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Abstract

In this paper we represent a new method for solving a symmetric fuzzy linear system by two crisp linear systems. Also necessary and sufficient conditions for the solution existence are given.

keywords: Symmetric fuzzy linear system; Fuzzy linear system; Nonnegative matrix

1 Introduction

Systems of simultaneous linear equations play a major role in various are as such as mathematics, physics, statistics, neural network and etc. A general model for solving an $n \times n$ fuzzy linear system which coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector was given by Friedman et al. [3]. They used the embedding method given in [2] and replace the original $n \times n$ fuzzy linear system by an $2n \times 2n$ crisp function linear system. In this paper we represent a new method for solving a $n \times n$ fuzzy linear system whose coefficients matrix is crisp and the right hand side column is an arbitrary symmetric fuzzy number vector. For solving an $n \times n$ fuzzy linear system we solve two $n \times n$ crisp function linear systems.

2 preliminaries

Here we recall the basic notation of symmetric fuzzy numbers and s symmetric fuzzy linear system.

Definition 1. A symmetric fuzzy number is a fuzzy set $u : \mathbb{R} \to I = [0, 1]$ which satisfies:

(i) u is upper semi continuous.

(ii) $u(x) = 0$ outside some interval $[c, d]$.

$iii)$ There are real numbers a, b such that $c \le a \le b \le d$ where

1. $u(x)$ is monotonic increasing on $[c, a]$.

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2. $u(x)$ is monotonic decreasing on $[b, d]$.
3. $u(x) = 1, a \leq x \leq b$.
4. If $u^c = \frac{a+b}{2}$ then $u(u^c + x) = u(u^c - x)$ for $\forall x \in \mathbb{R}$.

Definition 2. [2] An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$ which satisfy the following requirements:

1) $\underline{u}(r)$ is a bounded left continuous non decreasing over $[0, 1]$.

2) $\overline{u}(r)$ is a bounded left continuous non increasing over $[0, 1]$.

3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Also a fuzzy number is said to be symmetric if $\frac{\underline{u}(r)+\overline{u}(r)}{2} = u^c$, where u^c is a real constant for all $0 \leq r \leq 1$.

Definition 3. The $n \times n$ linear system

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n = y_1, \\ a_{21}x_1 + ... + a_{2n}x_n = y_2, \\ \quad . \\ \quad . \\ \quad . \\ a_{n1}x_1 + ... + a_{nn}x_n = y_n, \end{cases} \tag{1}$$

where the coefficient matrix $A = (a_{ij})$, $1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and y_i is symmetric fuzzy number, is called a symmetric fuzzy linear system($SFLS$).

Definition 4. A fuzzy number vector $(x_1, x_2, ..., x_n)^t$ given by $x_i = (\underline{x}_i(r)$, $\overline{x}_i(r))$, $1 \leq i \leq n$, $0 \leq r \leq 1$, is called a solution of the SFLS if

$$\begin{cases} \underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{y}_i. \end{cases} \tag{2}$$

Consider the fuzzy linear system $AX = Y$, like Eq.(1), we take

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

where B contains the positive entries of A, i.e., if $A_{i,j} > 0$ then $B_{i,j} = A_{i,j}$ else is zero, C is the absolute value of the negative entries of A, i.e., if $A_{i,j} < 0$ then $(B_{i,j}) = |A_{i,j}|$ else is zero, Where $1 \leq i, j \leq n$ it is clear that $A = B - C$ and the matrix S is $2n \times 2n$, crisp matrix.

Theorem 1. [3] If S^{-1} exists it must be have the same structure as S, i.e.

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}.$$

Theorem 2. Let X be a solution of Eq.(1) where $Y = (y_1, y_2, ..., y_n)^t$ is a symmetric fuzzy vector then X is a symmetric fuzzy vector.

Proof. Let $S^{-1} = (t_{ij})$, $1 \leq i,j \leq 2n$, by refer to [3] we see that for $1 \leq i \leq n$,

$$\underline{x}_i = \sum_{j=1}^{n} t_{ij}\underline{y}_j - \sum_{j=1}^{n} t_{i,n+j}\overline{y}_j,$$

$$\overline{x}_i = -\sum_{j=1}^{n} t_{i,n+j}\underline{y}_j + \sum_{j=1}^{n} t_{ij}\overline{y}_j.$$

Since y_i is symmetric, there exists $h_i(r) \in \mathbb{R}$ such that

$$\underline{y}_i(r) = y_i^c - h_i(r),$$

$$\overline{y}_i(r) = y_i^c + h_i(r),$$

for $r \in [0,1]$. Hence

$$\underline{x}_i = \sum_{j=1}^{n} t_{ij}y_j^c - \sum_{j=1}^{n} t_{ij}h_j - \sum_{j=1}^{n} t_{i,n+j}y_j^c - \sum_{j=1}^{n} t_{i,n+j}h_j,$$

$$\overline{x}_i = -\sum_{j=1}^{n} t_{i,n+j}y_j^c + \sum_{j=1}^{n} t_{i,n+j}h_j + \sum_{j=1}^{n} t_{ij}y_j^c + \sum_{j=1}^{n} t_{ij}h_j,$$

and then

$$\frac{\underline{x}_i + \overline{x}_i}{2} = \sum_{j=1}^{n} t_{ij}y_j^c - \sum_{j=1}^{n} t_{i,n+j}y_j^c,$$

which is constant and we take it x_i^c. Therefore x_i is a symmetric fuzzy number, and in particular if $\underline{y}_j(1) = \overline{y}_j(1)$ then $\underline{x}_i(1) = \overline{x}_i(1)$. Since x_i is symmetric then it is enough we find x_i^c and $\overline{x}_i(0) - \underline{x}_i(0)$. □

3 Fuzzy solution

Consider the fuzzy linear system (2), then we have

$$\begin{aligned} \sum_{a_{ij}\geq 0} a_{ij}\underline{x}_j + \sum_{a_{ij}<0} a_{ij}\overline{x}_j &= \underline{y}_i \\ \sum_{a_{ij}\geq 0} a_{ij}\overline{x}_j + \sum_{a_{ij}<0} a_{ij}\underline{x}_j &= \overline{y}_i, \end{aligned} \tag{3}$$

and hence

$$\sum_{a_{ij}\geq 0} a_{ij}(\overline{x}_j - \underline{x}_j) + \sum_{a_{ij}<0} a_{ij}(\underline{x}_j - \overline{x}_j) = \overline{y}_i - \underline{y}_i.$$

Let $w_j = \overline{x}_j - \underline{x}_j$ and $v_j = \overline{y}_i - \underline{y}_i$ then it concludes that

$$\sum_{a_{ij}\geq 0} a_{ij}w_j - \sum_{a_{ij}<0} a_{ij}w_j = v_i, \quad i = 1, ..., n,$$

and in the matrix form

$$(B+C)W=V,$$

where $W=(w_1,w_2,...,w_n)^t$ and $V=(v_1,v_2,...,v_n)^t$.

Theorem 3. Let X be a fuzzy solution of Eq.(1) where Y is a symmetric fuzzy vector then $AX^c=Y^c$.

Proof. From Theorem 2, X is a symmetric fuzzy vector. Due to Eq.(3),

$$\sum_{a_{ij}\geq 0}(a_{ij}\frac{(\overline{x}_j+\underline{x}_j)}{2})+\sum_{a_{ij}<0}(a_{ij}\frac{(\overline{x}_j+\underline{x}_j)}{2})=\frac{(\overline{y}_i+\underline{y}_i)}{2},$$

$$\sum_{a_{ij}\geq 0}a_{ij}x_j^c+\sum_{a_{ij}<0}a_{ij}x_j^c=y_i^c,$$

which concludes the proof. $\square$

By using above theorems, in finding the solution of Eq.(1), when the input vector is a symmetric fuzzy vector we must solve the following crisp linear systems,

$$\begin{cases}(B+C)W=V,\\(B-C)X^c=Y^c.\end{cases}\tag{4}$$

Theorem 4. The unique solution X of Eq.(1) is a symmetric fuzzy vector for arbitrary symmetric fuzzy vector Y if and and only if $(B+C)^{-1}$ and $(B-C)^{-1}$ exist and $(B+C)^{-1}$ be nonnegative.

Proof. Since X is a solution of Eq.(2) and Y is symmetric, then according to previous theorems, X is symmetric vector and due to (4), $X^c=(B-C)^{-1}Y^c=A^{-1}Y^c$, we represent the elements of $(B+C)^{-1}$ by (f_{ij}), then $w_i=\sum_{j=1}^n f_{ij}v_j$, i.e.

$$\overline{x}_j-\underline{x}_j=\sum_{j=1}^n f_{ij}(\overline{y}_j-\underline{y}_j).$$

Then the necessary and sufficient condition for $w_j=\overline{x}_j-\underline{x}_j\geq 0$, is $f_{ij}\geq 0$, for all $i,\ j$. Since $\overline{y}_i$ is monotonic decreasing and $\underline{y}_i$ is monotonic increasing for all i then $v_i(r)=\overline{y}_i(r)-\underline{y}_i(r)$ is monotonic decreasing due to Eq.(4), and hence $w_j(r)=\overline{x}_j(r)-\underline{x}_j(r)$ is monotonic decreasing and therefore $\underline{x}_i$, and $\overline{x}_i$ are increasing and decreasing functions with respect to $r\in[0,1]$, respectively, because

$$\underline{x}_i=x_i^c-\frac{1}{2}w_i(r),\quad \overline{x}_i=x_i^c+\frac{1}{2}w_i(r).$$

$\square$

Example 1. Consider the 2×2 symmetric fuzzy system

$$\begin{cases} x_1 - x_2 = (r, 2-r), \\ x_1 + 3x_2 = (4+2r, 8-2r). \end{cases}$$

Hence

$$\underline{x}_1 - \overline{x}_2 = r, \qquad \underline{x}_1 + 3\underline{x}_2 = 4 + 2r,$$

$$\overline{x}_1 - \underline{x}_2 = 2 - r, \qquad \overline{x}_1 + 3\overline{x}_2 = 8 - 2r,$$

and therefore

$$\begin{cases} (\overline{x}_1 - \underline{x}_1) + (\overline{x}_2 - \underline{x}_2) = 2 - 2r, \\ (\overline{x}_1 - \underline{x}_1) + 3(\overline{x}_2 - \overline{x}_2) = 4 - 4r, \end{cases} \tag{5}$$

which is equivalent to

$$\begin{cases} w_1 + w_2 = v_1, \\ w_1 + 3w_2 = v_2, \end{cases} \tag{6}$$

where $v_1 = 2 - 2r$, $v_2 = 4 - 4r$. Another crisp system is

$$\begin{cases} x_1^c - x_2^c = 1 = y_1^c, \\ x_1^c + 3x_2^c = 6 = y_2^c. \end{cases} \tag{7}$$

By solving (6) and (7), we have $w_1 = 1 - r$, $w_2 = 1 + r$, $x_1^c = \frac{9}{4}$, $x_2^c = \frac{5}{4}$ and therefore

$$\underline{x}_1 = \frac{9}{4} - \frac{1}{2}(1-r), \quad \overline{x}_1 = \frac{9}{4} + \frac{1}{2}(1-r),$$

$$\underline{x}_2 = \frac{5}{4} - \frac{1}{2}(1-r), \quad \overline{x}_2 = \frac{5}{4} + \frac{1}{2}(1-r).$$

Here $\underline{x}_1 \leq \overline{x}_1, \underline{x}_2 \leq \overline{x}_2$ and $\overline{x}_1, \overline{x}_2$ are monotonic increasing and $\underline{x}_1, \underline{x}_2$ are monotonic decreasing functions.

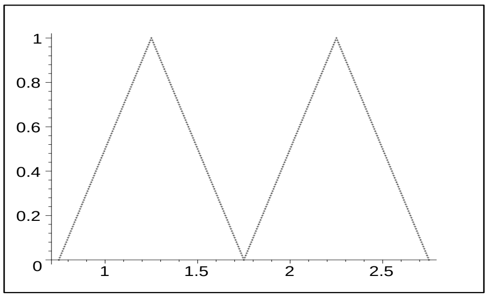

Figure 1 : Example 1. from left x_2, x_1

4 Weak fuzzy solution

Theorem 5. [1] The inverse of nonnegative matrix A is nonnegative if and only if A is a generalized permutation matrix. By virtue of Theorem 4, since $(B + C)$ is nonnegative, $(B + C)^{-1}$ may be negative, in this case w_j may be negative for some j and there for $\overline{x}_j - \underline{x}_j < 0$. The fact that x_j is not a fuzzy number and we define a fuzzy number vector

$$U = \{(\underline{u}_j, \overline{u}_j), 1 \leq j \leq n\},$$

where

$$\underline{u}_j(r) = \min\{\underline{x}_j(r), \overline{x}_j(r)\},$$
$$\overline{u}_j(r) = \max\{\underline{x}_j(r), \overline{x}_j(r)\}.$$

If $(\underline{x}_j(r), \overline{x}_j(r))$, $1 \leq j \leq n$, are all fuzzy numbers then $\underline{u}_j(r) = \underline{x}_j(r)$, $\overline{u}_j(r) = \overline{x}_j(r)$, $1 \leq j \leq n$, and U is called a strong fuzzy solution. Otherwise, U is called a weak fuzzy solution. Indeed, U is a symmetric fuzzy number, because $(\underline{x}_j(r), \overline{x}_j(r))$ are. In Example 1, the obtained solution was strong.
Example 2. Consider the 3×3 fuzzy system

$$\begin{cases} x_1 + x_2 - x_3 = (1 + r, 3 - r), \\ x_1 - 2x_2 + x_3 = (2 + (3/2)r, 5 - (3/2)r), \\ 2x_1 + x_2 + 3x_3 = (3 + r, 5 - r). \end{cases} \tag{8}$$

The two crisp linear systems are

$$\begin{cases} w_1 + w_2 + w_3 = 2 - r, \\ w_1 + 2w_2 + w_3 = 3 - 3r, \\ 2w_1 + w_2 + 3w_3 = 2 - 2r, \end{cases} \tag{9}$$

and

$$\begin{cases} x_1^c + x_2^c - x_3^c = 2, \\ x_1^c - 2x_2^c + x_3^c = \frac{7}{2}, \\ 2x_1^c + x_2^c + 3x_3^c = 4. \end{cases} \tag{10}$$

The solution vectors are $W = [2-2r, 1-r, -1+r]^t$ and $X^c = [2.461, -0.576, -0.115]^t$, then

$$x_1 = (2.461 - 0.5(2 - 2r), 2.461 + 0.5(2 - 2r)),$$
$$x_2 = (-0.576 - 0.5(1 - r), -0.576 + 0.5(1 - r)),$$
$$x_3 = (-0.115 - 0.5(r - 1), -0.115 + 0.5(r - 1)).$$

The fact that x_3 is not fuzzy number, the fuzzy solution in this case is a weak solution given by

$$u_1 = (2.461 - 0.5(2 - 2r), 2.461 + 0.5(2 - 2r)),$$
$$u_2 = (-0.576 - 0.5(1 - r), -0.576 + 0.5(1 - r)),$$
$$u_3 = (-0.115 + 0.5(r - 1), -0.115 - 0.5(r - 1)).$$

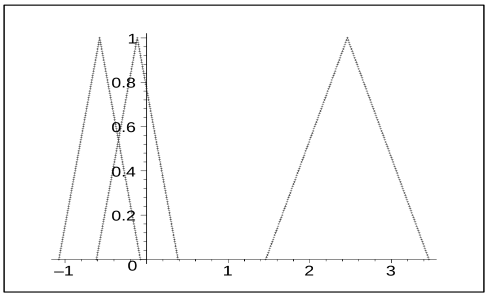

Figure 2 : Example 2. from left u_2 u_3 u_1

5 Conclusions

In this work we propose an efficient method for solving a system of n fuzzy linear equations with n variables with symmetric fuzzy numbers in right hand side. The original system with matrix A is replaced by a $2n \times 2n$ crisp linear system with a matrix S. The new system is then solved by two $n \times n$ crisp systems. The solution vector defines either a strong or weak fuzzy solution and in any case the solution is symmetric, again.

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