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Abstract

This paper deals the numerical solution of integral equations of the first kind

$$\int_a^b k(s,t)z(t)\,dt = u(s) \quad c \le s \le d$$

where $k(s,t), u(s)$ are known $L^2-functions$ and $z(s)$ is the unknown solution with using regularization method based on solving the related well-posed problem:

$$\underset{z}{Minimize} \quad \{\| Az - u \|^2 \ + \Omega[z]\}$$

where

$$(Az)(s) \ = \ \int_a^b k(s,t)z(t)\,dt$$

and Ω is a stabilizing functional whose choice determines the variant of the method. There are many stopping rules based on discrepancy principle or ν -method discussed in [3]. Here a new stopping rule is described which uses SVD (Singular Value Decomposition) and condition number of matrices . Finally, we give a number of numerical examples showing that the method works well in practice.

Keywords: Fredholm integral equations, Tikhonov regularization method, Singular Value Decomposition, Condition numbers, Ill-posed problems.

1 Introduction

We consider the numerical solution of the linear first kind integral equation

$$\int_a^b k(s,t)z(t)\,dt = u(s) \quad c \le s \le d \tag{1}$$

where $k(s,t), u(s)$ are known $L^2 - functions$ and $z(s)$ is the unknown solution. This problem is in general ill-posed and can be solved by regularization method [3,4,5] and augmented Galerkin method [1]. The method of Tikhonov regularization is based on solving the related well-posed problem:

$$\underset{z}{Minimize} \quad \{\| Az - u \|^2 \ + \Omega[z]\} \tag{2}$$

where

$$(Az)(s) \ = \ \int_a^b k(s,t)z(t)\,dt$$

and Ω is a stabilizing functional whose choice determines the variant of the method.

For example

$$\Omega[z] = \int_a^b \sum_{r=0}^{n} q_r(s)[\frac{d^r z}{ds^r}]^2\,ds \qquad where\ q_r(s) > 0 \quad for \ \ a \leq s \leq b$$

and n indicates the order of regularization. Here, we consider zero order. In this case problem (2) has a solution which is also a solution of:

$$\alpha z(s) \ + \ \int_a^b k^\star k(s,t)z(t)\,dt \ = \ G(s) \tag{3}$$

where

$$k^\star k(s,t) \ = \ \int_c^d \overline{k(y,s)}k(y,t)\,dy$$

$$G(s) \ = \ \int_c^d \overline{k(y,s)}u(y)\,dy$$

and hence (2) reduces to (3), which is a well-conditioned second kind integral equation.

2 Updating α

Suppose $z_T(s)$ and $\tilde{z}(t)$ be the exact and approximate solution of (1), using regularization method. Solving equation (3) with $\alpha = 0$ is equivalent to solving equation (1) because

$$\int_a^b k^\star k(s,t)z_T(t)\,dt = \int_a^b \int_c^d \overline{k(y,s)}k(y,t)z_T(t)\,dt\,dy =$$

$$\int_c^d \overline{k(y,s)}\,dy \int_a^b k(y,t)z_T(t)\,dt = \int_c^d \overline{k(y,s)}u(y)\,dy = G(s)$$

But equation (3) is ill-conditioned for $\alpha = 0$ and we can solve it only for small values of α [5].

Suppose we want ϵ approximation, i.e. ,

$$\| \tilde{z}(s) \ - \ z_T(s) \|_\infty = \sup_{a \leq s \leq b} \ | \tilde{z}(s) \ - \ z_T(s) | \leq \ \epsilon$$

hence we must have

$$\alpha \mid \tilde{z}(s) \mid \;\leq\; \epsilon \int_a^b \mid k^\star k(s,t) \mid \, dt \tag{4}$$

Knowing values of $\mid \tilde{z}(s) \mid$ at some points, an upper bound for α is obtained from (4), with this relation and a quadrature rule like trapezoid rule, value of α can be updated using the following iterative scheme.

First we choose $\alpha = 1$ (or 0 with moderate value for h in trapezoid rule) and solve equation (3) using trapezoid rule, i.e.

$$\alpha z(s_i) \;+\; h \sum_{j=0}^{n}{}'' \; k^\star k(s_i, s_j) z(s_j) \;=\; G(s_i) \quad i = 0, 1, \ldots, n \tag{5}$$

where $a = s_0 < s_1 < \ldots < s_n = b$. Solving this system of equations the values of $\tilde{z}(s_0), \ldots, \tilde{z}(s_n)$ are known and we should have

$$\alpha \quad \lessapprox \quad \min_{\substack{0 \leq i \leq n \\ \tilde{z}(s_i) \not\equiv 0}} \frac{\epsilon \int_a^b \mid k^\star k(s_i, t) \mid \, dt}{\mid \tilde{z}(s_i) \mid}$$

This upper bound for α is used as new α, and this process can be repeated.

3 Stopping rule

This rule is based on ill-conditioning of equations (5). This system can be written as

$$A_\alpha \, Z \;=\; G$$

where A_α is a square matrix which is ill-conditioned for small α. Computing condition number of A_α by direct method is easy. This number, $C(A_\alpha)$, equals $\parallel A_\alpha \parallel . \parallel A_\alpha^{-1} \parallel$, for example with $\parallel . \parallel_2$. Another way of evaluating condition number is to evaluate singular values of A_α. We know that if A_α is not singular then condition number is

$$\frac{Largest\ \ Singular\ Value}{Smallest\ Singular\ Value}$$

We denoted this ratio by $C'(A_\alpha)$. Hence if A_α is non-singular , $C(A_\alpha) = C'(A_\alpha)$, but for small α this identity does not hold numerically. We use this discrepancy for stopping further iterations on α.

4 Examples

Two of the following examples have also been considered by [1,2], our results are better.

Problem 1

$$\int_0^1 (\frac{x+y}{2} \ + \ xy \ + \ \frac{1}{3}) f(y)\, dy \ = \ x \ + \ \frac{7}{12} \qquad for \quad 0 \le x \le 1$$

with solution $f(x) \equiv 1$.

Problem 2

$$\int_0^1 (\frac{2(x+y)}{3} \ + \ 2xy \ + \ \frac{2}{3}) f(y)\, dy \ = \ x \ + \ \frac{5}{9} \qquad for \quad 0 \le x \le 1$$

with solution $f(x) \equiv x$.

Problem 3

$$\int_0^1 e^{xy} f(y)\, dy \ = \ \frac{e^{x+1}-1}{1+x} \qquad for \quad 0 \le x \le 1$$

with solution $f(x) \equiv e^x$.

Problem 4

$$\int_{-1}^{+1} (x+y)^2 f(y)\, dy \ = \ 2\frac{x^2}{3} \ + \ \frac{2}{5} \qquad for \quad -1 \le x \le 1$$

with solution $f(x) \equiv x^2$.

In these examples $\epsilon = \epsilon_{mach}$ and $h = 0.1$. Value of

$$\| \ \mathbf{E} \ \|_\infty = \max_x \ | \ f_T(x) \ - \ \widetilde{f}(x) \ |$$

and successive α's and $C'(A_\alpha) - C(A_\alpha)$ are shown in tables 1-5. The last row of each table gives the solution of system (5) with $\alpha = 0$. It shows that regularization method was necessary.

TABLE 1 (Example 1)

α	$\| \mathbf{E} \|_\infty$	$C'(A_\alpha) - C(A_\alpha)$
1.00E+00	7.13E-01	-4.44E-16
2.35E-10	3.50E-06	8.90E+00
6.74E-11	6.15E-06	6.75E+03
0.00E+00	1.29E+02	4.24E+18

TABLE 2 (Example 2)

α	$\Vert \mathbf{E} \Vert_\infty$	$C'(A_\alpha) - C(A_\alpha)$
1.00E+00	3.74E-01	0.00E+00
8.40E-10	4.18E-07	-9.03E+00
5.26E-10	1.36E-06	-5.01E+02
0.00E+00	5.91E+01	-4.24E+17

TABLE 3 (Example 3)

α	$\Vert \mathbf{E} \Vert_\infty$	$C'(A_\alpha) - C(A_\alpha)$
1.00E+00	1.60E+00	-8.88E-16
1.06E-10	3.24E-05	2.84E+02
4.35E-11	2.73E-05	-6.33E+02
4.35E-11	2.30E-05	4.51E+02
4.35E-11	2.30E-05	3.44E+03
0.00E+00	8.81E+01	2.41E+16

TABLE 4 (Example 3)

α	$\Vert \mathbf{E} \Vert_\infty$	$C'(A_\alpha) - C(A_\alpha)$
0.00E+00	8.81E+01	2.41E+16
1.08E-13	8.61E-03	5.58E+08
4.34E-11	2.08E-05	-3.34E+03
4.35E-11	2.12E-05	3.46E+03
4.35E-11	1.99E-05	2.66E+03
4.35E-11	3.03E-05	-1.66E+02
4.35E-11	1.79E-05	3.49E+03
4.35E-11	2.01E-05	-3.08E+03
4.35E-11	1.70E-05	1.95E+03
4.35E-11	2.15E-05	2.92E+03
4.35E-11	3.70E-05	2.98E+03
4.35E-11	1.97E-02	3.25E+03
4.35E-11	1.83E-05	-2.18E+03
4.35E-11	2.14E-05	2.97E+03
4.35E-11	1.90E-05	-7.53E+02

TABLE 5 (Example 4)

α	$\parallel \mathbf{E} \parallel_\infty$	$C'(A_\alpha) - C(A_\alpha)$
1.00E+00	4.73E-01	-1.78E-15
7.43E-10	4.06E-04	2.08E+01
3.91E-10	4.07E-04	8.17E+01
3.91E-10	4.05E-04	-2.41E+02
3.91E-10	4.08E-04	1.67E+01
3.91E-10	4.07E-04	2.67E+01
3.91E-10	4.06E-04	1.13E+02
0.00E+00	1.47E+02	-2.16E+18

5 Conclusions

The calculations were carried out using MATLAB with double precision. Our experience shows that we should stop further iterations on α , and accept the solution which corresponds to the previous value of it, when the absolute value of the difference between the two computed condition numbers is large.

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