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Interpolation of fuzzy data by using quartic piecewise polynomials induced from *E*(3) cubic splines

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Abstract

Purpose: In this paper, we will consider the interpolation of fuzzy data by using the fuzzy-valued piecewise quartic polynomials $Q_{y_0,y_1,\ldots,y_n}(x)$ induced from $E(3)$ cubic spline functions.

Method: It has been many years since researchers have attended to the problem of interpolation of fuzzy data. Here, for Lagrange interpolation of fuzzy data, we will use the piecewise quartic polynomial induced from $E(3)$ cubic spline functions to interpolate the fuzzy data. To do this, we will apply the extension principle to construct the membership function of $Q_{y_0,y_1,\ldots,y_n}(x)$.

Results: By using piecewise quartic polynomials, a new set of fuzzy spline functions was defined to interpolate given fuzzy data.

Conclusions: In our previous study, we used $E(3)$ cubic spline to construct $E(3)$ fuzzy cubic spline. In this article, we added one extra term to this spline to compute the piecewise quartic polynomials and hence the fuzzy-valued piecewise quartic polynomials.

Keywords: $E(3)$ cubic spline, Quartic piecewise polynomials, Fuzzy interpolation, Extension principle

MSC: 94D05; 26E50

Introduction

The following problem was first posed by L.A. Zadeh (see, for example, [1]). Suppose that we have $n+1$ distinct real numbers $x_0, x_1, \ldots, x_n$ and for each of these numbers, a fuzzy value in $\mathbb{R}$, rather than a crisp value, is given. Zadeh asked the question whether it is possible to construct some kind of smooth function on $\mathbb{R}$ to fit with the collection of fuzzy data at these $n+1$ points.

Lagrange interpolation of fuzzy data was first investigated by Lowen [1]. Later, Kaleva [2] avoided the well-known computational troubles associated with crisp Lagrange interpolation by using linear spline and not-a-knot cubic spline approximations. If the fuzzy data are not convex, then a technical difficulty arises, and in this case, the Bernstein approximation can be constructed (see, for example, Diamond and Ramer [3]). The interpolation of fuzzy data by using spline functions of odd degree was considered in [4] with complete splines, in [5] with natural splines, in [6] with fuzzy splines, and finally in [7] with $E(3)$ cubic splines. Constructing consistent fuzzy surfaces from fuzzy data in the sense of Lagrange polynomials, linear splines and not-a-knot cubic splines were described in [8]. As it has been mentioned by Behforooz [9], the convergence of the $E(3)$ cubic spline is higher than that of the not-a-knot cubic spline and the natural cubic spline, and also, it has superconvergence properties that the other two cubic splines do not have. These superconvergence properties of the $E(3)$ cubic spline suggested to construct a $E(3)$ cubic fuzzy spline to approximate the fuzzy data (see [7]).

In this paper, we will use piecewise quartic polynomials induced from a $E(3)$ cubic spline to approximate given fuzzy data. In the 'Interpolation of fuzzy data by quartic polynomials induced from $E(3)$ cubic spline' section, we will introduce these quartic polynomials induced from $E(3)$ cubic fuzzy spline to interpolate the fuzzy data.

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Finally, in the 'Numerical examples' section, some numerical examples will be presented to compare our new results with the results of other studies in [4-7].

Methods

It is clear that the $E(3)$ cubic spline $S(x)$ is more accurate than natural and not-a-knot cubic splines, particularly at the end subintervals. So, the authors of [7] constructed a new set of fuzzy splines called '$E(3)$ fuzzy cubic spline' by using the $E(3)$ cubic spline. In this work, we will add one extra term to $S(x)$ to compute the piecewise quartic polynomials $Q(x)$ and hence the fuzzy-valued piecewise quartic polynomials $Q_{y_0,y_1,\ldots,y_n}(x)$. First of all, we recall some fundamental results of fuzzy numbers and fuzzy interpolations (see also [10]).

Preliminaries

Definition 1. A fuzzy number is a mapping $u : \mathbf{R} \to I = [0, 1]$ with the following properties [10]:

(i) u is an upper semicontinuous function on $\mathbf{R}$.
(ii) $u(x) = 0$ outside of some interval $[c, d] \subset \mathbf{R}$.
(iii) There exist real numbers a, b such that $c \le a \le b \le d$ and

  (1) $u(x)$ is a monotonic increasing function on $[c, a]$,
  (2) $u(x)$ is a monotonic decreasing function on $[b, d]$,
  (3) $u(x) = 1$ for all x in $[a, b]$.

Definition 2. A fuzzy number $u = (m, \alpha, \beta)_{LR}$ of type LR is a function from the reals into the interval $[0, 1]$ satisfying

$$u(x) = \begin{cases} L(\dfrac{m-x}{\alpha}), & m-\alpha \le x \le m, \\ R(\dfrac{x-m}{\beta}), & m \le x \le m+\beta, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where L and R are decreasing and continuous functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

The set of all fuzzy numbers is denoted by $\mathcal{F}$. A popular type of fuzzy number is the set of triangular fuzzy number $u = (c, \alpha, \beta)$ defined by

$$u(x) = \begin{cases} \frac{x-c+\alpha}{\alpha}, & c-\alpha \le x \le c, \\ \frac{c+\beta-x}{\beta}, & c \le x \le c+\beta, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. Note that the triangular fuzzy numbers are special cases of $L-L$ fuzzy numbers (see [11]).

Definition 3. If $u \in \mathcal{F}$, then the α-level set of u is denoted by $[u]^\alpha$ and defined by $[u]^\alpha = \{x \in \boldsymbol{R} | \, u(x) \ge \alpha\}$, where $0 < \alpha \le 1$. Also, $[u]^0$ is called the support of u and it is given by $[u]^0 = \overline{\bigcup_{\alpha\in(0,1]}[u]^\alpha}$. It follows that the level sets of u are closed and bounded intervals in $\boldsymbol{R}$.

It is well known that the addition and multiplication operations of real numbers can be extended to $\mathcal{F}$. In other words, for any $0 < \alpha \le 1$, $\lambda \in \boldsymbol{R}$, and $u, v \in \mathcal{F}$, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha \ \ and \ \ [\lambda u]^\alpha = \lambda [u]^\alpha .$$

Consider $n+1$ distinct real numbers $x_0 \le x_1 \le x_2 \le \ldots \le x_n$. For each x_i, we associate a fuzzy number $u_i \in \mathcal{F}$. To solve Zadeh's problem, we must find a continuous function $F : \boldsymbol{R} \to \mathcal{F}$ such that $F(x_i) = u_i$ for $i = 0, 1, \ldots, n$.

Let $P_{y_0,y_1,\ldots,y_n}(x)$ be the Lagrange interpolation polynomial of degree n which interpolates the data (x_i, y_i), where $i = 0, 1, \ldots, n$. According to the extension principle in [11], we can write the membership function $F(x)$ for each $x \in \boldsymbol{R}$ as follows:

$$\begin{aligned} \mu_{F(x)}(t) &= \sup_{\substack{y_0,y_1,\ldots,y_n \\ t=P_{y_0\ldots y_n}(x)}} \ \min_{i=0,1,\ldots,n} \mu_{u_i}(y_i) \quad \text{if } P^{-1}_{y_0\ldots y_n}(t) \ne \emptyset, \\ &= \quad 0, \qquad\qquad \text{otherwise}, \end{aligned}$$

where μ_{u_i} is the membership function of u_i.

For each $\alpha \in (0,1]$ and $i = 0, 1, \ldots, n$, let $J_i^\alpha = [u_i]^\alpha = \mu_{u_i}^{-1}[\alpha, 1]$ and $F^\alpha(x)$ be the α-level sets of u_i and $F(x)$, respectively. Hence,

$$\begin{aligned} F^\alpha(x) &= \{t \in \mathbf{R} \mid \mu_{F(x)}(t) \ge \alpha\} \\ &= \{t \in \mathbf{R} \mid \exists\, y_0, y_1, \ldots, y_n : \mu_{u_i}(y_i) \ge \alpha, \\ &\qquad i = 0, 1, \ldots, n \text{ and } P_{y_0,y_1,\ldots,y_n}(x) = t\} \\ &= \{t \in \mathbf{R} \mid \exists\, \overline{y} \in \textstyle\prod_{i=0}^{n} J_i^\alpha : P_{y_0,y_1\ldots,y_n}(x) = t\}, \end{aligned}$$

where $\overline{y} = (y_0, y_1, \ldots, y_n) \in \mathbf{R}^{n+1}$. Now, we have

$$\mu_{F(x)}(t) = \sup\{\alpha \in (0,1] \mid \exists\, \overline{y} \in \prod_{i=0}^{n} J_i^\alpha : P_{y_0,y_1,\ldots,y_n}(x) = t\},$$

where, as mentioned by Lowen in [1], the supremum is attained; hence, from Nguyen [12], we have

$$F^\alpha(x) = \{y \in \mathbf{R} | \, y = P_{y_0,y_1,\ldots,y_n}(x), \ y_i \in J_i^\alpha\}.$$

However, from the Lagrange interpolation formula, we have

$$F^{\alpha}(x) = \sum_{i=0}^{n} L_i(x) J_i^{\alpha},$$

where $L_i(x)$ represents the Lagrange polynomials.

Piecewise quartic polynomials induced from *E*(3) cubic splines

Years ago, Behforooz and Papamichael [13] have introduced a set of piecewise quartic polynomials $Q(x)$ induced from certain cubic splines $S(x)$ with certain end conditions, and they have shown that for each $Q(x)$, the order of convergence of $Q(x)$ is higher than the order of convergence of the corresponding cubic spline $S(x)$. Here, we consider a $E(3)$ cubic spline with joint points (x_i, y_i) with equally spaced knots $x_i = x_0 + ih$, where $i = 0, 1, \ldots, n$, defined as follows.

Definition 4. For a given data $\{(x_i, y_i)\}_{i=0}^{n}$ with equally spaced points $x_i = x_0 + ih$, where $i = 0, 1, \ldots, n$, the $E(3)$ cubic spline with the knots x_i is a piecewise polynomial function S that possesses the following conditions:

(a) $S \in C^2[x_0, x_n]$,
(b) $S(x)$ is a polynomial of degree 3 for $x \in [x_{i-1}, x_i]$, where $i = 1, 2, \ldots, n$,
(c) $S(x_i) = y_i$, where $i = 0, 1, \ldots, n$ (interpolation conditions),
(d) $m_0 + 3m_1 = \frac{1}{6h}\{-17y_0 + 9y_1 + 9y_2 - y_3\}$ (left end condition),
(e) $m_n + 3m_{n-1} = -\frac{1}{6h}\{-17y_n + 9y_{n-1} + 9y_{n-2} - y_{n-3}\}$ (right end condition),

where $m_i = S'(x_i)$. The $n-1$ first derivative consistency relation

$$m_{i-1} + 4m_i + m_{i+1} = \frac{3}{h}\{y_{i+1} - y_{i-1}\}, \text{ where } i = 1, 2, \ldots, n-1,$$

together with the above two end conditions (d) and (e), is enough to compute $n+1$ parameter m_i, where $i = 0, 1, \ldots, n$. Then, we can compute $S(x)$ by using the following cubic Hermite interpolation polynomial formula:

$$S(x) = \phi_1(x)\, y_{i-1} + \phi_2(x)\, y_i + \phi_3(x) m_{i-1} + \phi_4(x)\, m_i, \quad \text{where } x \in [x_{i-1}, x_i],$$

$$\begin{cases} \phi_1 = \dfrac{1}{h^3}\{3h(x - x_i)^2 + 2(x - x_i)^3\}, \\ \phi_2 = \dfrac{1}{h^3}\{3h(x - x_{i-1})^2 - 2(x - x_{i-1})^3\}, \\ \phi_3 = \dfrac{1}{h^2}\{h(x - x_i)^2 + (x - x_i)^3\}, \\ \phi_4 = \dfrac{1}{h^2}\{-h(x - x_{i-1})^2 + (x - x_{i-1})^3\}. \end{cases}$$

In general, for most of the interpolation cubic spline functions $S(x)$, the order of convergence of S and its first derivative S' are $|S(x) - y(x)| = O(h^4)$ and $|S'(x) - y'(x)| = O(h^3)$, respectively.

However, under certain end conditions like the periodic cubic spline, the D1 cubic spline with end conditions $m_0 = y'_0$ and $m_n = y'_n$, or the $E(3)$ cubic spline with end conditions (d) and (e), the order of convergence of the first derivatives at the knots x_i is 4 rather than 3, i.e., $|S'(x_i) - y'(x_i)| = O(h^4)$. By using these types of cubic splines with these types of end conditions and adding one extra term to $S(x)$, the authors of [6] constructed a set of interpolation quartic polynomials $Q(x)$ such that $|Q(x) - y(x)| = O(h^5)$. It means that by simply adding only one extra term to $S(x)$, the order of convergence increases from 4 to 5. In this paper, we consider only quartic polynomials $Q(x)$ induced from $E(3)$ cubic splines. Also, we can use, for example, the $D1$ cubic spline or periodic cubic spline as well to construct another set of quartic polynomials $Q(x)$.

Results and discussion

In this section, we recall the quartic polynomials $Q(x)$ induced from the $E(3)$ cubic spline. Then, we use $Q(x)$ to construct fuzzy-valued piecewise quartic polynomials to interpolate fuzzy data.

Interpolation of fuzzy data by quartic polynomials induced from *E*(3) cubic spline

Suppose that $S(x)$ is the $E(3)$ cubic spline defined in the 'Piecewise quartic polynomials induced from $E(3)$ cubic splines' section.

Definition 5. The quartic polynomial $Q(x)$ induced from the $E(3)$ cubic spline is given by

$$Q(x) = S(x) + \Delta_i (x - x_{i-1})^2 (x - x_i)^2, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n,$$

where Δ_i, $i = 1, 2, \ldots, n-1$ are given by the following divided differences:

$$\Delta_i = S[x_{i-1}, x_{i-1}, x_i, x_i, x_{i+1}] = \frac{1}{4h^4}\{y_{i+1} + 4y_i - 5y_{i-1} - h[4m_i + 2m_{i-1}]\},$$

and in the last interval $[x_{n-1}, x_n]$, consider $\Delta_n = \Delta_{n-1}$.

In this section, we will use the $Q(x)$ to interpolate the fuzzy data. We denote the family of these quartic polynomials by $Q_4(x_0, x_n)$. If the base spline $q_i \in Q_4(x_0, x_n)$ such that $q_i(x_j) = 1$ for $i = j$ and $q_i(x_j) = 0$ for $i \neq j$,

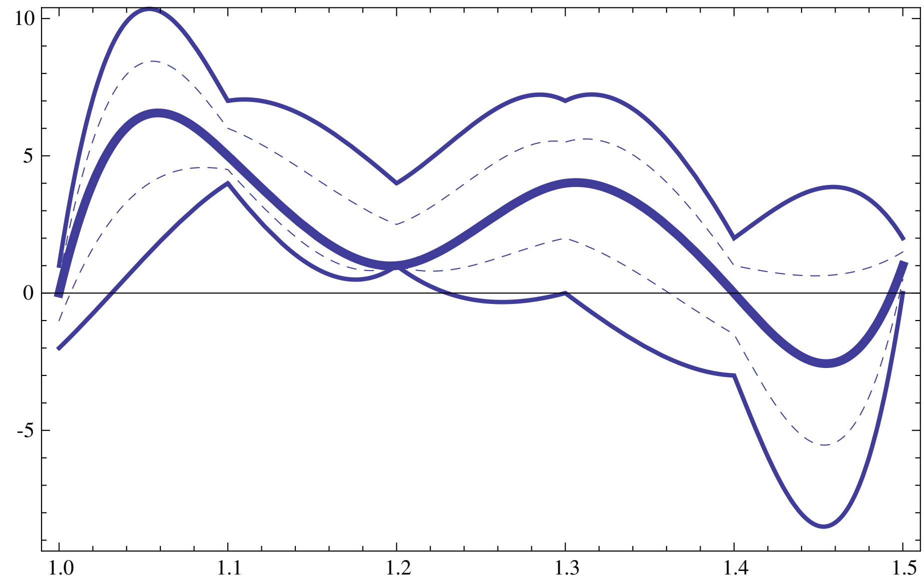

Figure 1 Endpoints of the 0-, 0.5-, and 1-level sets of fuzzy-valued piecewise quartic polynomials (Example 1). The solid line represents the support, the dashed line represents the 0.5-level set, and the thick line represents the 1-level set.

then similar to the Lagrange interpolation polynomial, the fuzzy spline

$$Q_{y_0,y_1,\ldots,y_n}(x) = \sum_{i=0}^{n} q_i(x)y_i$$

interpolates (x_i, y_i), where $i = 0, 1, \ldots, n$. From the 'Preliminaries' section, we have

$$F^{\alpha}(x) = \{t \in \boldsymbol{R} | \exists \overline{y} \in \prod_{i=0}^{n} J_i^{\alpha} : Q_{y_0,y_1,\ldots,y_n}(x) = t\}$$
$$= \sum_{i=0}^{n} q_i(x) J_i^{\alpha}$$

and

$$F(x) = \sum_{i=0}^{n} q_i(x)u_i.$$

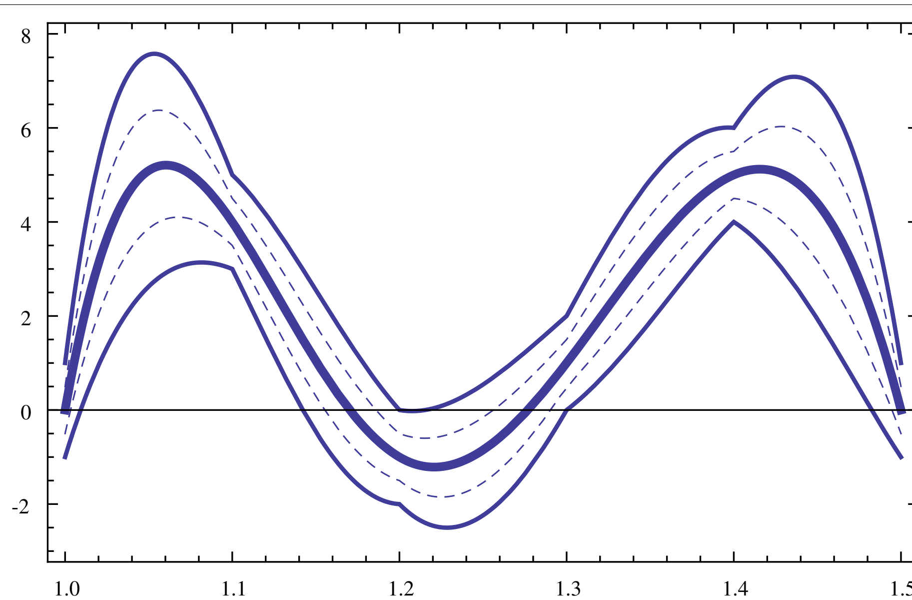

Figure 2 Endpoints of the 0-, 0.5-, and 1-level sets of fuzzy-valued piecewise quartic polynomials (Example 2). The solid line represents the support, the dashed line represents the 0.5-level set, and the thick line represents the 1-level set.

Hence, if all u_i are LL fuzzy numbers, then for each $x \in [x_0, x_n]$, $F(x)$ is an LL fuzzy number.

Numerical examples

Let $J_i^\alpha = [a_i^\alpha, b_i^\alpha]$. Then, the upper end point of $F^\alpha(x)$ is the solution of the following problem:

$$\text{Maximize} \quad Q_{y_0, y_1, \ldots, y_n}$$

$$\text{subject to} \quad a_i^\alpha \le y_i \le b_i^\alpha, \text{where } i = 0, 1, \ldots, n,$$

where the optimal solution is

$$y_i = \begin{cases} b_i^\alpha, \text{if } q_i(x) \ge 0, \\ a_i^\alpha, \text{if } q_i(x) < 0. \end{cases} \tag{2}$$

Similarly, the lower end point of $F^\alpha(x)$ can be obtained. Hence, if $u_i = (m_i, \alpha_i, \beta_i)$ and $F(x) = (m(x), \alpha(x), \beta(x))$, then we will have

$$m(x) = \sum_{i=0}^{n} q_i(x) m_i$$

$$\alpha(x) = \sum_{q_i(x) \ge 0} q_i(x)\alpha_i - \sum_{q_i(x) < 0} q_i(x)\beta_i$$

$$\beta(x) = \sum_{q_i(x) \ge 0} q_i(x)\beta_i - \sum_{q_i(x) < 0} q_i(x)\alpha_i$$

which are the same results as those of Kaleva [2].

Example 1. In this example, we have the data (x_i, u_i), where $i = 1, 2, 3, 4, 5$, in the following table:

x_i	1	1.1	1.2	1.3	1.4	1.5
m_i	0	5	1	4	0	1
α_i	2	1	0	4	3	1
β_i	1	2	3	3	2	1

The endpoints of the 0-, 0.5-, and 1-level sets of the fuzzy-valued piecewise quartic polynomials can be seen in Figure 1.

Example 2. Here, we have $u_i = y_i + A$, where $i = 0, 1, 2, 3, 4, 5$ and $A = (0, 1, 1)$:

x_i	1	1.1	1.2	1.3	1.4	1.5
y_i	0	4	-1	1	5	0

Figure 2 shows the endpoints of the 0-, 0.5- and 1-level sets.

Conclusions

In the literature, there are some published articles on using spline functions to construct fuzzy splines. At the same time, we know that $E(3)$ cubic spline $S(x)$ is more accurate than natural and not-a-knot cubic splines, particularly at the end subintervals. Hence, using the $E(3)$ cubic spline to construct $E(3)$ fuzzy cubic spline was the main idea in [7]. In this article, we added one extra term to $S(x)$ to compute the quartic polynomial $Q(x)$ to easily improve the order of convergence. The above examples are our witnesses to our claims.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors carried out the proof and read and approved the final manuscript.

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