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Approximation of Fuzzy Functions by Distance Method

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Abstract

Approximation of functions in a given space is an old problem in applied mathematics. In this paper the problem is considered for fuzzy data and fuzzy functions using the defuzzification function introduced by Fortemps and Roubens. We introduce a fuzzy polynomial approximation as D-approximation of a fuzzy function on a discrete set of points and we present a method to compute it.

Keywords: Fuzzy Approximation; Fuzzy Linear Programming, Ambiguity.

1 Introduction

Approximation of fuzzy functions on a finite set of distinct points, has been studied by several authors. In [6], the problem of fuzzy interpolation is found. Kaleva [5], presented some properties of fuzzy Lagrange and fuzzy spline interpolating functions, and properties of natural and complete splines of odd degree, are introduced in [1, 3].

Let $\mathcal{X}$ be a set of m distinct points $x_1, x_2, \ldots, x_m$; of $\mathbb{R}$. In this work we approximate a given fuzzy function $\widetilde{f}$ defined on $\mathcal{X}$.

We introduce a fuzzy valued polynomial and we consider the problem of approximating a given fuzzy function $\widetilde{f}$, on a discrete point set $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, by a fuzzy valued polynomial $\widetilde{P}_n$ of degree at most n, where the integers m and n, are given, (throughout this paper we consider $n < m$.)

The paper is organized as follows. Section 1, contains the basic concepts and introduces a fuzzy valued polynomial. We also use the ranking method of Fortemps and Roubens on the set of all fuzzy numbers in this section.

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In Section 3, the D-approximation of a fuzzy function on a finite set of distinct points, $\mathcal{X}$, is introduced. Also as we demonstrate in this section, an approach to the problem of finding a D-approximation of a given fuzzy function $\widetilde{f}$ on $\mathcal{X}$, is to express the problem as a fuzzy linear programming problem.But the questions are that: Does the $D-$approximation of a fuzzy function always exist? and is it unique? We will answer these questions in Section 4. At last, some examples are given in Section 5. Throughout this paper we use standard difference for fuzzy numbers.

2 Basic concepts

Let $F(\mathbb{R})$ be the set of all real fuzzy numbers (which are normal, upper semi-continuous, fuzzy convex and compactly supported).

In [8], *parametric form* of a fuzzy number has been introduced and presented by $\widetilde{v} = (\underline{v}(r), \overline{v}(r))$, where functions $\underline{v}(r)$ and $\overline{v}(r)$; $0 \leq r \leq 1$ satisfying the following properties:

1. $\underline{v}(r)$ is monotonically increasing left continuous function;
2. $\overline{v}(r)$ is monotonically decreasing left continuous function;
3. $\underline{v}(r) \leq \overline{v}(r) \quad , \quad 0 \leq r \leq 1$;
4. $\overline{v}(r) = \underline{v}(r) = 0$ for $r < 0$ or $r > 1$.

A crisp number α, can be simply represented by $\overline{v}(r) = \underline{v}(r) = \alpha$, $0 \leq r \leq 1$.

Let $\widetilde{v} = (\underline{v}(r), \overline{v}(r)), \widetilde{u} = (\underline{u}(r), \overline{u}(r)) \in F(\mathbb{R})$. Some results of applying fuzzy arithmetic on fuzzy numbers $\widetilde{v}$ and $\widetilde{u}$ are as follows:

- $x > 0 : \, x\widetilde{v} = (x\underline{v}(r), x\overline{v}(r))$;
- $x < 0 : \, x\widetilde{v} = (x\overline{v}(r), x\underline{v}(r))$;
- $\widetilde{v} + \widetilde{u} = (\underline{v}(r) + \underline{u}(r), \overline{v}(r) + \overline{u}(r))$;
- $\widetilde{v} - \widetilde{u} = (\underline{v}(r) - \overline{u}(r), \overline{v}(r) - \underline{u}(r))$.

For example parametric form of a triangular fuzzy number $\widetilde{v} = (v_s, v_l, v_r)$ is $\widetilde{v} = (v_l(r-1) + v_s, v_r(1-r) + v_s)$.

A fuzzy polynomial of degree at most n is a function $\widetilde{P}_n$ from $\mathbb{R}$ to $F(\mathbb{R})$ such that $\widetilde{P}_n(x) = \sum_{j=0}^{n} x^j \widetilde{a}_j$. Denote by $\widetilde{\prod}_n$ the set of all fuzzy polynomials $\widetilde{P}_n(x) = \sum_{j=0}^{n} x^j \widetilde{a}_j$ of degree at most n. A fuzzy polynomial of degree at most n can be put in the following parametric form:

$$\underline{v}(r) = \sum_{x^j \geq 0} x^j \underline{a_j}(r) + \sum_{x^j < 0} x^j \overline{a_j}(r),$$

$$\overline{v}(r) = \sum_{x^j \geq 0} x^j \overline{a_j}(r) + \sum_{x^j < 0} x^j \underline{a_j}(r),$$

where $0 \leq r \leq 1$.

Definition 2.1 *For an arbitrary fuzzy number $\widetilde{v} = (\underline{v}(r), \overline{v}(r))$, the quantity*

$$D(\widetilde{v}) = \frac{1}{2}\{\int_0^1 \overline{v}(r)dr + \int_0^1 \underline{v}(r)dr\}, \tag{1}$$

defines a defuzzification [4, 9].

It means that $\widetilde{v} \leq_D \widetilde{u}$ if and only if $D(\widetilde{v}) \leq D(\widetilde{u})$, and we use $\widetilde{v} \succeq 0$ if and only if $D(\widetilde{v}) \geq 0$, in this case we say $\widetilde{v}$ is nonnegative. It should be mentioned that $\geq_D$ compares two fuzzy numbers and $\succeq$ compares a fuzzy number with singleton zero.

Lemma 2.2 *For a trapezoidal fuzzy number $\widetilde{v}$, Definition 2.1 implies that*

$$D(\widetilde{v}) = \frac{1}{4}\{\underline{v}(0) + \overline{v}(0) + \overline{v}(1) + \underline{v}(1)\}. \tag{2}$$

Lemma 2.3 *If $\widetilde{v}, \widetilde{u}, \widetilde{w} \in F(\mathbb{R})$, then*

1. $D(-\widetilde{v}) = -D(\widetilde{v})$,
2. $D(\widetilde{u} + \widetilde{w}) = D(\widetilde{u}) + D(\widetilde{w})$,
3. $D(\alpha\widetilde{v}) = \alpha D(\widetilde{v})$,
4. $\widetilde{v} \leq_D \widetilde{u} + \widetilde{w} \quad \Longleftrightarrow \quad \widetilde{v} - \widetilde{w} \leq_D \widetilde{u}$,
5. $\widetilde{v} \leq_D \widetilde{u} \quad \Longleftrightarrow \quad \widetilde{v} + \widetilde{w} \leq_D \widetilde{u} + \widetilde{w}$,
6. $\widetilde{v} \leq_D \widetilde{u} \quad \Longleftrightarrow \quad -\widetilde{u} \leq_D -\widetilde{v}$.

Let $\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_m \in F(\mathbb{R})$, then $\widetilde{a}_k = \max_{i=1,2,\ldots,m} \widetilde{a}_i$ if and only if $D(\widetilde{a}_k) = \max_{i=1,2,\ldots,m} D(\widetilde{a}_i)$.

Definition 2.4 *If for a fuzzy number $\widetilde{a}$, there exist a monotonic decreasing left continuous function $\mu(r)$ on $[0,1]$ and three real numbers α_1, α_2 and b where $\widetilde{a} = (-\alpha_1\mu(r) + b, \alpha_2\mu(r) + b)$, we call $\mu(r)$, a* ***source*** *function.*

For all triangular or trapezoidal fuzzy numbers $\mu(r) = (1 - r)$.

Two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ have the same type if source functions of both fuzzy numbers are the same.

Definition 2.5 *The weak absolute value of a fuzzy number $\widetilde{v}$, is*

$$|\widetilde{v}| = \begin{cases} \widetilde{v}, & \widetilde{v} \succeq 0, \\ -\widetilde{v}, & \widetilde{v} \prec 0. \end{cases} \tag{3}$$

Remark 2.6 *By Definition 2.5 and Lemma 2.3, it is clear that if* $\widetilde{v}, \widetilde{u} \in F(\mathbb{R})$*, then*

1. $\widetilde{v}$ and $|\widetilde{v}|$ have the same type.
2. $D(|\widetilde{v}|) = |D(\widetilde{v})|$,
3. $\widetilde{v} \leq_D |\widetilde{v}|$,
4. $|\widetilde{v}| \leq_D \widetilde{u}\ (\widetilde{u} \succeq 0) \quad \Longleftrightarrow \quad -\widetilde{u} \leq_D \widetilde{v} \leq_D \widetilde{u}$,
5. $|\widetilde{v}| \geq_D \widetilde{u}\ (\widetilde{u} \succeq 0) \quad \Longleftrightarrow \quad \widetilde{v} \leq_D -\widetilde{u}$ or $\widetilde{v} \geq_D \widetilde{u}$.

Remark 2.7 *It is important to know that the definition of the absolute value of a fuzzy number based on extension principle does not satisfy all properties of Remark 2.6, for example it does not satisfy property 2.*

Definition 2.8 *A continuous function* $s : [0,1] \longrightarrow [0,1]$ *is regular reducing function [11] if* $s(r)$ *is increasing,* $s(0) = 0$*,* $s(1) = 1$ *and* $\int_0^1 s(r)dr = \frac{1}{2}$.

Definition 2.9 *The Ambiguity of a fuzzy number* $\widetilde{v}$ *is defined as follows, [11],*

$$Amb(\widetilde{v}) = \int_0^1 s(r)[\overline{v}(r) - \underline{v}(r)]dr.$$

where $s(r)$ *is a regular reducing function.*

Remark 2.10 *If* $\widetilde{u}, \widetilde{v} \in F(\mathbb{R})$ *and* $\alpha \in \mathbb{R}$*, then*

1. $Amb(\widetilde{u} + \widetilde{v}) = Amb(\widetilde{u}) + Amb(\widetilde{v})$,
2. $Amb(\widetilde{u} - \widetilde{v}) = Amb(\widetilde{u}) + Amb(\widetilde{v})$,
3. $Amb(\alpha\widetilde{v}) = |\alpha| Amb(\widetilde{v})$.

Definition 2.11 *We define ambiguity of a fuzzy function* $\widetilde{f}(x)$ *with respect to a set of points* $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ *by*

$$\boldsymbol{A}_{\mathcal{X}}(\widetilde{f}) = \max_{1 \leq i \leq m} Amb(\widetilde{f}(x_i)),$$

Definition 2.12 *Let* $\widetilde{S}$ *be a subset of* $F(\mathbb{R})$*. We say* $\widetilde{S}$ *is bounded from below if and only if there exist a* $\widetilde{u} \in F(\mathbb{R})$*, such that* $\widetilde{u} \leq_D \widetilde{s}$ *for all* $\widetilde{s} \in \widetilde{S}$.

3 $D-$Approximation of a Fuzzy Function

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ be a set of m distinct points of $\mathbb{R}$, and the values of a fuzzy function $\widetilde{f}$ are $\widetilde{f}(x_1), \widetilde{f}(x_2), \ldots, \widetilde{f}(x_m)$ at these points.

Definition 3.1 *For every polynomial $\widetilde{P}_n(x) \in \widetilde{\prod}_n$, let $\widetilde{\beta}(\widetilde{P}_n) = (\underline{\beta}(r), \overline{\beta}(r))$ be a nonnegative fuzzy number which has the same type as $\widetilde{f}(x_i)$'s and*

$$\widetilde{\beta}(\widetilde{P}_n) = \max_{i=1,2,\ldots,m} |\widetilde{P}_n(x_i) - \widetilde{f}(x_i)|. \tag{4}$$

Definition 3.2 *The polynomial $\widetilde{P}^*_n(x) \in \widetilde{\prod}_n$ is a $D-$approximation of $\widetilde{f}$ at $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, if for every $\widetilde{P}_n(x) \in \widetilde{\prod}_n$ we have $\widetilde{\beta}(\widetilde{P}^*_n) \leq_D \widetilde{\beta}(\widetilde{P}_n)$. i.e.*

$$\widetilde{\beta}(\widetilde{P}^*_n) = \min_{\widetilde{P}_n \in \widetilde{\prod}_n} \widetilde{\beta}(\widetilde{P}_n). \tag{5}$$

It is obvious that $\widetilde{P}^*_n(x)$ exists when

$$\max_{i=1,2,\ldots,m} |\widetilde{P}^*_n(x_i) - \widetilde{f}(x_i)| = \min_{\widetilde{P}_n \in \widetilde{\prod}_n} \{\max_{i=1,2,\ldots,m} |\widetilde{P}_n(x_i) - \widetilde{f}(x_i)|\}. \tag{6}$$

Let's write $\widetilde{\beta} = \widetilde{\beta}(\widetilde{P}_n)$. Hence one must minimize $\widetilde{\beta}$.

By a simple computation, it can be shown that we should solve the following fuzzy linear programming problem

$$\begin{cases} \min \widetilde{\beta} \\ s.t. \\ \widetilde{\beta} + \sum_{j=0}^{n} x_i^j \widetilde{a}_j \geq_D \widetilde{f}(x_i), & i = 1, 2, \ldots, m, \\ \widetilde{\beta} - \sum_{j=0}^{n} x_i^j \widetilde{a}_j \geq_D -\widetilde{f}(x_i), & i = 1, 2, \ldots, m. \end{cases} \tag{7}$$

Defining

$$A = \begin{pmatrix} 1 & 1 & x_1 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & x_m & \ldots & x_m^n \\ 1 & -1 & -x_1 & \ldots & -x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & -1 & -x_m & \ldots & -x_m^n \end{pmatrix}_{2m\times(n+2)},$$

and taking

$$\begin{cases} \widetilde{X}^T = (\widetilde{\beta}, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_n), \\ \widetilde{C}^T = (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_m, -\widetilde{f}_1, -\widetilde{f}_2, \ldots, -\widetilde{f}_m), \\ b^T = (1, 0, 0, \ldots, 0), \end{cases}$$

our problem changes to the following form:

$$(FLP)\begin{cases} \min & \widetilde{z} = b^T\widetilde{X}, \\ s.t. & \\ & A\widetilde{X} \geq_D \widetilde{C}. \end{cases} \tag{8}$$

A method for solving such a linear programming by fuzzy variables is introduced in [7, 10]. Ranking function D, has the same properties of ranking function used in [7, 10]. According to this method we use the following *auxiliary* linear programming

$$(AFLP)\begin{cases} \max & \widetilde{w} = \widetilde{C}^T Y, \\ s.t. & \\ & A^T Y = b, \\ & Y \geq 0, \end{cases} \tag{9}$$

Lemma 3.3 *If $\widetilde{X}$ is any feasible solution for (FLP) and Y is any feasible solution for $(AFLP)$, then $\widetilde{C}^T Y \leq_D b^T\widetilde{X}$.*

Lemma 3.4 *If $\widetilde{X}_0$ is a feasible solution for (FLP) and Y_0 is a feasible solution for $(AFLP)$, such that $\widetilde{C}^T Y_0 = b^T\widetilde{X_0}$, then $\widetilde{X}_0$ is an optimal solution of (FLP) and Y_0 is an optimal solution of $(AFLP)$*

Theorem 3.5 *If the auxiliary problem $(AFLP)$ has an optimal solution, then problem (FLP) has a fuzzy optimal solution.*

To solve $(AFLP)$, we must solve the following (LP)

$$(LP)\begin{cases} \max & w = C^T Y, \\ s.t. & \\ & A^T Y = b, \\ & Y \geq 0, \end{cases} \tag{10}$$

where $C^T = (c_1, c_2, \ldots, c_{2m})$ and $c_j = D(\widetilde{c}_j)$, $j = 1, 2, \ldots, 2m$.

By solving (LP), we get an optimal solution with basis B from matrix A^T. Variables and objective coefficients are named according to basis, Y_B^* and C_B, respectively. Thus $Y_B^* = B^{-1}b$ and optimal value of (LP) is $w^* = C_B^T Y_B^*$. Therefore the optimal solution of $(AFLP)$ is Y_B^* too, but the optimal value of $(AFLP)$ is $\widetilde{w}^* = \widetilde{C}_B^T Y_B^*$. It is a fuzzy number because components of $\widetilde{C}_B$ are fuzzy numbers. Consequently optimal solution of (FLP) is $\widetilde{X}^{*^T} = \widetilde{C}_B^T B^{-1}$, and optimal value of (FLP), is $\widetilde{\beta}^* = \widetilde{C}_B^T B^{-1} b$.

4 Existence and Uniqueness of $D-$Approximation of a Fuzzy Function

Since we want to compute $n+2$ variables($\widetilde{a}_j$'s for $j = 0, \ldots, n$, and $\widetilde{\beta}$), rank of matrix A should be $n+2$. If $n < m$, then $rank(A) = n+2$, see [2]. Thus we consider $n < m$.

Theorem 4.1 *$D-$approximation of a fuzzy function exists.*

Proof. Let $\widetilde{f}$ be an arbitrary fuzzy function whose values on points $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ are $\widetilde{f}(x_1), \widetilde{f}(x_2), \ldots, \widetilde{f}(x_m)$. It is clear that $\widetilde{X}_0^T = (\widetilde{\beta}_0, 0, 0, \ldots, 0)$ is a feasible solution of (FLP) such that

$$\widetilde{\beta}_0 = \sum_{\widetilde{f}(x_i) \succeq 0} \widetilde{f}(x_i) - \sum_{\widetilde{f}(x_i) \prec 0} \widetilde{f}(x_i),$$

because

$$A\widetilde{X} = \widetilde{\beta}_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \geq_D \begin{pmatrix} \widetilde{f}(x_1) \\ \widetilde{f}(x_2) \\ \vdots \\ \widetilde{f}(x_m) \end{pmatrix} = \widetilde{C}.$$

Now, $\widetilde{z} \succeq 0$ (by definition of $\widetilde{\beta}$) and the set of all $\widetilde{z}$ for feasible points, is bounded from below, therefore (FLP) has a solution and it means that $D-$approximation of a fuzzy function exists.

However by an example we show that $D-$approximation of a fuzzy function, is not unique. Let $\mathcal{X} = \{1, 2, 3\}$ and the values of a fuzzy function $\widetilde{f}$, on $\mathcal{X}$ are $\{(r, -r + 2), (r + 1, -r + 3), (r + 2, -r + 4)\}$ respectively. Then

$$\widetilde{P}_2(x) = x^2(3r - 3, -3r + 3) + x(11r - 10, -11r + 12) + (8r - 8, -8r + 8),$$

is a $D-$approximation of $\widetilde{f}$ on $\mathcal{X}$, and $\widetilde{\beta} = (r - 1, -r + 1)$. But it can be shown that $\widetilde{P}_2'(x) = x^2(2r - 2, -2r + 2) + x(8r - 7, -8r + 9) + (7r - 7, -7r + 7)$ is also a $D-$approximation of $\widetilde{f}$ on $\mathcal{X}$, and $\widetilde{\beta}' = (r - 1, -r + 1)$. However if we can find all of the optimal solutions of (8), after solving equation (12), which is presented at the end of this section, we can choose a polynomial with small fuzziness.

Definition 4.2 *$\widetilde{a}$ is a symmetric L-R fuzzy number(**SLR**) if there exists a source function $\mu(r)$ and two real numbers α and b where $\widetilde{a} = (-\alpha\mu(r) + b, \alpha\mu(r) + b)$.*

Definition 4.3 *A fuzzy polynomial $\widetilde{P}(x) = \sum_{j=0}^{n} x^j \widetilde{a}_j$ with symmetric coefficients is **SC** polynomial, if all coefficients $\widetilde{a}_j$'s are **SLR** with the same source function $\mu(r)$.*

Theorem 4.4 *Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ and $\widetilde{f} \in \widetilde{\prod}_n$, be a **SC** polynomial with source function $\mu(r)$, then the D-approximation to $\widetilde{f}$ on $\mathcal{X}$, is a **SC** polynomial $\widetilde{P}_n^*$ with the same source function $\mu(r)$, such that, for all $x \in \mathbb{R}$,*

$$[\widetilde{P}_n^*(x)]^1 = [\widetilde{f}(x)]^1.$$

Proof : Taking

$$\widetilde{f}(x)=\sum_{j=0}^{n}x^{j}\widetilde{a}_{j}=\sum_{j=0}^{n}x^{j}(-a_{j}\mu(r)+a_{j_{s}},a_{j}\mu(r)+a_{j_{s}}).$$

we have

$$(x_{i},\widetilde{f}(x_{i}))=(x_{i},(\underline{v}_{i}(r),\overline{v}_{i}(r))),$$

such that

$$\underline{v}_{i}(r)=-(\sum_{j=0}^{n}|x_{i}^{j}|a_{j})\mu(r)+(\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}}),$$

$$\overline{v}_{i}(r)=(\sum_{j=0}^{n}|x_{i}^{j}|a_{j})\mu(r)+(\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}}),$$

where $0\leq r\leq 1$. Now we proof this theorem in two steps. For the first step, suppose $a_{j}=0$, $j=0,1,\ldots,n$, then $\widetilde{f}(x)=P_{n}(x)\in\prod_{n}$is a crisp polynomial and it has been proved that $P_{n}^{*}(x)=P_{n}(x)$. On the other hand

$$(x_{i},\widetilde{f}(x_{i}))=(x_{i},(\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}},\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}})),\quad i=1,2,\ldots,m,$$

and $D(\widetilde{f}(x_{i}))=\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}}$, for $i=1,2,\ldots,m$, and it is a common crisp problem. Suppose B as an optimal basis (The basis corresponding to an optimal solution) for problem, then answer of this problem is $(\widetilde{C}_{B}^{T}B^{-1})^{T}=((C_{B}^{T}B^{-1})^{T},(C_{B}^{T}B^{-1})^{T})$, where for $j=0,1,\ldots,n+1$,

$$(\widetilde{C}_{B}^{T}B^{-1})_{j}^{T}=((C_{B}^{T}B^{-1})_{j}^{T},(C_{B}^{T}B^{-1})_{j}^{T}),$$

is a ***SLR***. Therefore $\widetilde{P}_{n}^{*}(x)=\widetilde{f}(x)$.

For the second step, suppose a_{j} as a nonnegative number for $j=0,1,\ldots,n$. Then

$$(x_{i},\widetilde{f}(x_{i}))=(x_{i},(-\theta_{i}\mu(r)+\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}},\theta_{i}\mu(r)+\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}})),$$

which $\theta_{i}=\sum_{j=0}^{n}|x_{i}^{j}|a_{j}$. Thus $D(\widetilde{f}(x_{i}))=\sum_{j=0}^{n}x_{i}^{j}a_{j_{s}}$ and the problem is similar to the first case and for $j=0,1,\ldots,n+1$,

$$(\widetilde{C}_{B}^{T}B^{-1})_{j}^{T}=(-\lambda_{j1}\mu(r)+(C_{B}^{T}B^{-1})_{j}^{T},\lambda_{j2}\mu(r)+(C_{B}^{T}B^{-1})_{j}^{T}).$$

That is

$$(\widetilde{C}_{B}^{T}B^{-1})^{T}=(-\Lambda_{1}\mu(r)+(C_{B}^{T}B^{-1})^{T},\Lambda_{2}\mu(r)+(C_{B}^{T}B^{-1})^{T}),$$

where $\Lambda_k^T = (\lambda_{0k}, \lambda_{1k}, \ldots, \lambda_{n+1,k})$, $k = 1, 2$. Also it can be shown that $\Lambda_1 = \Lambda_2$. Thus

$$\widetilde{P}_n^*(x) = \sum_{j=0}^{n} x^j \widetilde{a}_j' = \sum_{j=0}^{n} x^j (-b_j \mu(r) + a_{j_s}, b_j \mu(r) + a_{j_s}),$$

and $[\widetilde{P}_n^*(x)]^1 = [\widetilde{f}(x)]^1$.

Let $\mathcal{G}$ be the set of all $D-$approximations of $\widetilde{f}$. Since we have used standard difference, fuzziness of our approximation $\widetilde{P}_n(x)$ is not less than fuzziness of function $\widetilde{f}(x)$. Thus we find a fuzzy polynomial $\widetilde{P}_n^S$ among all $D-$approximations of $\widetilde{f}$ named *strong $D-$approximation* of $\widetilde{f}$, such that it satisfies the following equation.

$$\max_{1 \le i \le m} Amb(\widetilde{f}(x_i) - \widetilde{P}_n^S(x_i)) = \min_{\widetilde{P}_n \in \mathcal{G}} \max_{1 \le i \le m} Amb(\widetilde{f}(x_i) - \widetilde{P}_n(x_i)). \tag{11}$$

If $\widetilde{P}_n(x)$ is a $D-$ approximation of $\widetilde{f}$ on $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, then

$$\max_{1 \le i \le m} Amb(\widetilde{f}(x_i) - \widetilde{P}_n(x_i)) = Amb(\widetilde{f}(x_i)) + \max_{1 \le i \le m} Amb(\widetilde{P}_n(x_i))$$

$$= Amb(\widetilde{f}(x_i)) + \mathbf{A}_{\mathcal{X}}(\widetilde{P}_n).$$

Thus $\widetilde{P}_n^S$ is a strong $D-$approximation of $\widetilde{f}$, if it satisfies the following equation:

$$\mathbf{A}_{\mathcal{X}}(\widetilde{P}_n^S) = \min_{\widetilde{P}_n \in \mathcal{G}} \mathbf{A}_{\mathcal{X}}(\widetilde{P}_n). \tag{12}$$

Remark 4.5 *Unfortunately $D-$approximation of a fuzzy function is not unique, but all of $D-$approximations of a fuzzy function have a same $\widetilde{\beta}$-distance, like points on circumference of a circle which are equidistant from center, and using (12), we can select one of them.*

5 Numerical Examples

Example 5.1 *Let $m = 3$ and $n = 1$,*

x	1	2	3
$\widetilde{f}(x)$	$(r, 2-r)$	$(r+1, 3-r)$	$(r+2, 4-r)$

$$\begin{array}{lcl}
\widetilde{P}_1(x) = x(r, 2-r) + (3r-3, 3-3r) & , & \widetilde{\beta} = (r-1, -r+1), \\
\widetilde{P}_1'(x) = x(2r-1, 3-2r) + (6r-6, 6-6r) & , & \widetilde{\beta} = (r-1, -r+1), \\
\widetilde{P}_1''(x) = x(\frac{2}{5}r + \frac{3}{5}, \frac{7}{5} - \frac{2}{5}r) + (\frac{6}{5}r - \frac{6}{5}, \frac{6}{5} - \frac{6}{5}r) & , & \widetilde{\beta} = (r-1, -r+1), \\
\widetilde{P}_1'''(x) = x(\frac{1}{2}r + \frac{1}{2}, \frac{3}{2} - \frac{1}{2}r) + (\frac{3}{2}r - \frac{3}{2}, \frac{3}{2} - \frac{3}{2}r) & , & \widetilde{\beta} = (r-1, -r+1),
\end{array}$$

$\mathbf{A}_{\mathcal{X}}(\widetilde{P}_1) = 2$, $\mathbf{A}_{\mathcal{X}}(\widetilde{P}_1') = 4$, $\mathbf{A}_{\mathcal{X}}(\widetilde{P}_1'') = 0.8$ and $\mathbf{A}_{\mathcal{X}}(\widetilde{P}_1''') = 1$. According to (12), $\widetilde{P}_1''(x)$ is strong $D-$approximation of $\widetilde{f}$.

Example 5.2 *Let* $m = 6$ *and* $n = 2$*,*

x	$\widetilde{f}(x)$
-1	(15+r,17-r)
-0.9	(11.96+r,13.96-r)
-0.8	(9.24+r,11.24-r)
0.8	(9.24+r,11.24-r)
0.9	(11.96+r,13.96-r)
1	(15+r,17-r)

$$\widetilde{P}_2(x) = x^2(16 - \frac{25}{9}(1-r), 16 + \frac{25}{9}(1-r)) + x(r-1, -r+1) + (-\frac{5}{2}(1-r), \frac{5}{2}(1-r)),$$

$$\widetilde{\beta} = (r-1, -r+1).$$

The $D-$approximation function of the last example is compared with linear spline and Lagrange interpolations on the given points, in one and zero levels. See Figures 1-3.

The maximum error of D-approximation function is $er_D(1) = (7.28r - 7.78, 7.28 - 7.28r)$. Also maximum errors of linear spline interpolation and Lagrange interpolation are $er_S(0) = (10.24r - 8.24, 12.24 - 10.24r)$ and $er_L(0) = (41.63r - 41.63, 41.63r - 41.63)$ respectively, where $er_D(x)$, $er_S(x)$ and $er_L(x)$ are error functions of D-approximation, Spline interpolation and Lagrange interpolation respectively.

6 Conclusions

In this work we proposed a method to find an approximation of a fuzzy function on a set of points. Unfortunately the $D-$approximation of a fuzzy function is not unique, but all of them have the same $\widetilde{\beta}$-distance, like points on circumference of a circle which are equidistant from center but we can consider only one of them with smallest ambiguity.

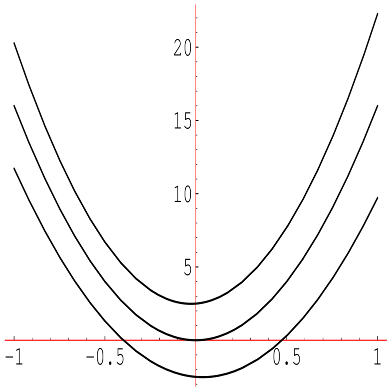

Figure 1 : $D-$approximation function

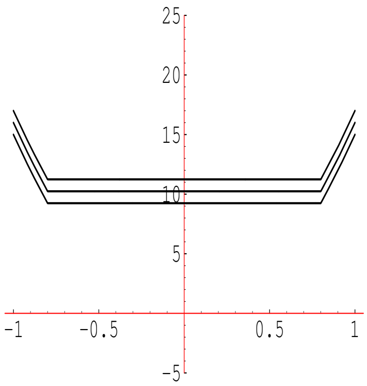

Figure 2 : Linear spline interpolation

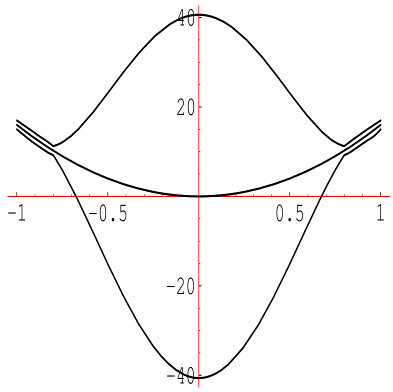

Figure 3 : Lagrange interpolation

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