

Spring 2011

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Received: 3 April 2010

Accepted: 29 July 2010

Abstract

In this paper, we investigate the existence of a positive solution of fully fuzzy linear equation systems where fuzzy coefficient matrix is a positive matrix. This paper mainly discusses a new decomposition of a nonsingular fuzzy matrix, a symmetric matrix times to a triangular (ST) decomposition. By this decomposition, every nonsingular fuzzy matrix can be represented as a product of a fuzzy symmetric matrix S and a fuzzy triangular matrix T.

Keywords: Fuzzy Number, Fuzzy Linear System, Symmetric Positive Definite and Triangular Decomposition.

1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [31] and Dubois and Prade [13]. We refer the reader to [25] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces [10] to control chaotic systems [20,24], fuzzy metric spaces [28], fuzzy differential equations [3], fuzzy linear and nonlinear systems [1,2,5,9] and particle physics [15,16,17,18,19,27,29].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems and fully fuzzy linear systems and several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them.

Fridman et al. [21] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is a fuzzy vector of positive fuzzy numbers.

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They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $Ax = Bx + y$ where A and B are real $n \times n$ matrices, the unknown vector x is vector consisting of n fuzzy numbers and the constant vector y is consisting of n fuzzy numbers [22]. In [1,2,9] the authors presented conjugate gradient and LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. The numerical methods for fuzzy linear systems were proposed by Allahviranloo [6, 7, 8]. Also, Wang et al. [30] presented an iterative algorithm for solving dual linear system of the form $x = Ax + u$, where A is real $n \times n$ matrix, the unknown vector x and the constant vector u are all vectors consisting of fuzzy numbers and Abbasbandy et al. [4] investigated the existence of a minimal solution of general dual fuzzy linear equations system of the form $Ax + f = Bx + c$, where A and B are real $m \times n$ matrix, the unknown vector x is vector consisting of n fuzzy numbers and the constant vectors f and c are consisting of m fuzzy numbers. Recently, Muzziloi et al. [26] considered fully fuzzy linear systems of the form $A_1x + b_1 = A_2x + b_2$ with A_1, A_2 are square matrices of fuzzy entries and b_1 and b_2 fuzzy number vectors and Dehghan et al. [11] considered fully fuzzy linear systems of the form $Ax = b$ where A and b are a fuzzy matrix, the unknown vector x is vector consisting of n fuzzy numbers and the constant b are vectors consisting of n fuzzy numbers.

In this paper we intend to solve the fuzzy linear system $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$, where \tilde{A}_1 and \tilde{A}_2 are fuzzy $n \times n$ matrices consisting of positive fuzzy numbers, the unknown vector \tilde{x} is a vector consisting of n positive fuzzy numbers and the constant \tilde{b} are vectors consisting of n positive fuzzy numbers. This paper mainly discusses a new decomposition of a nonsingular fuzzy matrix, the symmetric times triangular (ST) decomposition. By this decomposition every nonsingular fuzzy matrix can be represented as a product of a fuzzy symmetric matrix S and a fuzzy triangular matrix T.

2 Fully Fuzzy Linear System

Definition 2.1 A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of \tilde{A} is a fuzzy number [14]. \tilde{A} will be positive (negative) and denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$) if each element of \tilde{A} be positive (negative). Similarly, non-negative and non-positive fuzzy matrices may be defined

Let the elements of \tilde{A} be an LR fuzzy numbers. We may represent $\tilde{A} = (\tilde{a}_{ij})$ that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ and thus $\tilde{A} = (A, M, N)$, where A, M and N are three crisp matrices with the same size of \tilde{A} , such that $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are called the center matrix and the right and left spread matrices, respectively where $M > 0$ and $N > 0$.

Definition 2.2 A square fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})$ is an upper (lower) triangular fuzzy matrix, if

$$\tilde{a}_{ij} = \tilde{0} = (0,0,0) \quad , \forall i > j (\forall i < j).$$

Definition 2.3 Let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be $m \times n$ and $n \times p$ fuzzy matrices respectively. We define $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ to be $m \times p$ matrix with

$$\tilde{c}_{ij} = \sum_{k=1}^n \tilde{a}_{ik} \otimes \tilde{b}_{kj}.$$

From here, we use Dubois and Prade's approximate multiplication \otimes .

Definition 2.4 Consider the $n \times n$ linear system of equation and let $\tilde{x} = (x, y, z)$ and $\tilde{b} = (b, g, h)$ be the unknown and known vectors respectively therefore we have

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{cases}$$

The matrix form of the above equation is

$$\tilde{A} \otimes \tilde{x} = \tilde{b}, \tag{1}$$

or simply $\tilde{A}\tilde{x} = \tilde{b}$ where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij}) = (A, M, N), 1 \leq i, j \leq n$ is an $n \times n$ positive fuzzy matrix and $\tilde{x}_i = (x_i, y_i, z_i), \tilde{b}_i = (b_i, h_i, g_i), 1 \leq i \leq n$ are positive fuzzy vectors. This system is called a fully fuzzy linear system (FFLS). Also if \tilde{A} and \tilde{b} are positive LR fuzzy numbers, we call the system (1) a positive FFLS. In many applied problems, engineers have some information about the range of fuzzy solution. In these cases with fixed y and z as the left and right spread. The original problem is transformed to finding a vector x which satisfies in the following systems:

$$\begin{cases} Ax = b, \\ Mx + Ay = g, \\ Nx + Az = h. \end{cases}$$

Definition 2.5 Consider the positive FFLS (1). \tilde{x} is a solution, if and only if

$$\begin{cases} Ax = b, \\ Mx + Ay = h, \\ Nx + Az = g. \end{cases}$$

In addition, if $y \geq 0, z \geq 0$ and $x - y \geq 0$ we say $\tilde{x} = (x, y, z)$ is a consistent solution of positive FFLS or for abbreviation consistent solution and if $y < 0$ or $z < 0$, the system has not fuzzy solution.

3 Gneral fully fuzzy linear system

Usually, there is no inverse with respect to addition element for an arbitrary fuzzy number $\tilde{u} \in E^1$, i.e., there exists no element $\tilde{v} \in E^1$ such that

$$\tilde{u} \oplus \tilde{v} = 0$$

Actually, for all non crisp fuzzy numbers $u \in E^1$ we have

$$\tilde{u} \oplus (-\tilde{u}) \neq 0.$$

Therefore, the fully fuzzy linear system of equations

$$\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$$

Can not be equivalently replaced by the fully fuzzy linear equation system

$$(\tilde{A}_1 - \tilde{A}_2) \otimes \tilde{x} = \tilde{b}$$

which had been investigated. In the sequal, we will call the fully fuzzy linear system, a general dual fully fuzzy linear system

$$\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$$

where $\tilde{A}_1 = (\tilde{a}_{1ij})$, $\tilde{A}_2 = (\tilde{a}_{2ij})$ for $1 \leq i, j \leq n$ are positive fuzzy matrices and \tilde{x} , \tilde{b} are positive fuzzy vectors.

For FFLS (1), we define $\tilde{A}_1 = (A_1, M_1, N_1)$, $\tilde{A}_2 = (A_2, M_2, N_2)$, $\tilde{b} = (b, h, g)$ and $M = M_1 - M_2 > 0$; $N = N_1 - N_2 > 0$ where $M_1, M_2, N_1, N_2, A_1, A_2, x$ are positive therefore we will solve $\tilde{A}\tilde{x} = \tilde{b}$ by using definition 5 we have

$$\begin{cases} A_1 \otimes x = A_2 \otimes x \oplus b, \\ M_1 \otimes x \oplus A_1 \otimes y = M_2 \otimes x \oplus A_2 \otimes y \oplus g, \\ N_1 \otimes x \oplus A_1 \otimes z = N_2 \otimes x \oplus A_2 \otimes z \oplus h. \end{cases}$$

Thus we easily have

$$\begin{cases} x = (A_1 - A_2)^{-1}b, \\ y = (A_1 - A_2)^{-1}(g - M(A_1 - A_2)^{-1}b), \\ z = (A_1 - A_2)^{-1}(h - N(A_1 - A_2)^{-1}b). \end{cases}$$

Theorem 1 Let $\tilde{A}_1, \tilde{A}_2 \geq 0$ and \tilde{b} is a non-negative arbitrary fuzzy vector. Let $A_1 - A_2$ be the product of a permutation matrix by a diagonal matrix with positive diagonal entries. Also, let $h \geq M(A_1 - A_2)^{-1}b$, $g \geq N(A_1 - A_2)^{-1}b$ and $((M(A_1 - A_2)^{-1} + I)b) \geq h$. Then the system $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$ has a non-negative fuzzy solution.

Proof. Our hypothesis on $A_1 - A_2$, imply that $(A_1 - A_2)^{-1}$ exists and is a non-negative matrix [12].

So, $x = (A_1 - A_2)^{-1}b \geq 0$. On the other hand, $h \geq M(A_1 - A_2)^{-1}b$ and $g \geq N(A_1 - A_2)^{-1}b$. Thus with $y = (A_1 - A_2)^{-1}h - (A_1 - A_2)^{-1}Mx$ and $z = (A_1 - A_2)^{-1}g - (A_1 - A_2)^{-1}Nx$, we

have $y \geq 0$ and $z \geq 0$.

So $\tilde{x} = (x, y, z)$ is a fuzzy vector which satisfies $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$. Since $x - y = (A_1 - A_2)^{-1} [b - h + M(A_1 - A_2)^{-1} b]$; the positivity property of \tilde{x} can be obtained from $(I + M(A_1 - A_2)^{-1}) b \geq h$.

4 ST method for solving FFLS

We shall present our main results on symmetric and triangular decomposition in this section .

Theorem 2 [23] For every nonsingular and nonsymmetric $n \times n$ matrix A, whose leading principal submatrices are nonsingular, there exists a decomposition $A = ST$ where S is symmetric and T is unit triangular.

Proof. We shall prove the triangular matrix T is unit upper triangular .For $n = 2$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We can obtain

$$S = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} - a_{21}t_{12} \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & t_{12} \\ 0 & 1 \end{pmatrix}$$

Such that $A = ST$, where $t_{12} = (a_{12} - a_{21})/a_{11}$. Here, S is symmetric and nonsingular form the nonsingularity of A.

Suppose that $A_n = S_n T_n$ holds for $n = k$. Now we like to show that it is still true for $n = k + 1$. For $n = k + 1$ we write

$$A = \begin{pmatrix} A_k & a_{k+1} \\ \hat{a}_{k+1}^T & \alpha_{k+1} \end{pmatrix}$$

and take

$$S = \begin{pmatrix} S_k & S_{k+1} \\ S_{k+1}^T & \beta_{k+1} \end{pmatrix}$$

$$T = \begin{pmatrix} T_k & t_{k+1} \\ 0 & 1 \end{pmatrix}$$

It follows from $A = ST$ that

$$\begin{cases} S_k T_k = A_k, \\ S_{k+1}^T T_k = \hat{a}_{k+1}^T, \\ S_k t_{k+1} + S_{k+1} = a_{k+1}, \\ \beta_{k+1} + S_{k+1}^T t_{k+1} = \alpha_{k+1}. \end{cases}$$

Since A_k is nonsingular, S_k and T_k are nonsingular by the induction assumption.

Hence, we get the unique solution from above equations as follows:

$$\begin{cases} S_{k+1} = T_k^{-T} \hat{a}_{k+1}, \\ t_{k+1} = S_k^{-1} (a_{k+1} - S_{k+1}), \\ \beta_{k+1} = \alpha_{k+1} - S_{k+1}^T t_{k+1}. \end{cases}$$

Therefore, $A = ST$ is well defined for $n = k + 1$. From the nonsingularity of A , it is easy to check that S is nonsingular. \square

Therefore we will solve $\tilde{A}\tilde{x} = \tilde{b}$, by using definition 2.5 we have

$$\begin{cases} A_1 \otimes x = A_2 \otimes x \oplus b, \\ M_1 \otimes x \oplus A_1 \otimes y = M_2 \otimes x \oplus A_2 \otimes y \oplus g, \\ N_1 \otimes x \oplus A_1 \otimes z = N_2 \otimes x \oplus A_2 \otimes z \oplus h. \end{cases}$$

By replacing $A = ST$ we get

$$\begin{cases} x = T^{-1}S^{-1}b, \\ y = T^{-1}S^{-1}(g - MT^{-1}S^{-1}b), \\ z = T^{-1}S^{-1}(h - NT^{-1}S^{-1}b). \end{cases}$$

Theorem 3 Let $\tilde{A}_1 = (A_1, M_1, N_1)$; $\tilde{A}_2 = (A_2, M_2, N_2)$ and $\tilde{b} = (b, h, g)$ be non-negative fuzzy matrices and non-negative fuzzy vector respectively. Let $A_1 - A_2$ be the product of a permutation matrix by a diagonal matrix with positive diagonal entries. Also there exist a decomposition $A = A_1 - A_2 = ST$ where S is symmetric and T is unit

triangular. Also, let $h \geq MT^{-1}S^{-1}b$, $g \geq NT^{-1}S^{-1}b$ and $(MT^{-1}S^{-1} + I)b \geq h$. Then the system $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$ has a positive fuzzy solution.

Proof. Our hypotheses on $A = ST$, imply that $A^{-1} = T^{-1}S^{-1}$, exists and is a nonnegative matrix [9]. So, $x = T^{-1}S^{-1}b \geq 0$. On the other hand, $h \geq MT^{-1}S^{-1}b$ and $g \geq NT^{-1}S^{-1}b$. Thus with $y = T^{-1}S^{-1}h - T^{-1}S^{-1}MT^{-1}S^{-1}b$ and $z = T^{-1}S^{-1}g - T^{-1}S^{-1}NT^{-1}S^{-1}b$, we have $y \geq 0$ and $z \geq 0$. So $\tilde{x} = (x, y, z)$ is a fuzzy vector which satisfies $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$. Since $x - y = T^{-1}S^{-1}(b - h + MT^{-1}S^{-1}b)$, the positivity property of \tilde{x} can be obtained from $(MT^{-1}S^{-1} + I)b \geq h$. \square

4 Numerical examples

Example 1 Consider the fully fuzzy linear system in the following form:

$$\begin{cases} (6,5,6) \otimes x_1 \oplus (9,2,8) \otimes x_2 = (2,2,1) \otimes x_1 \oplus (3,0,1) \otimes x_2 \oplus (24,23,26), \\ (6,4,9) \otimes x_1 \oplus (8,5,9) \otimes x_2 = (1,1,3) \otimes x_1 \oplus (3,3,2) \otimes x_2 \oplus (25,23,27). \end{cases}$$

Therefore we have

$$A_1 = \begin{pmatrix} 6 & 9 \\ 6 & 8 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}, M_1 = \begin{pmatrix} 5 & 2 \\ 4 & 5 \end{pmatrix}, M_2 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 6 & 8 \\ 9 & 9 \end{pmatrix}, N_2 = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, b = \begin{pmatrix} 24 \\ 25 \end{pmatrix}, g = \begin{pmatrix} 23 \\ 23 \end{pmatrix}, h = \begin{pmatrix} 26 \\ 27 \end{pmatrix}.$$

Therefore by applying ST decomposition of A we obtain :

$$S = \begin{pmatrix} 4 & 5 \\ 5 & 15/4 \end{pmatrix}, T = \begin{pmatrix} 1 & 1/4 \\ 0 & 1 \end{pmatrix}.$$

Furthermore we have:

$$S^{-1} = \begin{pmatrix} -3/8 & 1/2 \\ 1/2 & -2/5 \end{pmatrix}, T^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix}.$$

By using ST decomposition, we have:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Therefore the solution of fully fuzzy linear system is a fuzzy vector.

Example 2 Consider the fully fuzzy linear system in the follow form:

$$\begin{cases} (8,5,8) \otimes x_1 \oplus (7,6,9) \otimes x_2 = (2,1,8) \otimes x_1 \oplus (2,3,3) \otimes x_2 \oplus (43,27,51), \\ (5,4,6) \otimes x_1 \oplus (9,7,9) \otimes x_2 = (2,2,2) \otimes x_1 \oplus (2,4,1) \otimes x_2 \oplus (44,26,32). \end{cases}$$

Therefore we have

$$A_1 = \begin{pmatrix} 8 & 7 \\ 5 & 9 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, M_1 = \begin{pmatrix} 5 & 6 \\ 4 & 7 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 8 & 9 \\ 6 & 9 \end{pmatrix}, N_2 = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, b = \begin{pmatrix} 43 \\ 44 \end{pmatrix}, g = \begin{pmatrix} 27 \\ 26 \end{pmatrix}, h = \begin{pmatrix} 51 \\ 32 \end{pmatrix}.$$

Therefore with apply ST decomposition of A we obtain:

$$S = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}, T = \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix},$$

Furthermore we have:

$$S^{-1} = \begin{pmatrix} 2/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix}, T^{-1} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1 \end{pmatrix}.$$

By using ST decomposition, we have

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore the solution of the fully fuzzy linear system is a crisp vector.

5 Summary and conclusions

In this paper, we propose decomposition for the nonsymmetric or the symmetric indefinite of coefficient matrix [23] of fully fuzzy linear systems. We obtain a fuzzy solution for fully fuzzy linear system by decomposing coefficient matrix to the symmetric times triangular (ST) where S is the symmetric matrix and T is the triangular matrix.

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