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Abstract

In this paper, a numerical method for finding minimal solution of a $m \times n$ fully fuzzy linear system of the form $\tilde{A}\tilde{x} = \tilde{b}$ based on pseudo inverse calculation, is given when the central matrix of coefficients is row full rank or column full rank, and where $\tilde{A}$ is a non-negative fuzzy $m \times n$ matrix, the unknown vector $\tilde{x}$ is a vector consisting of n non-negative fuzzy numbers and the constant $\tilde{b}$ is a vector consisting of m non-negative fuzzy numbers.

Key words: Fuzzy number; Fuzzy linear system; Minimal solution.

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1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [25], Dubois and Prade [14]. We refer the reader to [21] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces [11] to control chaotic systems [16,20], fuzzy metric spaces [23], fuzzy differential equations [3], fuzzy linear and nonlinear systems [1,2,5,9]

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems and fully fuzzy linear systems, several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them.

Friedman *et al.* [17] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $A\tilde{x} = B\tilde{x} + \tilde{y}$ where A , B are real $n \times n$ matrix, the unknown vector $\tilde{x}$ is vector consisting of n fuzzy numbers and the constant $\tilde{y}$ is vector consisting of n fuzzy numbers, in [18]. The $n \times n$ fuzzy linear systems has been studied by many authors [1,2,9,6,7,8] . Also, Wang *et al.* [24] presented an iterative algorithm for solving dual linear system of the form $\tilde{x} = A\tilde{x} + \tilde{u}$, where A is real $n \times n$ matrix, the unknown vector $\tilde{x}$ and the constant $\tilde{u}$ are all vectors consisting of fuzzy numbers and Abbasbandy *et al.* [4] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form $A\tilde{x} + \tilde{f} = B\tilde{x} + \tilde{c}$, where $A,\ B$ are real $m \times n$ matrices, the unknown vector $\tilde{x}$ is vector consisting of n fuzzy numbers and the constant $\tilde{f},\ \tilde{c}$ are vectors consisting of m fuzzy numbers. Recently, Muzziloi *et al.* [22] considered fully fuzzy linear systems of the form $\tilde{A}_1\tilde{x} + \tilde{b}_1 = \tilde{A}_2\tilde{x} + \tilde{b}_2$ with $\tilde{A}_1,\ \tilde{A}_2$ square matrices of fuzzy coefficients and $\tilde{b}_1,\ \tilde{b}_2$ fuzzy number vectors. Dehghan *et al.* [12,13] considered fully fuzzy linear systems of the form $\tilde{A} \otimes \tilde{x} = \tilde{b}$ where $\tilde{A}$ is a positive fuzzy $n \times n$ matrix, $\tilde{b}$ and $\tilde{x}$ are known and unknown positive fuzzy vectors.

In this paper we intend to solve the fully fuzzy linear system $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A}$ is a nonnegative fuzzy $m \times n$ matrix, the unknown vector $\tilde{x}$ is a vector consisting of n nonnegative fuzzy numbers and the constants $\tilde{b}$ is a vector consisting of m nonnegative fuzzy numbers.

2 Preliminaries

The minimal solution of an arbitrary linear system is formally defined such that [5]:

1. If the system is consistent and has a unique solution, then this solution is also the minimal solution.
2. If the system is consistent and has a set solution, then the minimal solution is a member of this set that has the least Euclidean norm.
3. If the system is inconsistent and has a unique least squares solution, then this solution is also the minimal solution.
4. If the system is inconsistent and has a least squares set solution, then the minimal solution is a member of this set that has the least Euclidean norm.

Definition 1. [19] A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \longrightarrow I = [0, 1]$ such that

i. $u(x)$ is upper semi-continuous.

ii. $u(x) = 0$ outside some interval $[a, d]$.

iii. There are real numbers b and c, $a \leq b \leq c \leq d$, for which

1. $u(x)$ is monotonically increasing on $[a, b]$,
2. $u(x)$ is monotonically decreasing on $[c, d]$,

3. $u(x) = 1, b \leq x \leq c$.

The set of all the fuzzy numbers (as given in definition 1) is denoted by E^1.

A popular fuzzy number is the triangular fuzzy number $\tilde{u} = (u_m, u_l, u_r)$ where u_m denotes the modal value and the real values $u_l > 0$ and $u_r > 0$ represent the left and right fuzziness, respectively. The membership function of a triangular fuzzy number is defined by:

$$u(x) = \begin{cases} \frac{x-u_m}{u_l} + 1, & u_m - u_l \leq x \leq u_m, \\ \frac{u_m-x}{u_r} + 1, & u_m \leq x \leq u_m + u_r, \\ 0, & otherwise. \end{cases}$$

Definition 2. (LR fuzzy numbers) A fuzzy number $\tilde{A}$ is said to be LR fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}), & x \leq a, \ \alpha > 0, \\ R(\frac{x-a}{\beta}), & x \geq a, \ \beta > 0, \end{cases}$$

where a is the mean value of $\tilde{A}$ and α and β are left and right spreads, respectively; and the function $L(.)$, which is called left shape function, satisfying:
(1) $L(x) = L(-x)$,
(2) $L(0) = 1$ and $L(1) = 0$,
(3) $L(x)$ is nonincreasing on $[0, \infty)$.

The definition of a right shape function $R(.)$ is usually similar to that of $L(.)$.
The mean value, left and right spread, and the shape functions of an LR fuzzy number $\tilde{A}$ are symbolically shown as
$\tilde{A} = (a, \alpha, \beta)_{LR}$.

Clearly, $\tilde{A} = (a, \alpha, \beta)_{LR}$ is positive, if and only if, $a - \alpha > 0$ (since $L(1) = 0$).

Definition 3. A matrix $\tilde{A} = (\tilde{a_{ij}})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number [15].
$\tilde{A}$ will be positive (negative) fuzzy matrix and denoted by $\tilde{A} \succ 0$ ($\tilde{A} \prec 0$) if each element of $\tilde{A}$ be positive (negative). Similarly, nonnegative and nonpositive fuzzy matrices may be defined.

Let each element of $\tilde{A}$ be a LR fuzzy number. We may represent $\tilde{A} = (\tilde{a_{ij}})$ that $\tilde{a_{ij}} = (a_{ij}, m_{ij}, n_{ij})_{LR}$,with new notation $\tilde{A} = (A, M, N)$, where A, M and N are three crisp matrices, with the same size of $\tilde{A}$, such that$A = (a_{ij})$, $M = (m_{ij})$, and $N = (n_{ij})$ are called the *center matrix* and the *left* and *right spread matrices,* respectively.

Definition 4. [15] Let $\tilde{u}$, $\tilde{v}$ be two fuzzy numbers of LR type:

$$\tilde{u} = (u, \theta, \lambda)_{LR}, \ \tilde{v} = (v, \phi, \eta)_{LR}$$

then

1. $(u, \theta, \lambda)_{LR} \oplus (v, \phi, \eta)_{LR} = (u + v, \theta + \phi, \lambda + \eta)_{LR}$,
2. $(u, \theta, \lambda)_{LR} \otimes (v, \phi, \eta)_{LR} \approx (uv, u\phi + v\theta, u\eta + v\lambda)_{LR} \ \ for\ positive\ \tilde{u}, \tilde{v}.$

3 Fully fuzzy linear system

Consider the $m \times n$ linear system of equations:

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus ... \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus ... \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (\tilde{a}_{m1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{m2} \otimes \tilde{x}_2) \oplus ... \oplus (\tilde{a}_{mn} \otimes \tilde{x}_n) = \tilde{b}_m. \end{cases}$$

Using matrix notation we get

$$\tilde{A} \otimes \tilde{x} = \tilde{b}, \tag{3.1}$$

or simply $\tilde{A}\tilde{x} = \tilde{b}$ where the coefficient matrix $\tilde{A} = (\tilde{a_{ij}})$, $1 \leq i \leq m$ $1 \leq j \leq n$ is a $m \times n$ fuzzy matrix and $\tilde{x_j} = (x_j, y_j, z_j)$, $\tilde{b_i} = (b_i, g_i, h_i)$ are nonnegative fuzzy numbers. This system is called a nonnegative fully fuzzy linear system (FFLS). In these cases with fixed y and z as the left and right spreads.

Consider the FFLS (3.1) where $\tilde{A}$ is nonnegative fuzzy matrix, $\tilde{x}$ is unknown nonnegative fuzzy vector and $\tilde{b}$ is known arbitrary fuzzy vector.

For calculate $\tilde{A} \otimes \tilde{x}$, we use definition 3, therefore

$$Ax = b, \tag{3.2}$$

$$Ay + Mx = g, \tag{3.3}$$

$$Az + Nx = h. \tag{3.4}$$

Corollary 1. Let T be $p \times q$ real column full rank or row full rank. There exists a $p \times p$ orthogonal matrix U, a $q \times q$ orthogonal matrix V, and a $p \times q$ diagonal matrix Σ with $\langle\Sigma\rangle_{ij} = 0$ for $i \neq j$ and $\langle\Sigma\rangle_{ii} = \sigma_i > 0$ with

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_s > 0$, where $s = \min\{p, q\}$, such that the singular value decomposition

$$T = U\Sigma V^t,$$

is valid. And if Σ^+ is that $q \times p$ matrix whose only nonzero entries are $\langle \Sigma^+ \rangle_{ii} = 1/\sigma_i$ for $1 \leq i \leq s$, then $T^+ = V\Sigma^+ U^t$ is the unique pseudo-inverse of T.

We refer the reader to [10] for more information on finding pseudo-inverse of an arbitrary matrix, and when we work with full rank matrices, there are not any problem and all calculations are stable and well-posed.
Thus we easily have

$$x = A^+ b, \tag{3.5}$$

and then by equation (3.3) and (3.5), we have

$$y = A^+(g - MA^+ b), \tag{3.6}$$

by equations (3.4) and (3.5),

$$z = A^+(h - NA^+ b). \tag{3.7}$$

Remark 1. Consider the FFLS (3.1) where $\tilde{A}$ is a nonnegative fuzzy matrix, $\tilde{x}$ is a solution, if and only if

$$Ax = b,$$
$$Ay + Mx = g,$$
$$Az + Nx = h.$$

In addition, if $y \geq 0, z \geq 0$ and $x - y \geq 0$ we say $\tilde{x} = (x, y, z)$ is a nonnegative fuzzy solution of nonnegative FFLS.
Remark 2. Consider the FFLS (3.1) where $\tilde{A}$ is a nonnegative fuzzy matrix with nonnegative pseudo inverse, $\tilde{x}$ is a minimal solution, if satisfy in (3.5)-(3.7). In addition, if $y \geq 0, z \geq 0$ and $x - y \geq 0$ we say $\tilde{x} = (x, y, z)$ is a nonnegative fuzzy minimal solution of nonnegative FFLS.
Theorem 1. Let $\tilde{A} \succeq 0$, $\tilde{b}$ is an arbitrary nonnegative fuzzy vector a the pseudo inverse of A is a nonnegative matrix. Also let $h \geq NA^+ b$, $g \geq MA^+ b$ and $(I + MA^+)b \geq g$. Then the system $\tilde{A} \otimes \tilde{x} = b$ has a nonnegative minimal fuzzy solution.

Proof. By hypotheses $x = A^+b \geq 0$. On the other hand, $h \geq NA^+b$ and $g \geq MA^+b$. Thus, with $y = A^+g - A^+Mx$, $z = A^+h - A^+Nx$; we have $y \geq 0$ and $z \geq 0$.
So, $\tilde{x} = (x, y, z)$ is a fuzzy minimal solution. Since $x-y = A^+[b+MA^+b-g]$, the positivity property of $\tilde{x}$ can be obtained from $(I + MA^+)b \geq g$. $\square$

4 Comparison with other methods

This study would not be completed without comparing it with other existing methods. Some comparisons are as follows:

- Abbasbandy *et al.* [4] investigated the existence of a minimal solution of general dual fuzzy linear systems of the form $A\tilde{x} + \tilde{f} = B\tilde{x} + \tilde{c}$, where A and B are real $m \times n$ matrices. In this paper we intend to solve the fully fuzzy linear system of the form $\tilde{A}\tilde{x} = \tilde{b}$ where $\tilde{A}$ is a nonnegative fuzzy matrix.
- Dehghan *et al.* [12,13] considered fully fuzzy linear systems of the form $\tilde{A}\tilde{x} = \tilde{b}$ where $\tilde{A}$ is a positive fuzzy $n \times n$ matrix, but in this paper we intend to solve the fully fuzzy linear system $\tilde{A}\tilde{x} = \tilde{b}$ where $\tilde{A}$ is a nonnegative fuzzy $m \times n$ matrix.

5 Numerical examples

To illustrate the technique proposed in this paper, consider the following examples.
Example 1. Consider the 2×3 fully fuzzy linear system

$$\begin{cases} (0.3, 0.1, 0.2) \otimes \tilde{x}_1 \oplus (0.2, 0.1, 0.3) \otimes \tilde{x}_2 \oplus (0.1, 0.05, 0.2) \otimes \tilde{x}_3 = (2, 1, 3), \\ (0.3, 0.2, 0.1) \otimes \tilde{x}_1 \oplus (0.2, 0.1, 0.1) \otimes \tilde{x}_2 \oplus (0.1, 0.03, 0.3) \otimes \tilde{x}_3 = (3, 2, 1.5), \end{cases}$$

where

$$A = \begin{pmatrix} 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 \end{pmatrix},$$

$$M = \begin{pmatrix} 0.1 & 0.1 & 0.05 \\ 0.2 & 0.1 & 0.03 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.3 \end{pmatrix},$$

$$b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, g = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, h = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}.$$

By corollary 1 we have:

$$A^{+} = \begin{pmatrix} 1.0714 & 1.0714 \\ 0.7143 & 0.7143 \\ 0.3571 & 0.3571 \end{pmatrix}.$$

Therefore, by Eq.(3.5)-(3.7), the minimal solution of fully fuzzy linear system is

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} (5.3571, 0.5740, 0.6122) \\ (3.5714, 0.3827, 0.4082) \\ (1.7857, 0.1913, 0.2041) \end{pmatrix}.$$

Example 2. Consider the 3×1 fully fuzzy linear system

$$\begin{cases} (1, 0.1, 0.5) \otimes \tilde{x}_1 = (4, 0.2, 1), \\ (2, 0.6, 0.3) \otimes \tilde{x}_1 = (3, 1.3, 0.6), \\ (3, 0.5, 2) \otimes \tilde{x}_1 = (5, 1, 4), \end{cases}$$

where

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, M = \begin{pmatrix} 0.1 \\ 0.6 \\ 0.5 \end{pmatrix}, N = \begin{pmatrix} 0.5 \\ 0.3 \\ 2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, g = \begin{pmatrix} 0.2 \\ 1.3 \\ 1 \end{pmatrix}, h = \begin{pmatrix} 1 \\ 0.6 \\ 4 \end{pmatrix}.$$

By corollary 1 we have:

$$A^{+} = \begin{pmatrix} 0.0714 & 0.1429 & 0.2143 \end{pmatrix}.$$

Therefore, by Eq.(3.5)-(3.7), the minimal solution of fully fuzzy linear system is

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \end{pmatrix} = \begin{pmatrix} (1.7857, 0.0571, 0.1087) \end{pmatrix}.$$

6 Summary and conclusions

In this paper, we proposed a model for solving a system of m fuzzy linear equations with n fuzzy variables. The original system with matrices coefficient $\tilde{A}$ is replaced by three $m \times n$ crisp matrices A, M and N. This paper mainly to discuss a decomposition of a singular or nonsingular matrix. By this decomposition, every singular or nonsingular matrix can be represented as a product of a three matrices V, Σ and U. Also, a condition for the existence of a nonnegative minimal fuzzy solution to the fully fuzzy linear system, is presented.

References

[1] S. Abbasbandy, A. Jafarian and R. Ezzati, Conjugate gradient method for fuzzy symmetric positive definite system of linear equations, *Appl. Math. Comput.* **171** (2005) 1184-1191.

[2] S. Abbasbandy, R. Ezzati and A. Jafarian, LU decomposition method for solving fuzzy system of linear equations, Appl. Math. Comput. **172** (2006) 633-643.

[3] S. Abbasbandy, J.J. Nieto and M. Alavi, Tuning of reachable set in one dimensional fuzzy differential inclusions, *Chaos, Solitons & Fractals* **26** (2005) 1337-1341.

[4] S. Abbasbandy, M. Otadi and M. Mosleh, Minimal solution of general dual fuzzy linear systems, *Chaos Solitons & Fractals* **37** (2008) 1113-1124.

[5] S. Abbasbandy, M. Otadi and M. Mosleh, Numerical solution of a system of fuzzy polynomials by fuzzy neural network, *Inform. Sci.* **178** (2008) 1948-1960.

[6] T. Allahviranloo, Succesive over relaxation iterative method for fuzzy system of linear equations, *Appl. Math. Comput.* **162** (2005) 189-196.

[7] T. Allahviranloo, Numerical methods for fuzzy system of linear equations, *Appl. Math. Comput.* **155** (2004) 493-502.

[8] T. Allahviranloo, M. Afshar Kermani, Solution of a fuzzy system of linear equation,*Appl. Math. Comput.***175** (2006) 519-531.

[9] B. Asady, S. Abbasbandy and M. Alavi, Fuzzy general linear systems, *Appl. Math. Comput.* **169** (2005) 34-40.

[10] S. Barnet, *Matrix Methods and Applications*, Clarendon Press, Oxford, 1990.

[11] M. Caldas and S. Jafari, θ-Compact fuzzy topological spaces, *Chaos Solitons & Fractals* **25** (2005) 229-232.

[12] M. Dehghan, B. Hashemi, M. Ghatee, Solution of the fully fuzzy linear systems using iterative techniques, *Chaos Solitons & Fractals* **34** (2007) 316-336.

[13] M. Dehghan, B. Hashemi, M. Ghatee, Computational mathods for solving fully fuzzy linear systems, *Appl. Math. Comput.* **179** (2006) 328-343.

[14] D. Dubois and H. Prade, Operations on fuzzy numbers, *J. Systems Sci.* **9** (1978) 613-626.

[15] D. Dubois and H. Prade, Systems of linear fuzzy constraints. *Fuzzy Sets Sys.* **3** (1980) 37-48.

[16] G. Feng and G. Chen, Adaptive control of discrete-time chaotic systems: a fuzzy control approach, *Chaos Solitons & Fractals* **23** (2005) 459-467.

[17] M. Friedman, Ma Ming and A. Kandel, Fuzzy linear systems, *Fuzzy Sets Sys.* **96** (1998) 201-209.

[18] M. Friedman, Ma Ming and A. Kandel, Duality in fuzzy linear systems, *Fuzzy Sets and Sys.* **109** (2000) 55-58.

[19] R. Goetschel, W. Voxman, Elementary calculus, *Fuzzy Sets and Systems* **18** (1986) 31-43.

[20] W. Jiang and Q. Guo-Dong and D. Bin, H_∞ Variable universe adaptive fuzzy control for chaotic system, *Chaos Solitons & Fractals* **24** (2005) 1075-1086.

[21] A. Kaufmann and M. M. Gupta, *Introduction Fuzzy Arithmetic*, Van Nostrand Reinhold, New York, 1985.

[22] S . Muzzioli and H. Reynaerts, Fuzzy linear systems of the form $A_1x + b_1 = A_2x + b_2$, *Fuzzy Sets and Sys.* **157** (2006) 939-951.

[23] J. H. Park, Intuitionistic fuzzy metric spaces, *Chaos Solitons & Fractals* **22** (2004) 1039-1046.

[24] X. Wang, Z. Zhong and M. Ha, Iteration algorithms for solving a system of fuzzy linear equations, *Fuzzy Sets and Sys.* **119** (2001) 121-128.

[25] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Inform. Sci.* **8** (1975) 199-249.