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A Note on Symmetry in the Vanishing of Ext

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Abstract. In [1] Avramov and Buchweitz proved that for finitely generated modules M and N over a complete intersection local ring R, $\mathrm{Ext}^i_R(M,N)=0$ for all $i \gg 0$ implies $\mathrm{Ext}^i_R(N,M)=0$ for all $i \gg 0$. In this note we give some generalizations of this result. Indeed we prove the above mentioned result when (1) M is finitely generated and N is arbitrary, (2) M is arbitrary and N has finite length and (3) M is complete and N is finitely generated.

1. introduction

Throughout the paper, R is assumed to be a commutative Noetherian ring with unity and $\dim(R)<\infty$. When R is a local ring, for each R-module M, $\widehat{M}$ denotes the completion of M with respect to the maximal ideal.

In [1, Theorem III] Avramov and Buchweitz proved that for finitely generated modules M and N over a complete intersection local ring R, $\mathrm{Ext}^i_R(M,N)=0$ for all $i \gg 0$ implies $\mathrm{Ext}^i_R(N,M)=0$ for all $i \gg 0$. They were interested in determining a class of local rings which satisfy this property. Then Huneke and Jorgensen [5] defined a class of Gorenstein local rings, which they called AB rings, and they showed that AB rings satisfy the above mentioned property (see [5, Theorem 4.1]).

Using the notation of [2], for given nonzero R-modules M and N, we define $p^R(M,N)$ to be

$$p^R(M,N)=\sup\{i\in\mathbb{N} \mid \mathrm{Ext}^i_R(M,N)\neq 0\}.$$

According to the paper [5], define the Ext-index of the ring R, denoted by Ext-index(R), to be the supremum of finite values of $p^R(M,N)$ for finitely generated R-modules M and N. The authors in [5] also called R an AB ring if it is a Gorenstein local ring of finite Ext-index. Furthermore they showed that the class of AB rings is strictly larger than the class of complete intersection local rings.

In section 2 of this paper we introduce an especial class of AB rings and we show that every complete intersection local ring belongs to this class. Then we show the following theorem:

Theorem A. *Let R be a d-dimensional complete intersection local ring. Assume*

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that M and N are two R-modules such that M is finitely generated and N is arbitrary. Then

$$\operatorname{Ext}_R^i(M,N)=0 \textit{ for all } i\gg 0 \Longrightarrow \operatorname{Ext}_R^i(N,M)=0 \textit{ for all } i>d.$$

In sections 3 we are concerned with the property of symmetry in the vanishing of Ext over complete intersection local rings when the module appears in the left hand side is not necessarily finitely generated and the right hand side module is finitely generated. As we see in [7], it is a general feeling that completeness is a kind of finiteness condition. Therefore in this direction we prove the following theorem:

Theorem B. *Suppose that R is a d-dimensional complete intersection local ring and M, N are two R-modules. If either M is of finite length and N is arbitrary, or M is finitely generated and N is complete, then*

$$\operatorname{Ext}_R^i(N,M)=0 \textit{ for all } i\gg 0 \Longrightarrow \operatorname{Ext}_R^i(M,N)=0 \textit{ for all } i\gg 0.$$

2. Preliminaries and Theorem A

Let R be a Gorenstein local ring and M be a finitely generated R-module. Let M^* denote the dual R-module $\operatorname{Hom}_R(M,R)$. If M is a maximal Cohen-Macaulay (MCM for short) R-module, then there exists a long exact sequence

$$\mathcal{C}(M): \ldots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \xrightarrow{\partial_{-2}} F_{-3} \xrightarrow{\partial_{-3}} \ldots$$

of finitely generated free R-modules such that $M=Ker\partial_{-1}$. Define the non-negative and negative syzygies of M by $M_i=Ker\partial_{i-1}$ for every integer i. Now we recall [5, Lemma 1.1], but one should note that the given proof in [5] is also true when N is an arbitrary R-module, more precisely:

Lemma 2.1. *Let R be a Gorenstein local ring. Suppose that M is an MCM R-module and N is an arbitrary R-module. Then for fixed $t\geq 3$ and for $1\leq i\leq t-2$ we have*

$$\operatorname{Ext}_R^i(M_{-t},N)\cong \operatorname{Tor}_{t-i-1}^R(M^*,N).$$

Definition 2.2. Set $\xi(R)$ to be the supremum of finite values of $p^R(M,N)$ where M and N are R-modules and M is finitely generated, i.e.

$$\xi(R)=\sup\{\, p^R(M,N) \mid p^R(M,N)<\infty \text{ where } M \text{ is a finitely generated } R\text{-module}\,\}.$$

We say that the ring R has finite ξ (or is of finite ξ) if it satisfies $\xi(R)<\infty$.

As some obvious properties of this type of rings we point out to the following proposition.

Proposition 2.3.

(1) *Suppose that $(R,\mathfrak{m})$ is a local ring with $\xi(R)<\infty$. Assume that x is a nonzero divisor on R. Then $\xi(R/xR)<\infty$.*

(2) *If R is a d-dimensional Gorenstein local ring with $\xi(R) < \infty$, then $\xi(R) = d$.*
(3) *Every complete intersection local ring is of finite ξ.*
(4) *Every Gorenstein local ring with finite ξ is an AB ring.*
(5) *Suppose that R is a d-dimensional Gorenstein local ring with finite ξ. Then for every $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ is of finite ξ.*

Proof. The proofs of (1), (2) and (3) are completely similar to the proofs of [5, Propositions 3.3(1), 3.2 and Corollary 3.5] respectively. (4) is trivial.

(5) Suppose that M is a finitely generated $R_{\mathfrak{p}}$-module and N is an arbitrary $R_{\mathfrak{p}}$-module such that $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(M, N) = 0$ for all $i \gg 0$. Write $M = R_{\mathfrak{p}} y_1 + ... + R_{\mathfrak{p}} y_t$. Let $M' = Ry_1 + ... + Ry_t$. We have $M_{\mathfrak{p}} \cong M \cong M'_{\mathfrak{p}}$. Thus if $\mathbf{F.} \longrightarrow M' \longrightarrow 0$ is a free resolution for M' as an R-module, then $\mathbf{F.} \otimes_R R_{\mathfrak{p}} \longrightarrow M'_{\mathfrak{p}} \longrightarrow 0$ is a free resolution for $M'_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module. So, we have $\operatorname{Ext}^i_R(M', N) \cong \operatorname{Ext}^i_R(M', \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, N)) \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M, N)$. Therefore $\operatorname{Ext}^i_R(M', N) = 0$ for all $i \gg 0$. By the assumption and (2), $\operatorname{Ext}^i_R(M', N) = 0$ for all $i > d$. Thus $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(M, N) = 0$ for all $i > d$ and this shows that $\xi(R_{\mathfrak{p}}) \leq d$. □

Theorem 2.4. *Let R be a d-dimensional Gorenstein local ring with $\xi(R) < \infty$. Assume that M and N are two R-modules such that M is finitely generated and N is arbitrary. Then*

$$\operatorname{Ext}^i_R(M, N) = 0 \textit{ for all } i \gg 0 \Longrightarrow \operatorname{Ext}^i_R(N, M) = 0 \textit{ for all } i > d.$$

Proof. Let L be the dth syzygy of M in a free resolution. We know that L is an MCM R-module and $\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^{i-d}_R(L, N)$ for all $i > d$. This shows that $\operatorname{Ext}^i_R(L, N) = 0$ for all $i \gg 0$. Thus for each $t \geq 1$, $\operatorname{Ext}^i_R(L_{-t}, N) = 0$ for all $i \gg 0$. Since $\xi(R) < \infty$, then for each $t \geq 1$ and $i > d$, $\operatorname{Ext}^i_R(L_{-t}, N) = 0$. On the other hand $\operatorname{Ext}^i_R(L_{-t}, N) \cong \operatorname{Ext}^1_R(L_{i-t-1}, N)$ for all $i \geq 1$. Thus for each $t \geq 1$ and $i > d$, $\operatorname{Ext}^1_R(L_{i-t-1}, N) = 0$. Now by suitable changing of i and t, we will have $\operatorname{Ext}^1_R(L_{-t'}, N) = 0$ for each $t' \geq 1$. Therefore by Lemma 2.1, $\operatorname{Tor}^R_{t'-2}(L^*, N) = 0$ for each $t' \geq 3$.

Therefore if $\mathbf{F.} \longrightarrow N \longrightarrow 0$ is a free resolution for N, then $\mathbf{F.} \otimes_R L^* \longrightarrow N \otimes_R L^* \longrightarrow 0$ is an exact sequence. Also, L^* is an MCM R-module, thus for $i \geq 1$ and every free R-module F, $\operatorname{Ext}^i_R(F \otimes_R L^*, R) = 0$. So, for $i \geq 1$ we have $\operatorname{Ext}^i_R(N \otimes_R L^*, R) = \operatorname{H}^i(\operatorname{Hom}_R(\mathbf{F.} \otimes_R L^*, R))$. Hence for $i \geq 1$ we get the following isomorphisms

$$\begin{aligned} \operatorname{Ext}^i_R(N \otimes_R L^*, R) &\cong \operatorname{H}^i(\operatorname{Hom}_R(\mathbf{F.}, \operatorname{Hom}_R(L^*, R))) \\ &= \operatorname{H}^i(\operatorname{Hom}_R(\mathbf{F.}, L^{**})) \\ &\cong \operatorname{H}^i(\operatorname{Hom}_R(\mathbf{F.}, L)) \\ &= \operatorname{Ext}^i_R(N, L). \end{aligned}$$

But R is Gorenstein, so $\operatorname{Ext}^i_R(N \otimes_R L^*, R) = 0$ for $i > d$. Therefore $\operatorname{Ext}^i_R(N, L) = 0$ for $i > d$. Now since $\operatorname{id}(R) = d$ we easily obtain that $\operatorname{Ext}^i_R(N, M) = 0$ for $i > d$. □

3. Theorem B

Let $(R, \mathfrak{m})$ be a local ring and $E(R/\mathfrak{m})$ be the injective envelope of the residue class field $R/\mathfrak{m}$. Recall that the Matlis dual of an R-module T is $\operatorname{Hom}_R(T, E(R/\mathfrak{m}))$ and is denoted by $T^\vee$. We say that T is Matlis reflexive if $T^{\vee\vee} \cong T$. Note that if T has finite length, then T is Matlis reflexive. Furthermore we have the following isomorphisms for R-modules V and W:

$$\operatorname{Tor}_i^R(V, W)^\vee \cong \operatorname{Ext}_R^i(V, W^\vee)$$

and

$$\operatorname{Ext}_R^i(V, W)^\vee \cong \operatorname{Tor}_i^R(V, W^\vee) \text{ when } V \text{ is finitely generated.}$$

Proposition 3.1. *Suppose that $(R, \mathfrak{m})$ is a d-dimensional Gorenstein local ring with finite ξ. Then for every R-modules M and N, where M has finite length and N is arbitrary we have*

$$\operatorname{Ext}_R^i(N, M) = 0 \textit{ for all } i \gg 0 \Longrightarrow \operatorname{Ext}_R^i(M, N) = 0 \textit{ for all } i > d.$$

Proof. We have

$$\operatorname{Ext}_R^i(N, M) \cong \operatorname{Ext}_R^i(N, M^{\vee\vee}) \cong \operatorname{Tor}_i^R(N, M^\vee)^\vee \cong \operatorname{Ext}_R^i(M^\vee, N^\vee).$$

Thus by assumption and Theorem 2.4, $\operatorname{Ext}_R^i(N^\vee, M^\vee) = 0$ for all $i > d$. Since $\operatorname{Ext}_R^i(N^\vee, M^\vee) \cong \operatorname{Tor}_i^R(N^\vee, M)^\vee$, we have $\operatorname{Tor}_i^R(N^\vee, M) = 0$ for all $i > d$. On the other hand $\operatorname{Tor}_i^R(N^\vee, M) \cong \operatorname{Ext}_R^i(M, N)^\vee$. Therefore $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > d$. □

By 2.4 and 3.1 we have the following corollary.

Corollary 3.2. *Let R be an Artinian Gorenstein local ring with $\xi(R) < \infty$. Assume that M and N are two R-modules where M is finitely generated and N is arbitrary. Then*

$$\operatorname{Ext}_R^i(N, M) = 0 \textit{ for all } i \gg 0 \Longrightarrow \operatorname{Ext}_R^i(M, N) = 0 \textit{ for all } i > 0.$$

Theorem 3.3. *Suppose that R is a d-dimensional Gorenstein local ring with $\xi(R) < \infty$. Assume that M is a finitely generated R-module and N is a complete R-module. Then*

$$\operatorname{Ext}_R^i(N, M) = 0 \textit{ for all } i \gg 0 \Longrightarrow \operatorname{Ext}_R^i(M, N) = 0 \textit{ for all } i \gg 0.$$

To prove this theorem, we need the following definition, remark and lemma.

Definition 3.4. Let $(R, \mathfrak{m})$ be a local ring and N be an arbitrary R-module. Let $\tau_N : N \longrightarrow \widehat{N}$ be the natural morphism. We say that N is quasi-complete if τ_N is surjective and N is separated if τ_N is injective.

Remark 3.5. Suppose that $(R, \mathfrak{m})$ is a local ring and N is an arbitrary R-module. Let $0 \longrightarrow K \longrightarrow L \longrightarrow L/K \longrightarrow 0$ be an exact sequence of R-modules. From [6, §8], recall that

(1) N is separated if and only if $\bigcap_n \mathfrak{m}^n N = 0$ for all $n \in \mathbb{N} \cup \{0\}$.

(2) L/K is separated if and only if K is closed in L.

(3) Using [7, 1.2, Corollary] and (2), we get that if K is closed in L and L is quasi-complete then L/K is complete.

Also from [8, Definition 2.1.11 and Proposition 2.1.12(i)] we have

(4) For every flat R-module F there exists a free R-submodule $L \subseteq F$ such that the natural injection $\rho : L \longrightarrow F$ is pure (i.e. $\rho \otimes Id_H : L \otimes_R H \longrightarrow F \otimes_R H$ is injective for every R-module H) and L is dense in the $\mathfrak{m}$-adic topology of F (i.e. $\bigcap_{n\geq 1}(L+\mathfrak{m}^n F) = F$ or $L+\mathfrak{m}^n F = F$ for all n). This implies that $L/\mathfrak{m}^n L \cong F/\mathfrak{m}^n F$ for all n. Therefore when F is a complete flat R-module we have $F \cong \widehat{L}$. In other words every complete flat R-module is the completion of a free R-module (and conversely, see [7, 2.4]).

Lemma 3.6. *Let $(R, \mathfrak{m})$ be a local ring and M be a complete R-module in $\mathfrak{m}$-adic topology. Suppose that $x \in \mathfrak{m}$ is a nonzero divisor on both R and M. Let*

$$0 \longrightarrow T \longrightarrow F \longrightarrow M \longrightarrow 0$$

be an exact sequence of R-modules where F is a complete flat R-module. Then both T and T/xT are complete in their $\mathfrak{m}$-adic topology.

Before proving the lemma, we should remark that M/xM is not necessarily complete, because xM is not necessarily closed in M. The following is an example of A. M. Simon.

Example 3.7. Let $R = k[[X, Y, Z]]$, where k is a field. Put $M_n = R/(XY - Z^n)$ and let M be the completion of $\bigoplus_{n=1}^{\infty} M_n$ as described in [7, 9.4]. In fact

$$M = \{(m_n)_{n\geq 1} \in \prod_{n=1}^{\infty} M_n | \text{ for all } s, \text{ all but finitely many } m_n \text{ belong to } \mathfrak{m}^s M_n\}.$$

Thus $M \subset \prod_{n=1}^{\infty} M_n$. Note that X is regular on R and M. Denote the images of X, Y, Z in M_n with x_n, y_n, z_n. Let $w_t = (z_1, z_2^2, z_3^3, ..., z_t^t, 0, ...)$ for each t. We have that $w_t = X.v_t$, where $(v_t)_i = y_i$ if $i \leq t$ and $(v_t)_i = 0$ otherwise. Thus $w_t \in XM$. The Cauchy sequence w_t has its limit in $M - XM$; indeed we have

$$\lim_{t\to\infty} w_t = (z_1, z_2^2, z_3^3, ..., z_t^t, z_{t+1}^{t+1}, ...)$$

and $(z_1, z_2^2, z_3^3, ..., z_t^t, z_{t+1}^{t+1}, ...) = X(y_1, y_2, y_3, ..., y_t, ...)$ which is not in XM because by the above mentioned structure of M, $(y_1, y_2, y_3, ..., y_t, ...)$ is not an element of M.

However if F is a complete flat R-module, it is mentioned in Remark 3.5(4) that F is the completion of a free R-module and we observe that F/xF is complete, i.e. xF is closed in F. With this in hand we prove Lemma 3.6.

Proof. Since M is complete, T is closed in F and thus complete (see [7, 1.3, Proposition]). With our hypothesis, we also have an exact sequence

$$0 \longrightarrow T/xT \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0.$$

Thus $xT = T \cap xF$ and xT is closed in T because $T \longrightarrow F$ is continuous. Consequently by Remark 3.5(3), T/xT is complete. $\square$

Now we can give the proof of Theorem 3.3:

Proof. We proceed by induction on d. The case $d = 0$ has been proved in a stronger form in 3.2.

So, assume that $d \geq 1$. Suppose that $\mathbf{P.} \longrightarrow M \longrightarrow 0$ is a free resolution of M. Consider the short exact sequence $\Lambda : 0 \longrightarrow M_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$. Since $\mathrm{id}P_0 = d$, using the short exact sequence Λ and hypothesis, we have $\mathrm{Ext}^i_R(N, M_1) = 0$ for all $i \gg 0$.

On the other hand, by [7, 2.5, Proposition], there exists a complete flat resolution $\mathbf{F.} \longrightarrow N \longrightarrow 0$ for R-module N. By Lemma 3.6, N_2 is a complete R-module. Also by [4, page 85, Corollary 3.2.7], for all j we have $\mathrm{pd}F_j \leq d$. Using the exact sequences $\Omega_j : 0 \longrightarrow N_j \longrightarrow F_{j-1} \longrightarrow N_{j-1} \longrightarrow 0$ for $j = 1, 2$ and the fact that $\mathrm{pd}F_{j-1} \leq d$, we obtain $\mathrm{Ext}^i_R(N_2, M_1) = 0$ for all $i \gg 0$. Since $\mathrm{depth}(R) \geq 1$, there exists an element x of $\mathfrak{m}$ which is non-zero divisor on R, M_1 and N_2. Thus we have the long exact sequence

$$\mathrm{Ext}^i_R(N_2, M_1) \longrightarrow \mathrm{Ext}^{i+1}_R(N_2/xN_2, M_1) \longrightarrow \mathrm{Ext}^{i+1}_R(N_2, M_1)$$

obtained from the short exact sequence

$$0 \longrightarrow N_2 \xrightarrow{x.} N_2 \longrightarrow N_2/xN_2 \longrightarrow 0 \quad (\ddagger).$$

By hypothesis, we have $\mathrm{Ext}^i_R(N_2/xN_2, M_1) = 0$ for all $i \gg 0$. Therefore by [6, Page 140, Lemma 2], $\mathrm{Ext}^i_{R/xR}(N_2/xN_2, M_1/xM_1) = 0$ for all $i \gg 0$.

Now R/xR is a $(d-1)$-dimensional Gorenstein local ring with finite ξ (see Proposition 2.3). Also by Lemma 3.6, all N_i/xN_i are complete for $i \geq 2$ and consequently by inductive hypothesis we have $\mathrm{Ext}^i_{R/xR}(M_1/xM_1, N_2/xN_2) = 0$ for all $i \gg 0$. Therefore again by [6, Page 140, Lemma 2], $\mathrm{Ext}^i_R(M_1, N_2/xN_2) = 0$ for all $i \gg 0$. Using again the short exact sequence $(\ddagger)$, we obtain the long exact sequence

$$\mathrm{Ext}^i_R(M_1, N_2/xN_2) \longrightarrow \mathrm{Ext}^{i+1}_R(M_1, N_2) \xrightarrow{x.} \mathrm{Ext}^{i+1}_R(M_1, N_2) \longrightarrow \mathrm{Ext}^{i+1}_R(M_1, N_2/xN_2).$$

So, we have $\mathrm{Ext}^i_R(M_1, N_2) = x\mathrm{Ext}^i_R(M_1, N_2)$ for all $i \gg 0$. Therefore by [7, page 233], $\mathrm{Ext}^i_R(M_1, N_2) = 0$ for all $i \gg 0$. Now, because R is a Gorenstein ring, by [3, page 79, 3.3.4(ii)], $\mathrm{id}F_j \leq d$. So, using the exact sequences Ω_j $(j = 1, 2)$, we have $\mathrm{Ext}^i_R(M_1, N) = 0$ for all $i \gg 0$ and using again the exact sequence Λ, we obtain $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \gg 0$. $\square$

As another application of [7, page 233] with the same method as above, we close this note by proving the following proposition.

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a d-dimensional Gorenstein complete local ring with finite ξ. Set*

$$\xi'(R) = \sup \{\, p^R(N, M) \mid p^R(N, M) < \infty \text{ where } M \text{ is finitely generated and } N \text{ is arbitrary} \,\}.$$

Then we have $\xi'(R) = d$.

Proof. By Corollary 3.2 and Theorem 2.4, the claim obviously holds for $d = 0$. Suppose that $d > 0$. Since $\mathrm{id}(R) = d$, there exists an R-module L such that $\mathrm{Ext}^d_R(L, R) \neq 0$ and $\mathrm{Ext}^i_R(L, R) = 0$ for all $i > d$. Thus $\xi'(R) \geq d$.

Let M be a finitely generated R-module and N be an arbitrary R-module such that $\mathrm{Ext}^i_R(N, M) = 0$ for all $i \gg 0$. Since $\mathrm{id}(R) = d$, we can replace M and N by their first syzygies in their R-free resolutions. Thus there exists a nonzero divisor x on R, M and N. Also using the short exact sequence $0 \longrightarrow M \xrightarrow{x.} M \longrightarrow M/xM \longrightarrow 0$, we obtain $\mathrm{Ext}^i_R(N, M/xM) = 0$ for all $i \gg 0$. Therefore by [6, page 140, Lemma 2], $\mathrm{Ext}^i_{R/xR}(N/xN, M/xM) = 0$ for all $i \gg 0$. Now by inductive hypothesis we have $\mathrm{Ext}^i_{R/xR}(N/xN, M/xM) = 0$ for all $i > d - 1$. Therefore, using again the above exact sequence, we have $\mathrm{Ext}^i_R(N, M) = x\mathrm{Ext}^i_R(N, M)$ for all $i > d$. But M is a complete R-module, so by [7, page 233] $\mathrm{Ext}^i_R(N, M) = 0$ for all $i > d$. This shows that $\xi'(R) \leq d$. □

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