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Abstract. In this paper we present a method for solving fuzzy linear systems by two crisp linear systems. Also necessary and sufficient conditions for existence of solution are given. Some numerical examples illustrate the efficiency of the method.

1. Introduction

Systems of simultaneous linear equations play a major role in various areas as such as mathematics, physics, statistics, neural network and etc. A general model for solving an $n \times n$ fuzzy linear system which coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector was given by Friedman et al. [7]. They used the embedding method given in [5] and replace the original $n \times n$ fuzzy linear system by a $2n \times 2n$ crisp function linear system. Some other numerical procedures, for example, Jacobi, Gauss-Seidel, SOR iterative methods and Adomian decomposition method for solving fuzzy linear systems are designed by [1],[2],[3]. In this paper we present a method for solving $n \times n$ fuzzy linear system whose coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. For solving $n \times n$ fuzzy linear system we solve two $n \times n$ crisp function linear systems (in comparison with Friedman's procedure). Numerical examples are provided to illustrate the efficiency of the method.

2. Preliminaries

Here we recall the basic notations for symmetric fuzzy numbers and symmetric fuzzy linear systems.

Definition 2.1. [8] A fuzzy number is a map $u : \mathbb{R} \to I = [0, 1]$ which satisfies:
(i) u is upper semi-continuous.
(ii) $u(x) = 0$ outside some interval $[c, d] \subset \mathbb{R}$.
(iii) There exist real numbers a, b such that $c \le a \le b \le d$ where

- **1:** $u(x)$ is monotonic increasing on $[c, a]$.
- **2:** $u(x)$ is monotonic decreasing on $[b, d]$.
- **3:** $u(x) = 1,\ a \le x \le b$.

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Also u is called symmetric fuzzy number if $u(u^c + x) = u(u^c - x)$ for $\forall x \in \mathbb{R}$, where $u^c = \frac{a+b}{2}$.

An equivalent parametric definition of fuzzy numbers is given in [5, 4] as:

Definition 2.2. An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, which satisfy the following requirements:

1: $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$.
2: $\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0, 1]$.
3: $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Also $u = (\underline{u}, \overline{u})$ is called a symmetric fuzzy number in parametric form if

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$

is a real constant for all $0 \le r \le 1$. For example $u = (2 + r, 5 - 2r)$ is a fuzzy number and $v = (1 + r, 3 - r)$ is a symmetric fuzzy number in parametric form. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \le r \le 1$, [5].

Definition 2.3. The $n \times n$ linear system

$$\text{(1)} \qquad \begin{cases} a_{11}x_1 + ... + a_{1n}x_n = y_1, \\ a_{21}x_1 + ... + a_{2n}x_n = y_2, \\ \quad . \\ \quad . \\ \quad . \\ a_{n1}x_1 + ... + a_{nn}x_n = y_n, \end{cases}$$

where the coefficients matrix $A = (a_{ij})$, $1 \le i, j \le n$ is a crisp $n \times n$ matrix and each y_i, $1 \le i \le n$, is fuzzy number in parametric form, is called a fuzzy linear system in parametric form (FLS) [7]. One recalls [5] that for arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$ in parametric form and scalar k

1: $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.
2: $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.
3: $kx = (k\underline{x}(r), k\overline{x}(r))$ if k is nonnegative and $kx = (k\overline{x}(r), k\underline{x}(r))$ if k is negative.

Definition 2.4. A fuzzy number vector $X = (x_1, x_2, ..., x_n)^t$ given by $x_i = (\underline{x}_i(r), \overline{x}_i(r))$, $1 \le i \le n$, $0 \le r \le 1$, is called (in parametric form) a solution of the FLS (1) if

$$\text{(2)} \qquad \begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i. \end{cases}$$

Definition 2.5. For fuzzy linear system $AX = Y$, like FLS (1), let matrix B contains the positive entries of A and matrix C contains the absolute value of the negative entries of A. Then $A = B - C$ and we define $A^+ = B + C$.

Definition 2.6. The permutation matrix be a square matrix with one unit element in each row and column and all other entries zero.

Definition 2.7. An arbitrary matrix A is said to be absolutely permutation matrix if A^+ is a permutation matrix.

Remark 2.8. If A is an absolutely permutation matrix then A^{-1} is an absolutely permutation matrix and $A^{-1} = A^T$.

Theorem 2.9. [6] *The inverse of nonnegative matrix A is nonnegative if and only if A is a permutation matrix.*

3. **Fuzzy solution**

The i^{th} equation in (1) is representable in the following equivalent form:

$$\begin{aligned} \sum_{a_{ij}\geq 0} a_{ij}\underline{x}_j + \sum_{a_{ij}<0} a_{ij}\overline{x}_j &= \underline{y}_i, \\ \sum_{a_{ij}\geq 0} a_{ij}\overline{x}_j + \sum_{a_{ij}<0} a_{ij}\underline{x}_j &= \overline{y}_i, \end{aligned} \tag{3}$$

and hence

$$\sum_{a_{ij}\geq 0} a_{ij}(\overline{x}_j - \underline{x}_j) - \sum_{a_{ij}<0} a_{ij}(\overline{x}_j - \underline{x}_j) = \overline{y}_i - \underline{y}_i. \tag{4}$$

If $w_j = \overline{x}_j - \underline{x}_j$ and $v_i = \overline{y}_i - \underline{y}_i$ then (4) has the form

$$\sum_{a_{ij}\geq 0} a_{ij}w_j - \sum_{a_{ij}<0} a_{ij}w_j = v_i, \quad i = 1, \ldots, n,$$

and in the matrix form

$$(B + C)W = V,$$

where $W = (w_1, w_2, ..., w_n)^t$, $V = (v_1, v_2, ..., v_n)^t$ and $A = B - C$. Let $X^c = (x_1^c, x_2^c, ..., x_n^c)$ and $Y^c = (y_1^c, y_2^c, ..., y_n^c)$ where $x_i^c = (\underline{x}_i(r) + \overline{x}_i(r))/2$ and $y_i^c = (\underline{y}_i(r) + \overline{y}_i(r))/2$ for $1 \leq i \leq n$.

Theorem 3.1. *Let X be a fuzzy solution of FLS (1) where coefficients matrix A is nonsingular matrix and Y is a fuzzy number vector. Then $AX^c = Y^c$.*

Proof. Due to Eq.(3), we have for each i, $1 \leq i \leq n$

$$\sum_{a_{ij}\geq 0} (a_{ij}\frac{(\overline{x}_j(r) + \underline{x}_j(r))}{2}) + \sum_{a_{ij}<0} (a_{ij}\frac{(\overline{x}_j(r) + \underline{x}_j(r))}{2}) = \frac{(\overline{y}_i(r) + \underline{y}_i(r))}{2}$$

and hence

$$\sum_{a_{ij}\geq 0} a_{ij}x_j^c + \sum_{a_{ij}<0} a_{ij}x_j^c = y_i^c,$$

i.e., $(B - C)X^c = Y^c$, which completes the proof. □

Remark 3.2. In Theorem 3.1, if Y is symmetric fuzzy vector then X is symmetric fuzzy vector.

Remark 3.3. For finding the solution of FLS (1), we must solve the following crisp linear systems,

$$\begin{cases} (B+C)W = V, \\ (B-C)X^c = Y^c. \end{cases} \tag{5}$$

Because after solving (5), it is enough we take

$$\underline{x}_i = x_i^c - 0.5w_i$$
$$\overline{x}_i = x_i^c + 0.5w_i$$

for each i, $1 \le i \le n$.

Theorem 3.4. *The unique solution X of FLS (1) is a fuzzy vector for arbitrary fuzzy vector Y if and only if both $(B+C)^{-1}$ and $(B-C)^{-1}$ exist and $(B+C)^{-1}$ is a nonnegative matrix.*

Proof. The Eq.(5) has a unique solution if and only if both $(B+C)^{-1}$ and $(B-C)^{-1}$ exist. Now let we consider $D = 0.5[(B+C)^{-1} + (B-C)^{-1}]$ and $E = 0.5[(B+C)^{-1} + (B-C)^{-1}]$ and let we take

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

hence

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}.$$

By referring to Theorem 3.3 in [7], we observe that the unique solution of FLS (1) is a fuzzy vector (in parametric form) for arbitrary Y if and only if $S^{-1} \ge 0$, therefore it is sufficient to show that $S^{-1} \ge 0$ if and only if $(B+C)^{-1} \ge 0$. If $S^{-1} \ge 0$ then $D \ge 0$ and $E \ge 0$ and hence

$$(B+C)^{-1} + (B-C)^{-1} \ge 0,$$
$$(B+C)^{-1} - (B-C)^{-1} \ge 0,$$

which implies $(B+C)^{-1} \ge 0$. Now let $(B+C)^{-1} \ge 0$ therefore by Theorem 2.9, $(B+C)$ is a permutation matrix and hence $(B-C)$ is absolutely permutation matrix. It shows that

$$D_{ij} = 0.5[(B+C)^T_{ij} + (B-C)^T_{ij}] \ge 0,$$
$$E_{ij} = 0.5[(B+C)^T_{ij} - (B-C)^T_{ij}] \ge 0,$$

which completes the proof. □

Example 3.5. Consider the 2×2 symmetric fuzzy system

$$\begin{cases} x_1 - x_2 = (r, 2-r), \\ x_1 + 3x_2 = (4+2r, 8-2r). \end{cases}$$

Hence

$$\underline{x}_1 - \overline{x}_2 = r, \ \underline{x}_1 + 3\underline{x}_2 = 4 + 2r,$$
$$\overline{x}_1 - \underline{x}_2 = 2 - r, \ \overline{x}_1 + 3\overline{x}_2 = 8 - 2r,$$

and therefore

$$\begin{cases} (\overline{x}_1 - \underline{x}_1) + (\overline{x}_2 - \underline{x}_2) = 2 - 2r, \\ (\overline{x}_1 - \underline{x}_1) + 3(\overline{x}_2 - \overline{x}_2) = 4 - 4r, \end{cases}$$

which is equivalent to

$$\begin{cases} w_1 + w_2 = v_1, \\ w_1 + 3w_2 = v_2, \end{cases} \tag{6}$$

where $v_1 = 2 - 2r$, $v_2 = 4 - 4r$. Another crisp system is

$$\begin{cases} x_1^c - x_2^c = 1 = y_1^c, \\ x_1^c + 3x_2^c = 6 = y_2^c. \end{cases} \tag{7}$$

By solving (6) and (7), we have $w_1 = 1 - r$, $w_2 = 1 - r$, $x_1^c = \frac{9}{4}$, $x_2^c = \frac{5}{4}$ and therefore

$$\underline{x}_1 = \frac{9}{4} - \frac{1}{2}(1-r),\ \overline{x}_1 = \frac{9}{4} + \frac{1}{2}(1-r),$$

$$\underline{x}_2 = \frac{5}{4} - \frac{1}{2}(1-r),\ \overline{x}_2 = \frac{5}{4} + \frac{1}{2}(1-r).$$

Here $\underline{x}_1 \leq \overline{x}_1$, $\underline{x}_2 \leq \overline{x}_2$ and $\overline{x}_1$, $\overline{x}_2$ are monotonic non-increasing and $\underline{x}_1$, $\underline{x}_2$ are monotonic non-decreasing functions.

Remark 3.6. The unique solution X of FLS (1) is a fuzzy vector for arbitrary fuzzy vector Y if and only if $(B+C)^{-1}$ and $(B-C)^{-1}$ exist and $(B+C)$ is a permutation matrix.

Remark 3.7. The unique solution X of FLS (1) is a fuzzy vector for arbitrary fuzzy vector Y if and only if $(B+C)^{-1}$ and $(B-C)^{-1}$ exist and A is an absolutely permutation matrix.

4. **Weak fuzzy solution**

We now restrict the discussion to triangular fuzzy numbers, i.e., $\underline{y}_i(r)$, $\overline{y}_i(r)$ and consequently $\underline{x}_i(r)$, $\overline{x}_i(r)$ are all linear functions of r, $\underline{y}_i(1) = \overline{y}_i(1)$ and $\underline{x}_i(1) = \overline{x}_i(1)$ for all $1 \leq i \leq n$. By virtue of Theorem 3.4, since $(B+C)$ is nonnegative, $(B+C)^{-1}$ may be negative, in this case w_i may be negative for some i and therefore $\overline{x}_i - \underline{x}_i < 0$. The fact that x_i is not a fuzzy number and we define a fuzzy number vector

$$U = ((\underline{u}_1, \overline{u}_1), \cdots, (\underline{u}_n, \overline{u}_n))^t,$$

where

$$\underline{u}_i(r) = \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\},$$

$$\overline{u}_i(r) = \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}.$$

If $(\underline{x}_i(r), \overline{x}_i(r))$, $1 \leq i \leq n$, are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r)$, $\overline{u}_i(r) = \overline{x}_i(r)$, $1 \leq i \leq n$, and U is called a strong fuzzy solution. Otherwise, U is called a weak fuzzy solution. In Example 3.5, the obtained solution was strong.

Example 4.1. [7] Consider the 3×3 fuzzy system

$$\begin{cases} x_1 + x_2 - x_3 = (r, 2-r), \\ x_1 - 2x_2 + x_3 = (2+r, 3), \\ 2x_1 + x_2 + 3x_3 = (-2, -1-r). \end{cases}$$

The two crisp linear systems are

$$\begin{cases} w_1 + w_2 + w_3 = 2 - 2r, \\ w_1 + 2w_2 + w_3 = 1 - r, \\ 2w_1 + w_2 + 3w_3 = 1 - r, \end{cases}$$

and

$$\begin{cases} x_1^c + x_2^c - x_3^c = 1, \\ x_1^c - 2x_2^c + x_3^c = 0.5(5+r), \\ 2x_1^c + x_2^c + 3x_3^c = 0.5(-3-r). \end{cases}$$

The solution vectors in parametric form are $W = (7-7r, -1+r, -4+4r)^t$ and $X^c = (1.19 + 0.12r, -1.12 - 0.27r, -0.92 - 0.15r)^t$, then

$$x_1 = (-2.31 + 3.62r, 4.69 - 3.38r),$$

$$x_2 = (-0.62 - 0.77r, -1.62 + 0.23r),$$

$$x_3 = (1.08 - 2.15r, -2.92 + 1.85r).$$

The fact that x_2, x_3 are not fuzzy numbers because, W_2 and W_3 are negatives , the fuzzy solution in this case is a weak solution given by

$$u_1 = (-2.31 + 3.62r, 4.69 - 3.38r),$$

$$u_2 = (-1.62 + 0.23r, -0.62 - 0.77r),$$

$$u_3 = (-2.92 + 1.85r, 1.08 - 2.15r).$$

5. **Conclusions**

In this work we propose an efficient method for solving a system of n fuzzy linear equations with n variables. The original system with matrix A is replaced by two $n \times n$ crisp linear systems. The new system is then solved by two $n \times n$ crisp systems. The solution vector be symmetric solution if the right hand side vector be symmetric.

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