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## Abstract

This paper deals the numerical solution of integral equations of the first kind

$$\int_a^b k(s,t)z(t) dt = u(s) \quad c \leq s \leq d$$

where  $k(s,t)$ ,  $u(s)$  are known  $L^2$ -functions and  $z(s)$  is the unknown solution with using regularization method based on solving the related well-posed problem:

$$\text{Minimize } \{ \|Az - u\|^2 + \Omega[z] \}$$

where

$$(Az)(s) = \int_a^b k(s,t)z(t) dt$$

and  $\Omega$  is a stabilizing functional whose choice determines the variant of the method. There are many stopping rules based on discrepancy principle or  $\nu$ -method discussed in [3]. Here a new stopping rule is described which uses SVD (Singular Value Decomposition) and condition number of matrices. Finally, we give a number of numerical examples showing that the method works well in practice.

**Keywords:** Fredholm integral equations, Tikhonov regularization method, Singular Value Decomposition, Condition numbers, Ill-posed problems.

## 1 Introduction

We consider the numerical solution of the linear first kind integral equation

$$\int_a^b k(s,t)z(t) dt = u(s) \quad c \leq s \leq d \tag{1}$$

where  $k(s, t), u(s)$  are known  $L^2$  – functions and  $z(s)$  is the unknown solution. This problem is in general ill-posed and can be solved by regularization method [3,4,5] and augmented Galerkin method [1]. The method of Tikhonov regularization is based on solving the related well-posed problem:

$$\text{Minimize } \{ \| Az - u \|^2 + \Omega[z] \} \quad (2)$$

where

$$(Az)(s) = \int_a^b k(s, t)z(t) dt$$

and  $\Omega$  is a stabilizing functional whose choice determines the variant of the method.

For example

$$\Omega[z] = \int_a^b \sum_{r=0}^n q_r(s) \left[ \frac{d^r z}{ds^r} \right]^2 ds \quad \text{where } q_r(s) > 0 \quad \text{for } a \leq s \leq b$$

and  $n$  indicates the order of regularization. Here, we consider zero order. In this case problem (2) has a solution which is also a solution of:

$$\alpha z(s) + \int_a^b k^* k(s, t)z(t) dt = G(s) \quad (3)$$

where

$$k^* k(s, t) = \int_c^d \overline{k(y, s)} k(y, t) dy$$

$$G(s) = \int_c^d \overline{k(y, s)} u(y) dy$$

and hence (2) reduces to (3), which is a well-conditioned second kind integral equation.

## 2 Updating $\alpha$

Suppose  $z_T(s)$  and  $\tilde{z}(t)$  be the exact and approximate solution of (1), using regularization method. Solving equation (3) with  $\alpha = 0$  is equivalent to solving equation (1) because

$$\begin{aligned} \int_a^b k^* k(s, t) z_T(t) dt &= \int_a^b \int_c^d \overline{k(y, s)} k(y, t) z_T(t) dt dy = \\ \int_c^d \overline{k(y, s)} dy \int_a^b k(y, t) z_T(t) dt &= \int_c^d \overline{k(y, s)} u(y) dy = G(s) \end{aligned}$$

But equation (3) is ill-conditioned for  $\alpha = 0$  and we can solve it only for small values of  $\alpha$  [5].

Suppose we want  $\epsilon$  approximation, i.e. ,

$$\| \tilde{z}(s) - z_T(s) \|_\infty = \sup_{a \leq s \leq b} | \tilde{z}(s) - z_T(s) | \leq \epsilon$$

hence we must have

$$\alpha |\tilde{z}(s)| \leq \epsilon \int_a^b |k^*k(s,t)| dt \quad (4)$$

Knowing values of  $|\tilde{z}(s)|$  at some points, an upper bound for  $\alpha$  is obtained from (4), with this relation and a quadrature rule like trapezoid rule, value of  $\alpha$  can be updated using the following iterative scheme.

First we choose  $\alpha = 1$  (or 0 with moderate value for  $h$  in trapezoid rule) and solve equation (3) using trapezoid rule, i.e.

$$\alpha z(s_i) + h \sum_{j=0}^n k^*k(s_i, s_j) z(s_j) = G(s_i) \quad i = 0, 1, \dots, n \quad (5)$$

where  $a = s_0 < s_1 < \dots < s_n = b$ . Solving this system of equations the values of  $\tilde{z}(s_0), \dots, \tilde{z}(s_n)$  are known and we should have

$$\alpha \lesssim \min_{\substack{0 \leq i \leq n \\ \tilde{z}(s_i) \neq 0}} \frac{\epsilon \int_a^b |k^*k(s_i, t)| dt}{|\tilde{z}(s_i)|}$$

This upper bound for  $\alpha$  is used as new  $\alpha$ , and this process can be repeated.

### 3 Stopping rule

This rule is based on ill-conditioning of equations (5). This system can be written as

$$A_\alpha Z = G$$

where  $A_\alpha$  is a square matrix which is ill-conditioned for small  $\alpha$ . Computing condition number of  $A_\alpha$  by direct method is easy. This number,  $C(A_\alpha)$ , equals  $\|A_\alpha\| \cdot \|A_\alpha^{-1}\|$ , for example with  $\|\cdot\|_2$ . Another way of evaluating condition number is to evaluate singular values of  $A_\alpha$ . We know that if  $A_\alpha$  is not singular then condition number is

$$\frac{\text{Largest Singular Value}}{\text{Smallest Singular Value}}$$

We denoted this ratio by  $C'(A_\alpha)$ . Hence if  $A_\alpha$  is non-singular,  $C(A_\alpha) = C'(A_\alpha)$ , but for small  $\alpha$  this identity does not hold numerically. We use this discrepancy for stopping further iterations on  $\alpha$ .

### 4 Examples

Two of the following examples have also been considered by [1,2], our results are better.

Problem 1

$$\int_0^1 \left( \frac{x+y}{2} + xy + \frac{1}{3} \right) f(y) dy = x + \frac{7}{12} \quad \text{for } 0 \leq x \leq 1$$

with solution  $f(x) \equiv 1$ .

Problem 2

$$\int_0^1 \left( \frac{2(x+y)}{3} + 2xy + \frac{2}{3} \right) f(y) dy = x + \frac{5}{9} \quad \text{for } 0 \leq x \leq 1$$

with solution  $f(x) \equiv x$ .

Problem 3

$$\int_0^1 e^{xy} f(y) dy = \frac{e^{x+1} - 1}{1+x} \quad \text{for } 0 \leq x \leq 1$$

with solution  $f(x) \equiv e^x$ .

Problem 4

$$\int_{-1}^{+1} (x+y)^2 f(y) dy = 2\frac{x^2}{3} + \frac{2}{5} \quad \text{for } -1 \leq x \leq 1$$

with solution  $f(x) \equiv x^2$ .

In these examples  $\epsilon = \epsilon_{mach}$  and  $h = 0.1$ . Value of

$$\| \mathbf{E} \|_{\infty} = \max_x | f_T(x) - \tilde{f}(x) |$$

and successive  $\alpha$ 's and  $C'(A_{\alpha}) - C(A_{\alpha})$  are shown in tables 1-5. The last row of each table gives the solution of system (5) with  $\alpha = 0$ . It shows that regularization method was necessary.

TABLE 1 (Example 1)

$\alpha$	$\  \mathbf{E} \ _{\infty}$	$C'(A_{\alpha}) - C(A_{\alpha})$
1.00E+00	7.13E-01	-4.44E-16
2.35E-10	3.50E-06	8.90E+00
6.74E-11	6.15E-06	6.75E+03
0.00E+00	1.29E+02	4.24E+18

TABLE 2 (Example 2)

$\alpha$	$\ \mathbf{E}\ _\infty$	$C'(A_\alpha) - C(A_\alpha)$
1.00E+00	3.74E-01	0.00E+00
8.40E-10	4.18E-07	-9.03E+00
5.26E-10	1.36E-06	-5.01E+02
0.00E+00	5.91E+01	-4.24E+17

TABLE 3 (Example 3)

$\alpha$	$\ \mathbf{E}\ _\infty$	$C'(A_\alpha) - C(A_\alpha)$
1.00E+00	1.60E+00	-8.88E-16
1.06E-10	3.24E-05	2.84E+02
4.35E-11	2.73E-05	-6.33E+02
4.35E-11	2.30E-05	4.51E+02
4.35E-11	2.30E-05	3.44E+03
0.00E+00	8.81E+01	2.41E+16

TABLE 4 (Example 3)

$\alpha$	$\ \mathbf{E}\ _\infty$	$C'(A_\alpha) - C(A_\alpha)$
0.00E+00	8.81E+01	2.41E+16
1.08E-13	8.61E-03	5.58E+08
4.34E-11	2.08E-05	-3.34E+03
4.35E-11	2.12E-05	3.46E+03
4.35E-11	1.99E-05	2.66E+03
4.35E-11	3.03E-05	-1.66E+02
4.35E-11	1.79E-05	3.49E+03
4.35E-11	2.01E-05	-3.08E+03
4.35E-11	1.70E-05	1.95E+03
4.35E-11	2.15E-05	2.92E+03
4.35E-11	3.70E-05	2.98E+03
4.35E-11	1.97E-02	3.25E+03
4.35E-11	1.83E-05	-2.18E+03
4.35E-11	2.14E-05	2.97E+03
4.35E-11	1.90E-05	-7.53E+02

TABLE 5 (Example 4)

$\alpha$	$\  \mathbf{E} \ _{\infty}$	$C'(A_{\alpha}) - C(A_{\alpha})$
1.00E+00	4.73E-01	-1.78E-15
7.43E-10	4.06E-04	2.08E+01
3.91E-10	4.07E-04	8.17E+01
3.91E-10	4.05E-04	-2.41E+02
3.91E-10	4.08E-04	1.67E+01
3.91E-10	4.07E-04	2.67E+01
3.91E-10	4.06E-04	1.13E+02
0.00E+00	1.47E+02	-2.16E+18

## 5 Conclusions

The calculations were carried out using MATLAB with double precision. Our experience shows that we should stop further iterations on  $\alpha$ , and accept the solution which corresponds to the previous value of it, when the absolute value of the difference between the two computed condition numbers is large.

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