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## Abstract

In this paper a method for solving a general fuzzy linear system with crisp solution is considered. We consider the method in special case when the elements of the coefficient matrix and the right hand side are trapezoidal fuzzy numbers. The method in detail is discussed and followed by theorem and illustrated by solving some examples.

*Keywords:* Fuzzy system of linear equations, Fuzzy number, Trapezoidal fuzzy number .

## 1 Introduction

System of linear equations play major role in science such as mathematics, physics, statistics, engineering and so on. The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by Zadeh [10, 14, 15], Mizumoto and Tanaka [12], Dubois and Prade [8] and Nahmias [13]. One of the major applications using fuzzy number arithmetic is using linear systems whose all or some of the parameters are represented by fuzzy numbers [1, 2, 3, 4, 5, 6, 7, 8, 9, 11].

In this paper, we use a new method for solving an  $n \times n$  fuzzy system of linear equations, whose coefficients matrix and the right hand side column are trapezoidal fuzzy numbers. In Section 2, we present some basic definitions and results on FSLE. In Section 3, we propose a method for solving fuzzy linear systems. Numerical examples are given in Section 4, and concluding remarks in Section 5.

## 2 Preliminaries

**Definition 1.** A fuzzy number is a fuzzy set like  $u : \mathbb{R} \rightarrow I = [0, 1]$  which satisfies,

1.  $u$  is upper semi-continuous,

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2.  $u(x) = 0$  outside some interval  $[c, d]$ ,
3. There are real numbers  $a, b$  such that  $c \leq a \leq b \leq d$  and
  - 3.1  $u(x)$  is monotonic increasing on  $[c, a]$ ,
  - 3.2  $u(x)$  is monotonic decreasing on  $[b, d]$ ,
  - 3.3  $u(x) = 1, a \leq x \leq b$ .

The set of all these fuzzy numbers is denoted by  $E$ . An equivalent parametric form is also given in [11] as follows.

**Definition 2.** A fuzzy number  $u$  in parametric form is a pair  $(\underline{u}, \bar{u})$  of functions  $\underline{u}(r), \bar{u}(r), 0 \leq r \leq 1$ , which satisfy the following requirements:

1.  $\underline{u}(r)$  is a bounded monotonic increasing left continuous function,
2.  $\bar{u}(r)$  is a bounded monotonic decreasing left continuous function,
3.  $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$ .

A popular fuzzy number is the trapezoidal fuzzy number  $u = (x_0, y_0, \alpha, \beta)$  with interval defuzzifier  $[x_0, y_0]$  and left fuzziness  $\alpha$  and right fuzziness  $\beta$  where the membership function is

$$u(x) = \begin{cases} \frac{1}{\alpha}(x - x_0 + \alpha) & x_0 - \alpha \leq x \leq x_0, \\ 1 & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \leq x \leq y_0 + \beta, \\ 0 & \text{Otherwise.} \end{cases}$$

Its parametric form is

$$\underline{u}(r) = x_0 - \alpha + \alpha r, \quad \bar{u}(r) = y_0 + \beta - \beta r.$$

Let  $TF(\mathbb{R})$  be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows [9].

For arbitrary  $u = (\underline{u}, \bar{u}), v = (\underline{v}, \bar{v})$ , we define addition  $(u + v)$  and multiplication by scalar  $k > 0$  as

$$u + v = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)), \quad (1)$$

$$ku = \begin{cases} (k\underline{u}, k\bar{u}) & k \geq 0, \\ (k\bar{u}, k\underline{u}) & k < 0. \end{cases} \quad (2)$$

Also,  $u = v$  if and only if  $\underline{u} = \underline{v}$  and  $\bar{u} = \bar{v}$ .

### 3 Fuzzy linear systems

In this section we consider the fuzzy linear systems and propose a method for solving these problems.

**Definition 3.** The  $n \times n$  linear system of equations

$$\begin{cases} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 + \dots + \tilde{a}_{1n}x_n = \tilde{b}_1, \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 + \dots + \tilde{a}_{2n}x_n = \tilde{b}_2, \\ \vdots \\ \tilde{a}_{n1}x_1 + \tilde{a}_{n2}x_2 + \dots + \tilde{a}_{nn}x_n = \tilde{b}_n, \end{cases} \quad (3)$$

where the elements of the coefficients matrix  $\tilde{A} = (\tilde{a}_{ij})$ ,  $1 \leq i, j \leq n$ , are trapezoidal fuzzy numbers and  $\tilde{b}_i \in E$ ,  $1 \leq i \leq n$ . The system (3) is called a fuzzy system of linear equations (FSLE).

**Definition 4.** A vector  $X = (x_1, x_2, \dots, x_n)^t$  is called a solution of (3) if and only if

$$\sum_{j=1}^n (\underline{a}_{ij}, \bar{a}_{ij})x_j = (\underline{b}_i, \bar{b}_i), \quad 1 \leq i \leq n,$$

where  $\tilde{a}_{ij} = (\underline{a}_{ij}, \bar{a}_{ij})$  and  $\tilde{b}_i = (\underline{b}_i, \bar{b}_i)$  for  $i, j = 1, 2, \dots, n$ .

It is clear that, for  $1 \leq i \leq n$

$$\sum_{j=1}^n (\underline{a}_{ij}, \bar{a}_{ij})x_j = \begin{cases} (\sum_{j=1}^n \underline{a}_{ij}x_j, \sum_{j=1}^n \bar{a}_{ij}x_j) & \text{if } x_j \geq 0, \\ (\sum_{j=1}^n \bar{a}_{ij}x_j, \sum_{j=1}^n \underline{a}_{ij}x_j) & \text{Otherwise.} \end{cases}$$

Suppose  $\tilde{a}_{ij} = (z_{ij}, w_{ij}, \alpha_{ij}, \beta_{ij})$ ,  $\tilde{b}_i = (h_i, k_i, \gamma_i, \delta_i)$ . In this work, we assume that each arbitrary square matrix with elements in  $[z_{ij} - \alpha_{ij}, w_{ij} + \beta_{ij}]$  is invertible. Otherwise the fuzzy linear system (3) has not any solution.

**Example 1.** Consider the  $2 \times 2$  fuzzy linear system

$$\begin{cases} (2, 2, 1, 2)x_1 + (2, 2, 1, 0)x_2 = (-2, -2, 2, 5), \\ (1, 1, 1, 0)x_1 + (1, 1, 1, 2)x_2 = (10, 10, 4, 1). \end{cases}$$

The parametric form of this system is as follows:

$$\begin{cases} (1 + r, 4 - 2r)x_1 + (1 + r, 2)x_2 = (-4 + 2r, 3 - 5r), \\ (r, 1)x_1 + (r, 3 - 2r)x_2 = (6 + 4r, 11 - r). \end{cases}$$

It is clear that  $A^{-1}$  does not exist. Now for  $r = 1$  we have

$$\begin{cases} 2x_1 + 2x_2 = -2, \\ x_1 + x_2 = 10, \end{cases}$$

and by refer to Definition 4, this system has no solution.

**Theorem 1.** The fuzzy linear system (3) has a solution  $X = (x_1, x_2, \dots, x_n)^t$  if and only if the following linear systems

$$\begin{cases} (z_{11} + w_{11})x_1 + (z_{12} + w_{12})x_2 + \dots + (z_{1n} + w_{1n})x_n = (h_1 + k_1), \\ (z_{21} + w_{21})x_1 + (z_{22} + w_{22})x_2 + \dots + (z_{2n} + w_{2n})x_n = (h_2 + k_2), \\ \vdots \\ (z_{n1} + w_{n1})x_1 + (z_{n2} + w_{n2})x_2 + \dots + (z_{nn} + w_{nn})x_n = (h_n + k_n), \end{cases} \quad (4)$$

and

$$\begin{cases} (z_{11} - \alpha_{11} + w_{11} + \beta_{11})x_1 + (z_{12} - \alpha_{12} + w_{12} + \beta_{12})x_2 + \dots + \\ \quad (z_{1n} - \alpha_{1n} + w_{1n} + \beta_{1n})x_n = (h_1 - \gamma_1 + k_1 + \delta_1), \\ (z_{21} - \alpha_{21} + w_{21} + \beta_{21})x_1 + (z_{22} - \alpha_{22} + w_{22} + \beta_{22})x_2 + \dots + \\ \quad (z_{2n} - \alpha_{2n} + w_{2n} + \beta_{2n})x_n = (h_2 - \gamma_2 + k_2 + \delta_2), \\ \vdots \\ (z_{n1} - \alpha_{n1} + w_{n1} + \beta_{n1})x_1 + (z_{n2} - \alpha_{n2} + w_{n2} + \beta_{n2})x_2 + \dots + \\ \quad (z_{nn} - \alpha_{nn} + w_{nn} + \beta_{nn})x_n = (h_n - \gamma_n + k_n + \delta_n), \end{cases} \quad (5)$$

have unique solution  $X = (x_1, x_2, \dots, x_n)^t$  and in this case the fuzzy linear system (3) has unique solution.

**Proof.** Let  $x_j = x'_j - x''_j$ ,  $1 \leq j \leq n$ , be the unique solution of systems (4) and (5), where  $x'_j, x''_j > 0$ . Hence  $\forall r \in [0, 1]$  and  $1 \leq i \leq n$ ,

$$\begin{aligned} \sum_{j=1}^n (z_{ij} + w_{ij})(x'_j - x''_j) &= (h_i + k_i), \\ \sum_{j=1}^n (z_{ij} + w_{ij} - (\alpha_{ij} - \beta_{ij}))(x'_j - x''_j) &= (h_i + k_i - (\gamma_i - \delta_i)), \end{aligned} \quad (6)$$

therefore

$$\sum_{j=1}^n (\alpha_{ij} - \beta_{ij})(x'_j - x''_j) = (\gamma_i - \delta_i). \quad (7)$$

So by multiplying (7) by  $r$  and adding by (6) we have

$$\sum_{j=1}^n (z_{ij} + w_{ij} - (\alpha_{ij} - \beta_{ij}) + (\alpha_{ij} - \beta_{ij})r)(x'_j - x''_j) = (k_i + h_i - (\gamma_i - \delta_i) + (\gamma_i - \delta_i)r). \quad (8)$$

Now for solving (3), we replace the parametric form of  $\tilde{a}_{ij}$  and  $\tilde{b}_i$  from Section 2. If  $x = x' - x''$ ,  $x'$  and  $x'' \geq 0$  is the solution of system (3), then by noticing to Section 2, for  $i = 1, 2, \dots, n$ , we have

$$\begin{aligned} \sum_{j=1}^n (z_{ij} - \alpha_{ij} + \alpha_{ij}r, w_{ij} + \beta_{ij} - \beta_{ij}r)(x'_j - x''_j) &= (h_i - \gamma_i + \gamma_i r, k_i + \delta_i - \delta_i r), \\ \sum_{j=1}^n (z_{ij} - \alpha_{ij} + \alpha_{ij}r, w_{ij} + \beta_{ij} - \beta_{ij}r)x'_j - \sum_{j=1}^n (z_{ij} - \alpha_{ij} + \alpha_{ij}r, w_{ij} + \beta_{ij} - \beta_{ij}r)x''_j &= \\ &= (h_i - \gamma_i + \gamma_i r, k_i + \delta_i - \delta_i r). \end{aligned}$$

Hence we have the following two relations

$$\sum_{j=1}^n (z_{ij} - \alpha_{ij} + \alpha_{ij}r)x'_j - \sum_{j=1}^n (w_{ij} + \beta_{ij} - \beta_{ij}r)x''_j = (h_i - \gamma_i + \gamma_i r),$$

$$\sum_{j=1}^n (w_{ij} + \beta_{ij} - \beta_{ij}r)x'_j - \sum_{j=1}^n (z_{ij} - \alpha_{ij} + \alpha_{ij}r)x''_j = (k_i + \delta_i - \delta_i r),$$

and hence  $\forall r \in [0, 1]$

$$\sum_{j=1}^n (z_{ij} + w_{ij} - (\alpha_{ij} - \beta_{ij}) + (\alpha_{ij} - \beta_{ij})r)(x'_j - x''_j) = (k_i + h_i - (\gamma_i - \delta_i) + (\gamma_i - \delta_i)r). \quad (9)$$

The relations (8) and (9) are equivalent, also equivalent with (5) and (4) for  $r = 0, 1$ , respectively, which complete the proof. For uniqueness, it is clear that, if vectors  $X = (x_1, x_2, \dots, x_n)^t$  and  $Y = (y_1, y_2, \dots, y_n)^t$  be the solutions of (3), then  $X$  and  $Y$  are the solutions of (4) and (5) and hence  $X = Y$ .

## 4 Numerical Examples

**Example 2.** Consider the  $3 \times 3$  fuzzy linear system

$$\begin{cases} (1, 2, 2, 1)x_1 + (2, 4, 1, 3)x_2 = (5, 10, 4, 7), \\ (-1, 1, 1, 2)x_1 + (2, 3, 1, 2)x_3 = (1, 4, 2, 4), \\ (1, 2, 2, 4)x_2 + (2, 2, 2, 1)x_3 = (4, 6, 6, 9). \end{cases}$$

Therefore we must solve the following systems

$$\begin{cases} 2x_1 + 8x_2 = 18, \\ x_1 + 6x_3 = 7, \\ 5x_2 + 3x_3 = 13, \end{cases} \quad \begin{cases} 3x_1 + 6x_2 = 15, \\ 5x_3 = 5, \\ 3x_2 + 4x_3 = 10. \end{cases}$$

Consequently the solution is  $X = (1, 2, 1)^t$ .

**Example 3.** Consider the  $4 \times 4$  fuzzy linear system

$$\begin{cases} (2, 2, 3, 0)x_1 + (1, 1, 1, 2)x_2 + (2, 2, 2, 3)x_4 = (5, 5, 10, 4), \\ (3, 3, 1, 3)x_1 + (2, 2, 2, 1)x_3 + (1, 1, 1, 2)x_4 = (9, 9, 5, 9), \\ (1, 1, 3, 2)x_2 + (1, 1, 1, 2)x_3 + (1, 1, 2, 1)x_4 = (1, 1, 5, 6), \\ (2, 2, 1, 2)x_1 + (4, 4, 1, 0)x_2 + (1, 1, 1, 1)x_3 = (1, 1, 3, 6). \end{cases}$$

Now we must solve the following systems

$$\begin{cases} x_1 + 3x_2 + 5x_4 = 4, \\ 8x_1 + 3x_3 + 3x_4 = 22, \\ x_2 + 3x_3 + x_4 = 3, \\ 5x_1 + 7x_2 + 2x_3 = 5, \end{cases} \quad \begin{cases} 2x_1 + x_2 + 2x_4 = 5, \\ 3x_1 + 2x_3 + x_4 = 9, \\ x_2 + x_3 + x_4 = 1, \\ 2x_1 + 4x_2 + x_3 = 1. \end{cases}$$

Hence the solution is  $X = (2, -1, 1, 1)^t$ .

**Example 4.** Consider the  $3 \times 3$  fuzzy linear system

$$\begin{cases} (\frac{1}{2}, \frac{2}{3}, \frac{2}{5}, \frac{1}{3})x_1 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{14}, \frac{1}{2})x_2 = (1, \frac{7}{6}, \frac{5}{21}, \frac{5}{6}), \\ (1, 1, \frac{2}{3}, \frac{1}{3})x_1 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2})x_3 = (\frac{3}{2}, \frac{3}{2}, \frac{5}{6}, \frac{5}{6}), \\ (\frac{4}{3}, \frac{4}{3}, \frac{1}{3}, 1)x_2 + (\sqrt{2}, \sqrt{3}, \sqrt{2} - 1, \sqrt{5} - \sqrt{3})x_3 = \\ (2.7475, 3.0653, 0.7475, 1.5041). \end{cases}$$

Therefore we must solve the following systems

$$\begin{cases} \frac{11}{10}x_1 + \frac{10}{7}x_2 = \frac{58}{21}, \\ \frac{5}{3}x_1 + \frac{4}{3}x_3 = 3, \\ \frac{10}{3}x_2 + (1 + \sqrt{5})x_3 = 6.5694, \end{cases} \quad (10)$$

$$\begin{cases} \frac{7}{6}x_1 + x_2 = \frac{13}{6}, \\ 2x_1 + x_3 = 3, \\ \frac{8}{3}x_2 + (\sqrt{2} + \sqrt{3})x_3 = 5.8128. \end{cases} \quad (11)$$

The solutions of (10) and (11) are

$$X = (1.0000001831745677, 0.9999998290370702, 0.999997710317905),$$

and

$$X = (1.0000139346689914, 0.9999837428861764, 0.9999721306620172),$$

respectively. Consequently, the solution of this example is about  $X = (1.0000, 1.0000, 1.0000)$ .

## 5 Conclusions

In this work, we have suggested a new method for solving fuzzy linear systems provided the elements of the coefficients matrix be trapezoidal fuzzy numbers. Finally, examples were presented to illustrate the proposed method.

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