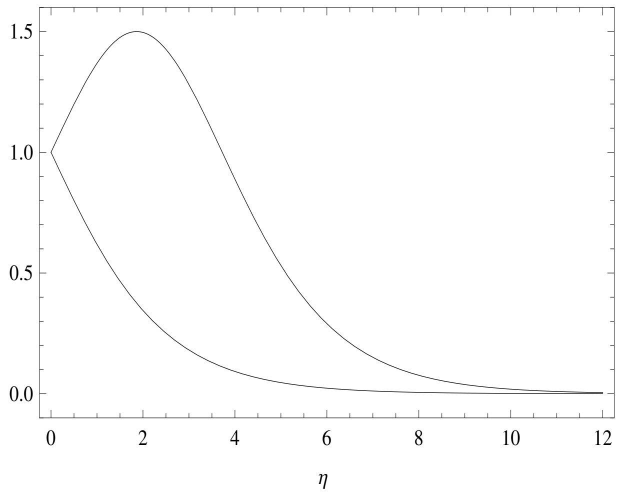


Fig. 3: Dual non-dimensional temperature profiles $\theta(\eta)$ via similarity variable η.

By the change of variable

$$\mu=1-\frac{1}{\sqrt[3]{\eta+1}}, \tag{34}$$

we have

$$\frac{\mathrm{d}u}{\mathrm{d}\eta}=\frac{1}{3}(1-\mu)^4\frac{\mathrm{d}u}{\mathrm{d}\mu}, \tag{35}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}^2\eta}=\frac{1}{9}(1-\mu)^8\frac{\mathrm{d}^2u}{\mathrm{d}^2\mu}-\frac{4}{9}(1-\mu)^7\frac{\mathrm{d}u}{\mathrm{d}\mu}. \tag{36}$$

Hence, Eqs. (31)-(33) are converted to

$$\frac{2}{9}(1-\mu)^8u''-\frac{8}{9}(1-\mu)^7u'+u-u^2=0 \tag{37}$$

$$u(0)=0\quad u'(0)=\delta \tag{38}$$

where $\delta=3\alpha$, with additional forcing condition

$$u(1)=1. \tag{39}$$

Now, the technique is applied on the Eqs. (37)-(38) and the parameter δ, which is played an important role to realize about multiplicity of solutions, will be obtained with the help of rule of multiplicity of solutions. It is straightforward to use the set of base functions

$$\{\mu^n, n=0,1,2,...\} \tag{40}$$

Under the rule of solution expression and according to the initial conditions (38), it is easy to choose $u_0(\mu,\delta)=\delta\mu$ as initial guess of solution $u(\mu)$, $H(\mu)=1$ as auxiliary function, and to choose auxiliary linear operator

$$\Lambda[\phi(\mu,\delta;p)]=\frac{\partial^2\phi(\mu,\delta;p)}{\partial\mu^2} \tag{41}$$

with the property

$$\mathrm{L}[c_1+c_2\mu]=0. \tag{42}$$

Therefore, after two subsequent integrations, the M th-order deformation Equation (14) yields for $M\geq 1$

$$u_m(\mu,\delta)=\chi_m u_{m-1}(\mu,\delta)+\hbar\int_0^{\mu}\int_0^{\xi}R_m(\vec{u}_{m-1},\tau,\delta)\,\mathrm{d}\tau\,\mathrm{d}\xi+c_1+c_2\mu \tag{43}$$

where from (17) and (37) (32)

$$R_m(\vec{u}_{m-1},\tau,\delta)=\frac{2}{9}(1-\mu)^8u''_{m-1}-\frac{8}{9}(1-\mu)^7u'_{m-1}+u_{m-1}-\sum_{j=0}^{m-1}u_ju_{m-1-j} \tag{44}$$

and integration constants c_1 and c_2 are obtained from the boundary conditions

$$u_m(0,\delta)=0,\; u'_m(0,\delta)=0 \tag{45}$$

In this way we obtain the functions $u_m(\mu,\delta)$ for $m=1,2,3,...$ from Eq. (43) successively. Finally, we can obtain M th order approximate solution

$$U_M(\mu,\delta,\hbar)=\sum_{m=0}^{M}u_m(\mu,\delta). \tag{46}$$

So Eq. (19), with the help of additional forcing condition (39), becomes

$$u(1)\approx U_M(1,\delta,\hbar)=1 \tag{47}$$

According to the above equation in Figure 4, δ (prescribed parameter) as a function of convergence controller parameter $\hbar$, has been plotted in the $\hbar$-range [-11.5, 0.5], for M=25. Two δ-plateaus can be identified in this Figure, namely $\delta=1.2246$ ($\alpha=0.4082$) and $\delta=-1.2246$ ($\alpha=-0.4082$) in the range [-10,-3] of $\hbar$. Accordingly, we conclude that HAM furnishes dual solutions, in a full agreement with the exact result shown in Figure 1 (It is worth mentioning here that Figure 1 indicate existence of two solutions for $b=-1$ so that, $u'(0)=0.4082$ for the first solution and $u'(0)=-0.4082$ for the second solution).

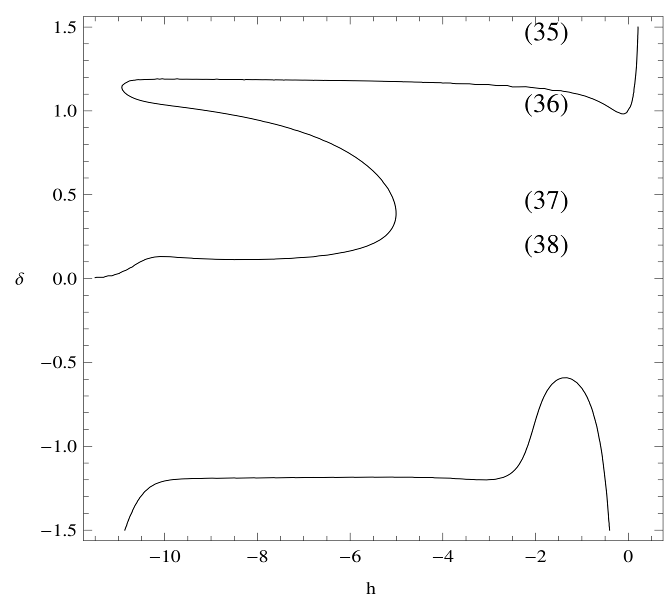


Fig. 4: Prescribed parameter δ via convergence controller parameter $\hbar$ in according to (47) with $M=25$.

3.3 Effective calculation of the two branches of solution

After finding out existence of dual solutions for problem (37)-(38) or equivalently problem (31)-(32), we turn to calculate the solutions. First method, that is straightforward, is to put ordered couples $(-7,1.2246)$ and $(-7,-1.2246)$ for $(\hbar,\delta)$ in Eq. (46) and then to use the change of variable (34), in this way we will calculate dual solutions for $u(\eta)=f^{'}(\eta)$ simultaneously, but we may need more iterations due to (34) and (44). Second method, that is simple, is just to use the original Eqs. (31)-(32) because in this time we know that the parameter α admit two values 0.4082 and -0.4082. We use the second method by applying the standard HAM procedure to calculate dual solutions. According to (31)-(32) suppose that

$$
\begin{aligned}
&\mathrm{H}(\eta)=1,\\
&\mathsf{L}[\phi(\eta;p)]=\frac{\partial^2\phi(\eta;p)}{\partial\eta^2},\quad \text{with the property } \mathsf{L}[c_0+c_1\eta]=0,\\
&\phi(\eta;0)=u_0(\eta)=0.4082\eta,\quad \text{for the first solution},\\
&\phi(\eta;0)=u_0(\eta)=-0.4082\eta,\quad \text{for the second solution},\\
&\mathsf{N}[\phi(\eta;p)]=2u^{''}_{m-1}+u_{m-1}-\sum_{j=0}^{m-1}u_j u_{m-1-j}.
\end{aligned}
$$

Following the standard HAM procedure we have plotted the $\hbar$-curves of third derivatives of upper and lower branches of solutions in Figures 5 and 7, respectively. It is easy to discover that popper valid region for convergence controller parameter $\hbar$ is the interval $[-0.9,-0.1]$. Furthermore, dual solutions of velocity and temperature profiles for different order approximations are shown in Figures 6 and 7, respectively and are compared with the corresponding dual exact solutions as well.

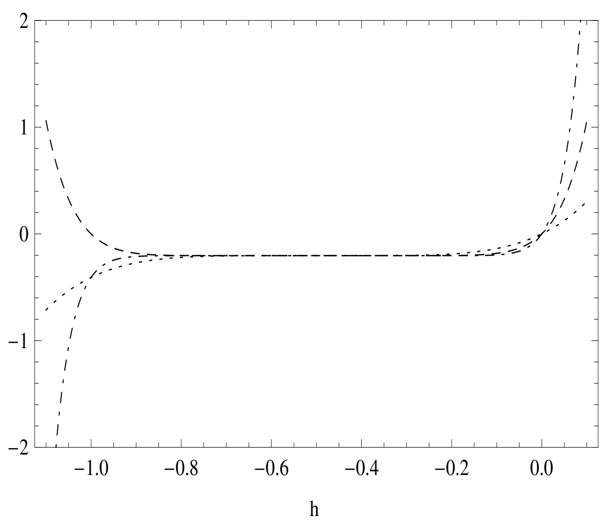
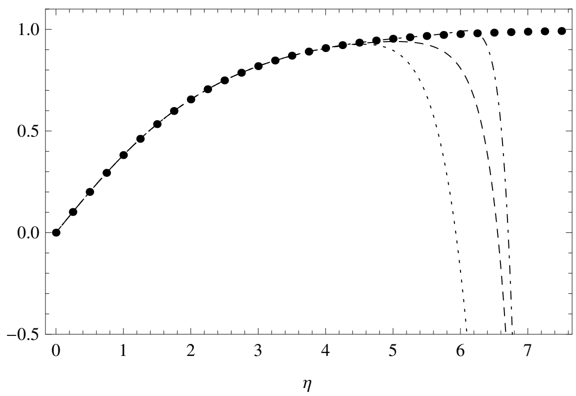
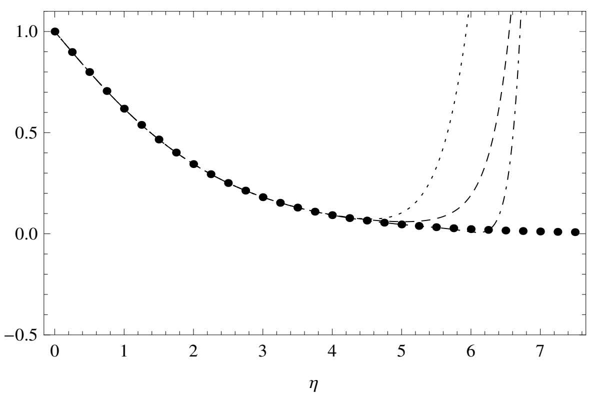


Fig. 5: $\hbar$-curves of $u^{'''}(0)$ for upper branch solution; Dotted line: 5th order approximation; Dashed line: 10th order approximation; DotDashd line: 15th order approximation.



(a)



(b)

Fig. 6: (a) The upper branch solution of velocity profile $f^{'}(\eta)$; Bold line: the exact; Dotted line: 5th order; Dashed line: 10th order; DotDashd line: 15th order. (b) The correspond temperature profile $\theta(\eta)$; Bold line: the exact; Dotted line: 5th order; Dashed line: 10th order; DotDashd line: 15th order.

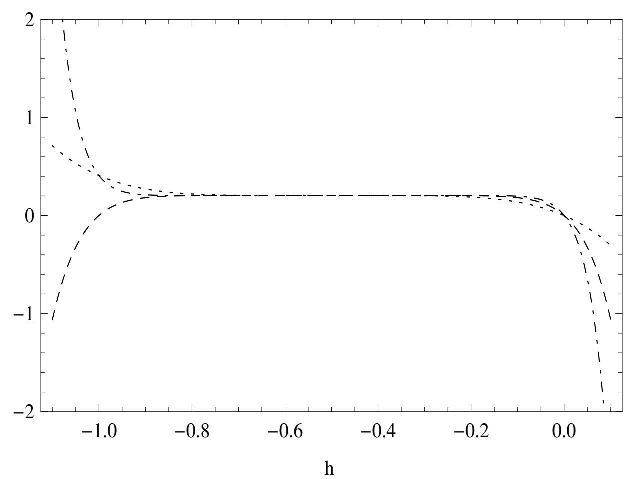
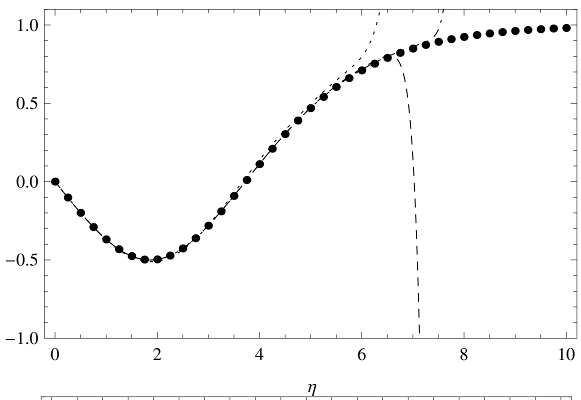
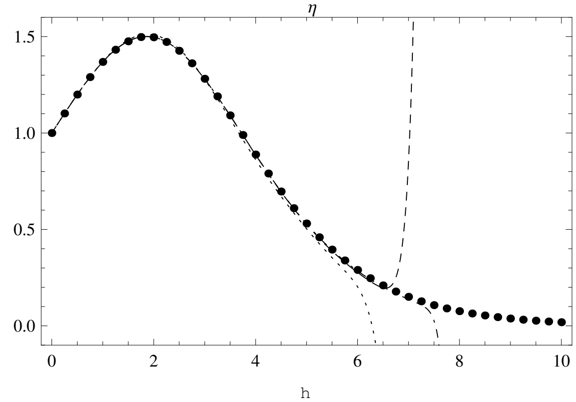


Fig. 7: $\hbar$-curves of $u^{'''}(0)$ for lower branch solution; Dotted line: 5th order approximation; Dashed line: 10th order approximation; DotDashd line: 15th order approximation.





(a) (b)

Fig. 8: (a) The lower branch solution of velocity profile $f^{'}(\eta)$; Bold line: the exact; Dotted line: 5th order; Dashed line: 10th order; DotDashd line: 15th order. (b) The correspond temperature profile $\theta(\eta)$; Bold line: the exact; Dotted line: 5th order; Dashed line: 10th order; DotDashd line: 15th order.

4 CONCLUSIONS

It is so important not to loose any solution of nonlinear differential equations with boundary conditions in engineering and physical sciences. In this regard, the present paper has introduced a novel technique to prevent this, in other words the proposed method is not only to predict existence of multiple solutions, but also to calculate all branches of solutions effectively by onely using one initial approximation guess, one auxiliary function and one auxiliary linear operator. In fact, this method presents a new point of view to the well-known homotopy analysis method, which says that the number of horizontal plateaus occurred in the plot of the so-called prescribed parameter δ as a function of convergence controller parameter $\hbar$ is indeed a good indicating the existence of multiple solutions (rule of multiplicity of solutions).

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