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A spectral model applied to homogeneous turbulence

The conditional dissipation rate of an initially binary scalar in homogeneous turbulence
The spectral relaxation model of the scalar dissipation rate in homogeneous turbulence

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A model for the effect of scalar spectral relaxation on the scalar dissipation rate of an inert, passive scalar \((Sc=1)\) in fully developed homogeneous turbulence is presented. In the model, wave-number space is divided into a finite number \([\text{the total number depending on the turbulence Reynolds number } Re_t \text{ and the Schmidt number } Sc]\) of intermediate stages whose time constants are determined from the velocity spectrum. The model accounts for the evolution of the scalar spectrum from an arbitrary initial shape to its fully developed form and its effect on the scalar dissipation rate for finite \(Re_t\) and \(Sc=1\). Corrsin's result \([AIChE J. 10, 870 (1964)\]) for the scalar mixing time is attained for large \(Re_t\) in the presence of a constant mean scalar gradient and a stationary, isotropic turbulence field. Comparisons with DNS results for stationary, isotropic turbulence and experimental data for decaying, homogeneous grid turbulence demonstrate the satisfactory performance of the model. © 1995 American Institute of Physics.

I. INTRODUCTION

The scalar dissipation rate is a key quantity in the modeling of both inert and reactive turbulent scalar fields. Indeed almost all currently employed models for scalar mixing ranging from simple moment closures to full probability density function (PDF) simulations require information concerning the coupling between the turbulence time scale and the scalar dissipation time scale for closure. Ample experimental and direct numerical simulation (DNS) data exist to show that the usual assumption of a direct proportionality between these time scales \(\text{(e.g., the eddy-breakup model)}\) is often unjustified and that, in reality, the velocity-to-scalar time-scale ratio varies widely according to the flow under consideration and with time, depending in particular on the initial scalar-to-mechanical integral-scale ratio.

The goal of the present study is to derive and verify a new model (the spectral relaxation model) of the scalar dissipation rate that accounts directly for the effect of the velocity spectrum on the relaxation of the scalar spectrum from an arbitrary initial form like in the model proposed by Newman et al., the scalar dissipation time scale appears as a dynamical variable. However, unlike in the model proposed these authors, the spectral relaxation model contains a range of turbulence time scales and thus accounts for the cascade of scalar energy from large to small scales. This division of processes according to their characteristic length scale allows us to remove all fitting parameters present in the model of Newman et al. and to replace them with physical constants that characterize the fundamental processes of turbulent stretching and molecular dissipation.

Due to the cascade description, the spectral relaxation model resembles, in some respects, the linear eddy model proposed by Kerstein. In Kerstein's model advection and diffusion of the scalar are treated simultaneously by solving a one-dimensional diffusion equation on a domain that is periodically rearranged and compressed to mimic the action of random turbulent eddies. Since it involves the numerical solution of a diffusion equation, for large \(Re_t\) and large \(Sc\) the linear eddy model will become intractable for inhomogeneous scalar fields due to the large number of nodes needed to resolve the field down to the Batchelor scale. In the spectral relaxation model, it is recognized that as long as the slower large-scale processes of inertial-range turbulent mixing are correctly accounted for, such small-scale processes as diffusion and scalar gradient stretching will be present in local “equilibrium” and can be lumped together in a single dynamic variable. The total number of dynamical variables is thus drastically reduced making the spectral relaxation model computationally tractable for simulating full three-dimensional inhomogeneous flows. The trade-off for reducing the number of variables is the loss of small-scale information concerning the shape of the scalar PDF. For full PDF simulations, an additional closure will thus be required to describe the relaxation of the scalar PDF to its final form.

The remainder of this work is arranged as follows. In Sec. II the fundamental equations of scalar mixing are reviewed along with the closure for the scalar dissipation proposed by Fox. In Sec. III, the spectral relaxation model is presented. The derivation of the time constants for each substage wave-number band from the velocity spectrum is contained in Sec. IV. The performance of the spectral relaxation model is verified by comparing its predictions to experimental and DNS data for four limiting cases in Sec. V: an inert, passive scalar field in stationary, isotropic turbulence with and without a mean scalar gradient, and in decaying, homogeneous grid turbulence with and without a mean scalar gradient. In all four cases, despite the lack of empirical fitting parameters used by competing models, the agreement is quite satisfactory. Conclusions are drawn in Sec. IV.

II. SCALAR MIXING

The most comprehensive yet computationally tractable treatment of turbulent reactive scalar mixing is perhaps the joint velocity-composition PDF approach proposed by Pope.
However, even at this level of closure, the scalar mixing term in the PDF balance equation must be modeled. Any successful model for this term must address two fundamental problems: first, an inert, passive scalar PDF should relax to a Gaussian form regardless of its initial condition; second, the effect of turbulent mixing on the scalar dissipation rate should be properly accounted for. Several models have been proposed to describe the relaxation of the scalar PDF to its limiting form. Nevertheless, none have proven to be entirely satisfactory and research to improve them continues.

Our focus here is on improving the description of the effect of a turbulent velocity field on the rate of scalar mixing. A new model for the scalar dissipation rate is proposed to explicitly account for relaxation of the scalar spectrum to its final self-similar form through the actions of turbulent mixing. The model builds on earlier work, wherein a fully developed scalar spectrum is assumed, by accounting for the transient period leading to the fully developed form. This period is particularly important for reacting flows where neglect of the transient behavior can lead to serious prediction errors, since most of the fast reactions take place in the developing regime. To date, most models for the scalar dissipation rate have dealt with this difficulty by introducing an empirical mixing constant, \( C_\phi \), that must be adjusted to account for the flow regime under consideration. Typically, for nonpremixed flows, \( C_\phi \) must be set as low as 50% of its theoretical value. In contrast, the model proposed in this work contains no empirical constants or fitting parameters.

A. Scalar mean and variance

We begin the model derivation by considering the moment equations for the mean and variance of an inert, passive scalar advected by an incompressible turbulent flow. The scalar field is governed by

\[
\frac{D\phi}{Dt} = \Gamma \nabla^2 \phi,
\]

from which Reynolds averaging yields

\[
\frac{D\phi}{Dt} = \frac{\partial (\phi)}{\partial t} + \frac{\partial}{\partial x_i} \langle u'_i \phi \rangle = \Gamma \nabla^2 \langle \phi \rangle,
\]

wherein repeated indices imply summation.

Likewise, Reynolds averaging of Eq. (1) multiplied by \( \phi' = \phi - \langle \phi \rangle \) yields the governing equation for the scalar variance:

\[
\frac{D\phi'^2}{Dt} = \frac{\partial (\phi'^2)}{\partial t} + \frac{\partial}{\partial x_i} \langle u'_i \phi'^2 \rangle - 2S_{\phi^2} - 2\Gamma \langle \chi'^2 \rangle,
\]

where \( \chi'^2 = \partial^2 \phi' / \partial x_i \partial x_i \) and the source term can be expressed as

\[
S_{\phi^2} = -\langle u'_i \phi' \rangle \frac{\partial \langle \phi \rangle}{\partial x_i}.
\]

The classical eddy-breakup model closes the moment equations at the level of the scalar variance by assuming that the last term in Eq. (3) has the form

\[
2\Gamma \langle \chi'^2 \rangle = \frac{C_{\phi}}{\tau_v} \langle \phi'^2 \rangle,
\]

where \( C_{\phi} \) is the empirical mixing constant discussed above and \( \tau_v \) is a characteristic time for the turbulence field, usually taken to be \( k/\varepsilon \). In this work, we avoid this assumption by formulating a model for the scalar dissipation, \( \Gamma \langle \chi'^2 \rangle \).

B. Scalar dissipation

The equation governing the magnitude of the scalar gradient follows from Eq. (1):

\[
\frac{D\chi'^2}{Dt} = 2\Gamma \chi'^2 - 2\chi' e_{ij} \chi_j,
\]

where \( \chi'_i = \partial \phi' / \partial x_i \) and \( e_{ij} = \partial u_j / \partial x_i \) is the strain rate tensor.

Reynolds averaging of Eq. (6) after decomposing the scalar gradient into its mean and fluctuating parts \( \chi'_i = \chi'_i + \langle \chi'_i \rangle \) yields

\[
\left\langle \frac{D\chi'^2}{Dt} \right\rangle = -2\Gamma \left[ \frac{\partial \chi'_i}{\partial x_j} - \frac{\partial \chi'_j}{\partial x_i} \right] - 2\langle \chi'_i e_{ij} \chi'_j \rangle - 2\langle \chi'_i e_{ij} \rangle \frac{\partial \langle \phi \rangle}{\partial x_j} - 2\langle \chi'_i e_{ij} \rangle \frac{\partial \langle \phi \rangle}{\partial x_i}.
\]

The scalar dissipation is defined as

\[
\varepsilon_{\phi} = \Gamma \chi'^2
\]

and its balance equation is found by multiplying Eq. (7) by \( \Gamma \). The scalar dissipation rate is defined by

\[
r_{\phi} = \frac{\varepsilon_{\phi}}{\langle \phi'^2 \rangle},
\]

and the scalar mixing time by

\[
\tau_{\phi} = \frac{1}{\langle r_{\phi} \rangle}.
\]

Note that both \( \varepsilon_{\phi} \) and \( r_{\phi} \) are random fields since they involve the fluctuating variable \( \chi'^2 \).

The second term on the right-hand side of Eq. (7) corresponds to turbulent stretching of the scalar gradient and the first term to its molecular dissipation. The last two terms are production terms: the first due to the mean scalar gradient and the second to the mean velocity gradient. Fox, based on studies of diffusion in random scalar fields and DNS of scalar mixing in isotropic stationary turbulence, proposed the following models for the molecular dissipation and turbulent stretching terms:

\[
C_{\phi} = C_{\phi} \left( \frac{\varepsilon_{\phi}}{\nu} \right) \langle \phi' \rangle
\]

and

\[
\Gamma \langle \chi'_i e_{ij} \rangle = -C_{\phi} \left( \frac{\varepsilon_{\phi}}{\nu} \right) \langle \phi' \rangle,
\]

where \( C_{\phi} = 0.54 \) (Refs. 23 and 24) is near Batchelor’s constant, and \( C_x = 3 \) for one-dimensional diffusion in random lamellar systems. Furthermore, if \( \langle \chi'_i \rangle \) is assumed to
be nearly isotropic, the source term due to the mean velocity gradient can be neglected for constant-density flows.

Following Pipino and Fox,\textsuperscript{26} the source term due to the mean scalar gradient can be modeled by

$$\Gamma' \chi I c_i j \frac{\partial \rho}{\partial x_j} = - \frac{C_\omega}{C_{\chi - 1}} \left( \frac{\epsilon}{\nu} \right)^{1/2} \left( \langle u_i' \phi' \rangle \right) \frac{\partial \rho}{\partial x_i} \tag{13}$$

or

$$S_{\rho S} = \frac{C_\omega}{C_{\chi - 1}} \left( \frac{\epsilon}{\nu} \right)^{1/2} S \phi^2. \tag{14}$$

Note that the source term for $\langle \chi I c_i j \rangle$ is proportional to $S_{\rho S}$ divided by the square of the Batchelor length scale, and thus represents a source for the maximum attainable scalar dissipation (i.e., with all scalar variance located at dissipative scales). The source term differs from that employed by Mantel and Borghi\textsuperscript{27} where the Taylor length scale is used in place of the Batchelor length scale in order to obtain the correct limiting behavior for $\langle r \phi \rangle$. In this work, we obtain the correct limiting behavior by modeling the relaxation of the scalar spectrum to its self-similar form.

C. Scalar moment equations

Dropping terms of order $Re_\chi^{-1}$, the moment equations for an inert, passive scalar field advected by an incompressible turbulent flow become

$$\frac{\partial \langle \phi \rangle}{\partial t} + \frac{\partial}{\partial x_i} \langle u_i' \phi \rangle = 0, \tag{15}$$

$$\frac{\partial \langle \phi \rangle^{1/2}}{\partial t} + \frac{\partial}{\partial x_i} \langle u_i' \phi^{1/2} \rangle = S_{\phi \phi} - \frac{1}{2} \langle \epsilon \phi \rangle, \tag{16}$$

$$\frac{\partial \langle \epsilon \phi \rangle}{\partial t} + \frac{\partial}{\partial x_i} \langle u_i' \epsilon \phi \rangle = 2S_{\phi \phi} - 2C_\chi \langle r \phi \rangle \langle \epsilon \phi \rangle + 2C_\omega \left( \frac{\epsilon}{\nu} \right)^{1/2} \langle \epsilon \phi \rangle. \tag{17}$$

The turbulent flux terms in these expressions are often closed by invoking a gradient-diffusion model:

$$\langle u_i' \Theta \rangle = -D_{ij} \frac{\partial \Theta}{\partial x_j}, \tag{18}$$

where $D_{ij}$ is the turbulent--diffusivity tensor which is usually assumed to be diagonal. This modeling assumption can be avoided by reformulating the problem in terms of the joint velocity-composition PDF.\textsuperscript{1}

While dimensionally correct, the model given above [Eq. (17)] for the scalar dissipation is physically inconsistent for at least two reasons. First, for the case of a constant mean scalar gradient, the steady-state mechanical-to-scalar time-scale ratio predicted by the model is

$$R = \frac{2k \langle r \phi \rangle}{\epsilon} = 0.35 \text{ Re}_\chi, \tag{19}$$

but in DNS of stationary turbulence\textsuperscript{4} $R \rightarrow 2$, independent of $Re_\chi$. Second, Eq. (17) mixes terms that occur at vastly different length scales. By definition, $S_{\phi \phi}$ is significant at low wave numbers near the integral-scale wave number of the velocity field. The other terms on the right-hand side involving $\langle \epsilon \phi \rangle$, on the other hand, are significant only at high wave numbers near the Batchelor scale.

In order to correctly account for the wave-number dependence of the terms appearing in Eq. (17), information concerning the entire scalar spectrum must be included in the model. Including additional variables to account for the various bands in wave-number space will also allow us to capture the transient behavior of the scalar spectrum that greatly influences the scalar dissipation rate in nonpremixed flows, particularly for low Reynolds numbers. For example, in DNS of stationary, isotropic turbulence with no mean scalar gradient,\textsuperscript{4} the transient behavior is strongly dependent on the initial scalar-to-velocity integral-scale ratio. The model should thus show a strong dependence on the initial values of the variables in the absence the source terms. These properties are all present in the spectral relaxation model described below.

III. SPECTRAL RELAXATION MODEL

The spectral relaxation model describes the cascade of scalar dissipation from large to small scales by introducing additional variables that correspond to “potential” scalar dissipation residing at wave numbers smaller than the Batchelor wave number. These variables are referred to as potential scalar dissipation because they are formed by dividing the scalar variance in a given band of wave numbers by the square of the Batchelor length scale. Thus their sum represents the maximum attainable scalar dissipation for a given scalar variance.

We denote the potential scalar dissipation in the $i$th stage of the cascade by $\langle \epsilon_i \rangle$ and in the $(i,j)$th substage by $\langle \epsilon_{i,j} \rangle$. The cascade will be composed of three principal stages representing (1) transport from wave numbers below the integral-scale wave number $k_0$ up to $k_0$ [$k_0$ is defined in Eq. (44) and proportional to $Re_\chi^{3/2} k_K$], (2) from $k_0$ to the Kolmogorov-scale wave number $k_K$ of the velocity field, and (3) transport from $k_K$ to the Batchelor-scale wave number $k_B$. Spectral transport of the scalar is assumed to be local and unidirectional from low to high wave numbers.

The spectral relaxation model for $Sc \gg 1$ is given by

$$\langle D \epsilon_i \rangle = - \frac{2}{\epsilon_i} \langle \epsilon_i \rangle, \tag{20}$$

$$\langle D \epsilon_{i,1} \rangle = \frac{2}{\epsilon_{i,1}} \langle \epsilon_i \rangle - \frac{2}{\epsilon_{i,2}} \langle \epsilon_{i,1} \rangle + 2S_{\phi \phi}, \tag{21}$$

$$\langle D \epsilon_{i,2} \rangle = \frac{2}{\epsilon_{i,2}} \langle \epsilon_{i,1} \rangle - \frac{2}{\epsilon_{i,2}} \langle \epsilon_{i,2} \rangle, \tag{22}$$

$$\langle D \epsilon_{i,n} \rangle = \frac{2}{\epsilon_{i,n}} \langle \epsilon_{i,n-1} \rangle - \frac{2}{\epsilon_{i,n}} \langle \epsilon_{i,n} \rangle, \tag{23}$$

where $D_{ij}$ is the turbulent--diffusivity tensor which is usually assumed to be diagonal. This modeling assumption can be avoided by reformulating the problem in terms of the joint velocity-composition PDF.\textsuperscript{1}
\[
\frac{D \epsilon_{3,1}}{Dt} = \frac{2}{t_{2,n_2}} \langle \epsilon_{2,n_2} \rangle - \frac{2}{t_{3,1}} \langle \epsilon_{3,1} \rangle,
\]

\[
\vdots
\]

\[
\frac{D \epsilon_{3,n_3}}{Dt} = \frac{2}{t_{3,n_3-1}} \langle \epsilon_{3,n_3-1} \rangle - \frac{2}{t_{3,n_3}} \langle \epsilon_{3,n_3} \rangle,
\]

\[
\frac{D \epsilon_{\phi}}{Dt} = \frac{2}{t_{3,n_3}} \langle \epsilon_{3,n_3} \rangle + 2C_\alpha \left( \frac{\epsilon}{\nu} \right)^{1/2} \langle \epsilon_{\phi} \rangle
\]
\[
- \frac{4C_\alpha}{2 + Sc^{-1}} \left( \sum_{i=1}^{n_2} r_{2,i} + \sum_{i=1}^{n_3} r_{3,i} + r_{\phi} \right)
\times \langle \epsilon_{\phi} \rangle,
\]

where \( \langle r_{i,j} \rangle \) are local characteristic scalar dissipation rates defined by

\[
\langle r_{i,j} \rangle = \frac{\langle \epsilon_{i,j} \rangle}{\langle \epsilon_{\phi} \rangle}.
\]

The time constants and wave-number bands for each substage are derived in Sec. IV from the velocity spectrum. The time constants have the following forms:

\[
t_1 = \frac{k}{\epsilon},
\]

\[
t_{2,1} = \frac{k}{\epsilon} \left( \frac{\nu}{\epsilon} \right)^{1/2},
\]

\[
\vdots
\]

\[
t_{3,j} = \frac{1}{3} t_{3,j-1},
\]

\[
\vdots
\]

\[
t_{2,n_2-1} = \frac{1}{3} t_{2,n_2-2},
\]

\[
t_{2,n_2} = \frac{1}{2} t_{2,n_2-1},
\]

\[
\vdots
\]

\[
t_{3,1} = \frac{1}{2n_3} \ln(Sc) \left( \frac{\nu}{\epsilon} \right)^{1/2},
\]

\[
\vdots
\]

\[
t_{3,n_3} = t_{3,1}.
\]

Note that the time constants are chosen such that each stage is faster than its predecessors (i.e., the rate-limiting stage is the one with the lowest wave number):

\[
t_1 > t_{2,1} > \cdots > t_{2,n_2} > t_{3,1} = \cdots = t_{3,n_3} \geq t_{\min},
\]

where \( t_{\min} \) is the minimum scalar dissipation time found when all energy in the scalar spectrum is above the Batchelor wave number as discussed in Sec. V. The values of \( n_2 \geq 2 \) and \( n_3 \geq 1 \) depend on the \( Re_\lambda \) and \( Sc \) and are chosen so that the inequalities hold whenever possible. Nevertheless, choosing \( n_2 \) and \( n_3 \) larger than their minimum values will not adversely affect the model's predictions since the excess stages will quickly relax to their quasi-steady-state values. This effect is illustrated in Figs. 1 and 2 where \( Sc=1 \), \( Re_\lambda=100 \) and \( n_2 \) is varied from 1 to 6. It can be seen that the limiting behavior is attained at \( n_2=4 \).

Note also that the sums of the time constants for stages 2 and 3 satisfy the following equalities:

\[
\sum_{j=1}^{n_2} t_{2,j} = \frac{3}{2} \frac{3}{2} \frac{\nu}{\epsilon}^{1/2}
\]

and

\[
\sum_{j=1}^{n_3} t_{3,j} = t_{3,1}.
\]
so that the characteristic transport time from $k_0$ to $k_B$ is

$$t_{3,j} = \frac{1}{2} \ln(\text{Sc}) \left( \frac{\nu}{\epsilon} \right)^{1/2}$$

(37)

as derived in Sec. IV.

The source term is placed at $k_0$ [Eq. (21)] since it represents scalar dissipation production due to the product of the scalar fluxes and the mean scalar gradients [Eq. (13)], both of which are integral-scale terms. When the source term is constant, the system will approach a dynamic equilibrium wherein $\bar{\epsilon}(r_i,j) = 0$ and the other variables are constant. In the absence of a source term, all variables decay to zero but the characteristic scalar dissipation rates $\langle r_i,j \rangle$ reach constant values characteristic of a fully developed, self-similar scalar spectrum.

The three terms on the right-hand side of Eq. (26) model, respectively, the cascade from smaller wave numbers, turbulent stretching in the viscous–diffusive subrange of the scalar spectrum and molecular dissipation. The turbulent stretching term is taken directly from the model proposed by Fox [Eq. (17)]. The characteristic dissipation rate in the molecular dissipation term has been modified to include the sum of both the actual and potential scalar dissipation rates, i.e., the scalar dissipation rate depends on the shape of the entire scalar spectrum and not just the portion at high wave numbers. This modeling assumption is justified for at least two reasons: (1) it yields the correct limiting behavior as $Re_\Lambda \to \infty$, and (2) $C_x(=3)$ can be taken as independent of the initial scalar-to-velocity integral-scale ratio and of $Re_\Lambda$.

IV. STAGE WAVE-NUMBER BANDS AND TIME CONSTANTS

The time constants and wave-number bands for the sub-stages are derived by assuming a fully developed velocity spectrum with $E_{\nu}(k) \propto k^{-5/3}$ for $k_0 \ll k \ll k_B$. For $k_B = k_B$, we will use Batchelor’s result: $E_{\nu}(k) \propto k^{-1}$. The rate of transport up the scalar spectrum, $\gamma(k)$, is defined by

$$\frac{1}{k} \frac{dk}{dt} = \gamma(k).$$

(39)

For $k_0 \ll k \ll k_B$, the form of $\gamma(k)$ follows from the velocity spectrum. For $0 \ll k \ll k_0$, the form is chosen to make the “velocity” in wave-number space constant, i.e., the one-dimensional velocity spectrum is assumed to be independent of $k$ below $k_0$. The resultant expression for $\gamma(k)$ is

$$\gamma(k) = \begin{cases} a_1 k^{-1}, & 0 \leq k \leq k_0, \\ a_2 k^{2/3}, & k_0 \leq k \leq k_B, \\ a_3, & k_B \leq k \leq k_B, \end{cases}$$

(40)

wherein the constants are fixed by forcing $\gamma(k)$ to be continuous and $a_3$ equal to the maximum strain rate:

$$a_1 = k_0^{5/3} k_B^{-2/3} a_3.$$  

(41)

The above choice for $\gamma(k)$ leads to Kolmogorov–Obukhov–Corrsin–Batchelor scaling [18,19,21,28] of the scalar spectrum in isotropic turbulence. A recent experimental study [29] of the temperature fluctuations in grid turbulence has demonstrated that the scalar spectrum exhibits this type of scaling even at relatively low values of $Re_\varepsilon$ provided that the velocity field is isotropic. In shear flows, Kolmogorov–Obukhov–Corrsin scaling is attained only at high Reynolds numbers; for example, the scaling exponent for the scalar spectrum at $Re_\varepsilon = 200$ is near $-1.3$ (Ref. 30) (as opposed to $-5/3$). Thus, for shear flows, $\gamma(k)$ can be modified in the range $k_0 \ll k \ll k_B$ to include the effect of $Re_\varepsilon$ on the scaling exponent [i.e., $2/3$ can be replaced by $f(Re_\varepsilon)$ where $-f(Re_\varepsilon)$ is the scaling exponent].

The characteristic transport time between any two wave numbers, $k_1 \ll k_2$, is found by integrating Eq. (39) and solving for $t$. For stage 1, $k_1 = 0$ and $k_2 = k_B$; thus, we find

$$t_1 = \frac{\left(\frac{k_B}{k_0}\right)^{2/3}}{\left(\frac{\nu}{\epsilon}\right)^{1/2}} = Re_\Lambda \left(\frac{\nu}{\epsilon}\right)^{1/2},$$

(42)

where

$$Re_\Lambda = \frac{k}{\sqrt{\nu \epsilon}} = 0.3873 Re_\Lambda$$

(43)

(in this expression $k$ is the turbulent kinetic energy). Note that $t_1$ agrees with Eq. (28).

For substage 2-1 where $k_1 = k_0$ and $k_2 = k_{2,1} \ll k_B$, we find

$$t_{2,1} = \frac{3}{2} \left[ \frac{1}{3} Re_\Lambda + \frac{2}{3} \right] k_B^{-3/2}$$

(44)

Comparing this result to Eq. (29), we see that the wave number corresponding to the end of substage 2-1 is

$$k_{2,1} = \left( \frac{1}{3} Re_\varepsilon + \frac{2}{3} \right)^{-3/2} k_B.$$  

(46)

In general, denoting the substage wave numbers by

$$k_{2,i} = \beta_{2,i}^{3/2} k_B,$$

we find the following recursion formula for $\beta_{2,i}$:

$$\beta_{2,i} = \beta_{2,i-1} - \frac{1}{3} \beta_{2,i-2}$$

(49)

for $1 \leq i \leq n_2 - 1$, where

$$\beta_{2,0} = 3 Re_\varepsilon - 2,$$

$$\beta_{2,1} = -Re_\varepsilon.$$  

(50)

Thus

$$\beta_{2,i} = \frac{1}{3^i} (Re_\varepsilon + 3^{i-1}).$$  

(52)

Note that $\beta_{2,i} = 1$ when $Re_\varepsilon = 1$ implying $k_{2,i} = k_B$. The final wave number for stage 2 is

$\beta_{2,1}^{3/2} k_B = k_B$.
\[ k_{2,n_2} = k_K. \]

For substage 3-1 where \( k_1 = k_K \) and \( k_2 = k_{3,1} = k_B \), we find
\[ t_{3,1} = \ln \left( \frac{k_{3,1}}{k_K} \right) \left( \frac{\nu}{\epsilon} \right)^{1/2}. \]

Comparing this result to Eq. (33), we see that the wave number for the end of substage 3-1 is
\[ k_{3,1} = \frac{2}{\epsilon} \frac{1}{2} \ln(\epsilon), \]

or, in general, for substage 3-i
\[ k_{3,i} = \frac{2}{\epsilon} \frac{1}{2} \ln(\epsilon), \]

so that the final wave number for stage 3 is
\[ k_{3,n_3} = k_B. \]

V. APPLICATION TO HOMOGENEOUS FLOWS

A. Constant mean scalar gradient, stationary turbulence

The limiting case of stationary, isotropic turbulence and an inert, passive scalar field with \( S_c^2 \) constant (constant mean scalar gradient) corresponds to the fully developed spectrum considered by Corrsin. When \( \text{Re}_\lambda \) is very large, the scalar dissipation rate is controlled by the rate of transport from large to small scales so that \( \langle \tau \rangle = \epsilon/k \) (Sc=1) as \( t \to \infty \). For Sc=1, the limiting behavior for the scalar mixing time has been determined by Corrsin:
\[ \tau_\phi = \frac{2}{2 + \text{Sc}^{-1}} \left( \frac{3}{2} \frac{k}{\epsilon} + \frac{1}{2} \ln(\text{Sc}) \right) \left( \frac{\nu}{\epsilon} \right)^{1/2}. \]

Setting the material derivatives equal to zero in Eqs. (16) and (20)-(26) yields:
\[ \langle \phi \rangle = S_c \phi, \]
\[ \langle \epsilon \rangle = 0, \]
\[ \langle \epsilon_{i,j} \rangle = t_{i,j} \epsilon = t_{i,j} \frac{C_o}{C_x} \left( \frac{\epsilon}{\nu} \right)^{1/2}, \]

\[ \left( \frac{\epsilon}{\nu} \right)^{1/2} = \frac{2}{2 + \text{Sc}^{-1}} \left( \frac{3}{2} \frac{\text{Re}_\lambda - 1}{\frac{3}{2} + \frac{1}{2} \ln(\text{Sc})} + \frac{C_x^{-1}}{C_o} \right) \times \langle \tau_\phi \rangle \langle \epsilon_\phi \rangle = 0. \]

The final equation can be solved to find the scalar mixing time:
\[ \tau_\phi = \frac{2}{2 + \text{Sc}^{-1}} \left( \frac{3}{2} \frac{k}{\epsilon} + \frac{1}{2} \ln(\text{Sc}) \right) \left( \frac{\nu}{\epsilon} \right)^{1/2} + 2 \frac{C_x - 2 C_w - 2}{C_o(2 + \text{Sc}^{-1})} \left( \frac{\nu}{\epsilon} \right)^{1/2}. \]

The transient behavior of \( \langle \phi^2 \rangle / S_c^2 \) and \( \langle \epsilon_\phi \rangle / S_c^2 \) for increasing \( \text{Re}_\lambda \) and \( \text{Sc}=1 \) are shown in Figs. 3 and 4. In general, the length of the transient regime decreases as \( \text{Re}_\lambda \) increases. For infinite \( \text{Re}_\lambda \), \( \langle \phi \rangle = 1 \). For finite \( \text{Re}_\lambda \), the spectrum relaxation model predicts a time lag for \( \langle \phi \rangle \) during which the scalar spectrum cascades down to dissipative scales.

The transient behavior of \( \langle \phi^2 \rangle / S_c^2 \) and \( \langle \epsilon_\phi \rangle / S_c^2 \) for increasing \( \text{Sc} \) and \( \text{Re}_\lambda = 100 \) is shown in Figs. 5 and 6. In general, the length of the transient regime increases slowly as \( \text{Sc} \) increases. For the same \( \text{Re}_\lambda \), the transient behavior of \( \langle \phi \rangle \) becomes noticeably “smoother” as \( \text{Sc} \) increases (cf. Fig. 4 for \( \text{Sc}=1 \)).
B. Zero mean scalar gradient, stationary turbulence

The limiting case of stationary, isotropic turbulence with no mean scalar gradient \((S_\varphi' = 0)\) is also of interest. For this case, the material derivatives in Eqs. (20)–(26) become total derivatives and the behavior of \(\langle r_\varphi \rangle\) depends on the initial values of \(\langle r_\varphi \rangle\). The equations can be made dimensionless by defining \(\tau^* = t/t_1\) and mechanical-to-scalar time-scalar ratios:

\[
R_1 = 2t_1 \langle r_1 \rangle,
\]

\[
R_{i,j} = 2t_1 \langle r_{i,j} \rangle,
\]

\[
R = 2t_1 \langle r_\varphi \rangle.
\]

The spectral relaxation model in dimensionless form becomes

\[
\frac{dR}{d\tau^*} = -2R_1 + RR_1,
\]

\[
\frac{dR_{2,1}}{d\tau^*} = 2R_1 - 2 \frac{t_1}{t_{2,1}} R_{2,1} + RR_{2,1},
\]

\[
\frac{dR_{i,j}}{d\tau^*} = 2 \frac{t_1}{t_{i,j}} R_{i,j-1} - 2 \frac{t_1}{t_{i,j}} R_{i,j} + RR_{i,j},
\]

\[
\frac{dR}{d\tau^*} = 2 \frac{t_1}{t_{2,n_3}} R_{2,n_3} + 2C_\omega \Re_1 R - \frac{2C_\omega (2 + S\zeta^{-1})}{2 + S\zeta^{-1}}
\]

\[
\times \left[ R_1 + \sum_{i=1}^{n_2} R_{2,i} + \sum_{i=1}^{n_3} R_{3,i} + R \right] R + R^2.
\]

Since Eq. (67) has the largest time constant, if \(R_1(0)>0\) the limiting value of \(R\) will be 2 as seen in DNS. In the other extreme, if \(R_1=R_{i,j}(0)=0\) and \(R(0)>0\), corresponding to an initial scalar spectrum with all its energy below the Batchelor length scale, then \(R\) has a limiting value proportional to \(R_\varphi\). Note, however, that regardless of the initial conditions the sum of potential and actual mechanical-to-scalar time-scale ratios quickly becomes constant:

\[
R_1 + \sum_{i=1}^{n_2} R_{2,i} + \sum_{i=1}^{n_3} R_{3,i} + R \rightarrow \frac{2C_\omega (2 + S\zeta^{-1})}{2 + S\zeta^{-1}} \Re_1 = \frac{2t_1}{t_{\min}}.
\]

Substage number dependence on \(\Re_\varphi\) and \(S\zeta\): The above expression defines \(t_{\min}\) and can be used to fix \(n_2\) and \(n_3\). For example, comparing \(t_{\min}\) to Eq. (33), we see that

\[
n_2 = 1, \quad \text{for } 1 < S\zeta \leq 1650,
\]

\[
n_2 = 2, \quad \text{for } 1650 < S\zeta \leq 2716390.
\]

\[
R_{i,j} = 2t_1 \langle r_{i,j} \rangle,
\]

\[
R = 2t_1 \langle r_\varphi \rangle.
\]

Hence, in practice, no more than two substages will likely be needed between \(k_0\) and \(k_B\). Similarly, for \(S\zeta=1\),

\[
n_2 = 2, \quad \text{for } 2.582 < \Re_\varphi \leq 31,
\]

\[
n_2 = 3, \quad \text{for } 31 < \Re_\varphi \leq 88,
\]

\[
n_2 = 4, \quad \text{for } 88 < \Re_\varphi \leq 261.
\]

For \(S\zeta>1\), the upper limits on \(\Re_\varphi\) are 59, 176, and 519, respectively. Thus many laboratory-scale flows will require no more than four substages between \(k_0\) and \(k_B\).
Comparison with DNS: A qualitative comparison of the spectral relaxation model’s predictions with the DNS results of Eswaran and Pope at Re₀=49.2 (Re₁=19) and Sc=0.7 can be made. Since Sc<1, we will drop stage 3 and use three substages for stage 2. Note, however, that for Sc<l the scalar spectrum displays a $k^{-7/3}$ range that is not included in the spectral relaxation model. The model must thus be applied with caution when Sc<l and, at best, may be applicable only when 0<Sc<1.

For Sc<1, the total transport time in stage 2 must be modified to account for the fact that $k_B<k_K$. Taking $k_1=k_0$ and $k_2=k_B=Sc^{1/2}k_K$, we find

$$t_2 = \frac{3}{2} \left[ \frac{Re_1 - \left(\frac{k_K}{k_B}\right)^{2/3}}{k_0} \right]^{1/2} = \frac{3}{2} \left[ \frac{Re_1 - Sc^{1/3}}{k_0} \right]^{1/2}. \quad (77)$$

The three substage time constants will thus be taken as

$$t_{2,1} = [1 - Sc^{-1/3} Re_1^{-1}] t_1, \quad (78)$$

$$t_{2,2} = \frac{1}{3} [1 - Sc^{-1/3} Re_1^{-1}] t_1, \quad (79)$$

$$t_{2,3} = \frac{1}{6} [1 - Sc^{-1/3} Re_1^{-1}] t_1. \quad (80)$$

For Re₀=49 and Sc=0.7 the spectral relaxation model reduces to

$$\frac{dR_1}{dt^*} = -2R_1 + RR_1, \quad (81)$$

$$\frac{dR_{2,1}}{dt^*} = 2R_1 - 2.126R_{2,1} + RR_{2,1}, \quad (82)$$

$$\frac{dR_{2,2}}{dt^*} = 2.126R_{2,1} - 6.378R_{2,2} + RR_{2,2}, \quad (83)$$

$$\frac{dR_{2,3}}{dt^*} = 6.378R_{2,2} - 12.756R_{2,3} + RR_{2,3}, \quad (84)$$

$$\frac{dR}{dt^*} = 12.756R_{2,3} + 20.52R - 1.75 \left[ R_{1} + \sum_{i=1}^{3} R_{2,i} + R \right] R + R^2. \quad (85)$$

The substage wave numbers are $k_{2,1}=3.74k_0$, $k_{2,2}=13.35k_0$, and $k_{2,3}=69.29k_0$. The maximum mechanical-to-scalar time-scale ratio is 23.76 and occurs when $R_1(0)=R_{2,1}(0)=R_{2,2}(0)=R_{2,3}(0)=0$.

The behavior of $R$ for other combinations of initial conditions will depend on the initial scalar spectrum $E_\phi(k,0)$. We will fix the initial conditions so that Eq. (71) is satisfied for all $t^*$. In principle, the initial conditions could then be found by integrating $E_\phi(k,0)$ over the wave-number band corresponding to each substage, e.g.,

$$R_1(0) = 27.36 \frac{\int_0^{k_0} E_\phi(k,0) dk}{\int_0^{k_0} E_\phi(k,0) dk}. \quad (86)$$

However, the exact form of $E_\phi(k,0)$ is not available for the DNS data. We will thus set the initial conditions to approximate the initial scalar-to-velocity integral-scale ratios used by Eswaran and Pope. The initial conditions are given in Table I for each scalar-to-velocity integral-scale wave-number ratio $(k_i/k_0)$ in terms of their ratios with respect to the limiting value, e.g., $R(0)/R(\infty)$. At steady state, all ratios reach a final value of unity so that initial ratios greater than 1 indicate that a larger fraction of the initial scalar spectrum is in the corresponding wave-number band than is present in the final self-similar scalar spectrum and vice versa. The model predictions for the scalar variance and the mechanical-to-scalar time-scale ratio are shown in Figs. 7 and 8, respectively, and show excellent agreement with the DNS data.

C. Zero mean scalar gradient, decaying turbulence

Another limiting case of interest is the decay of an inert, passive scalar field in decaying, homogeneous grid turbu-

![FIG. 7. Scalar variance $(\langle d\phi^2/\langle \phi^2 \rangle_0)$ versus dimensionless time $(t^* = \epsilon t/k)$ found from the spectral relaxation model with Re₀=49.2 ($Re_1=3$) and Sc=0.7 for stationary, isotropic turbulence. Curves correspond to different values of the initial scalar-to-velocity integral-scale wave-number ratio as in Eswaran and Pope. Lines $(k_i/k_0)$: solid (1), dashed (2), dot-dash (4), dotted (6), large dotted (8).]
We will use the velocity field of Sirivat and Warhaft\textsuperscript{3} with mean velocity \( \langle U \rangle = 3.4 \) m/s and \( \nu = 1.65 \times 10^{-5} \) m\(^2\)/s. For this velocity field, the turbulent kinetic energy and dissipation rate downstream from the grid (mesh size \( M = 0.025 \) m) are well approximated by

\[ k = 1.0766 \left( \frac{x}{M} \right)^{-1.3} \text{ m}^2/\text{s}^2 \]

and

\[ r_v = \frac{\varepsilon}{k} = 180.0 \left( \frac{x}{M} \right)^{-1} \text{ s}^{-1} \]

for \( x/M > 40 \).

In order to compare the spectral relaxation model's predictions with the experimental data, we shall relate the dimensionless axial position (\( x/M \)) to \( t \) using the mean velocity:

\[ \frac{x}{M} = 40 + \frac{\langle U \rangle}{M} (t - t_0), \]

where \( t_0 = 1.3/4.5 \) is the initial time. The turbulence fields can then be found by solving

\[ \frac{dk}{dt} = -r_v k \]

and

\[ \frac{dr_v}{dt} = \frac{1}{1.3} r_v^2 \]

with \( k(t_0) = 0.0089 \) and \( r_v(t_0) = 4.5 \). The resulting kinetic energy and dissipation fields, plotted in terms of \( x/M \), are shown in Figs. 9 and 10, respectively.

Since the scalar variance and dissipation fields are homogeneous, they depend only on \( t \) (we neglect streamwise transport due to \( u' \) which should be small compared to the mean convection term). The spectral relaxation model for this case (\( Sc = 1 \)) can be rewritten in terms of the scalar dissipation rates \( \langle r_{i,j} \rangle \):

\[ \frac{1}{\langle r_{i,j} \rangle} \frac{d\langle r_{i,j} \rangle}{dt} = \frac{1}{\langle \varepsilon_{i,j} \rangle} \frac{d\langle \varepsilon_{i,j} \rangle}{dt} + 2 \langle r_d \rangle, \]

or, in terms of the mechanical-to-scalar time-scale ratios \( (R_{i,j} = 2 \langle r_{i,j} \rangle / r_v ) \), as

\[ \frac{1}{R_{i,j}} \frac{dR_{i,j}}{dt} = \frac{1}{\langle \varepsilon_{i,j} \rangle} \frac{d\langle \varepsilon_{i,j} \rangle}{dt} + r_v \frac{r_v}{1.3}. \]

Since \( Re_x \leq 27 \), we will take \( n_2 = 2 \). The model equations become

\[ \frac{1}{R_{i,j}} \frac{dR_{i,j}}{dt} = \frac{1}{\langle \varepsilon_{i,j} \rangle} \frac{d\langle \varepsilon_{i,j} \rangle}{dt} + r_v \frac{r_v}{1.3}. \]
TABLE II. Initial (x/M=40) and final (x/M=4000) values of the mechanical-to-scalar time-scale ratios and their sum (Σ) found from the spectral relaxation model for comparison with the decaying, homogeneous grid turbulence data of Warhaft and Lumley. The values given in the first four rows of the table are the ratios with respect to z given at the bottom of each column. Note that the final value of Σ depends on the initial scalar-to-velocity integral-scale ratio.

<table>
<thead>
<tr>
<th>Time-scale ratio</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
<td>Final</td>
</tr>
<tr>
<td>R_1</td>
<td>0.064</td>
<td>0.067</td>
<td>1.00</td>
</tr>
<tr>
<td>R_{a1}</td>
<td>0.466</td>
<td>0.466</td>
<td>0.00</td>
</tr>
<tr>
<td>R_{s2}</td>
<td>0.419</td>
<td>0.416</td>
<td>0.00</td>
</tr>
<tr>
<td>R</td>
<td>0.051</td>
<td>0.051</td>
<td>0.00</td>
</tr>
<tr>
<td>Σ</td>
<td>26.96</td>
<td>23.63</td>
<td>20.96</td>
</tr>
</tbody>
</table>

\[
\frac{d\langle \phi^2 \rangle}{dt} = -r_0 R \langle \phi^2 \rangle, \tag{94}
\]

\[
\frac{dR_1}{dt} = r_v [R - 1.231] R_1, \tag{95}
\]

\[
\frac{dR_{a1}}{dt} = 2r_v (R + R_0 - 2 \frac{Re_1}{Re_1 - 1}) R_{a1}, \tag{96}
\]

\[
\frac{dR_{s2}}{dt} = 2r_v \frac{Re_1}{Re_1 - 1} R_{s2} + r_v \left( R + 0.769 - 4 \frac{Re_1}{Re_1 - 1} \right) R_{s2}, \tag{97}
\]

and

\[
\frac{dR}{dt} = 4r_v \frac{Re_1}{Re_1 - 1} R_{s2} + r_v \left( R + 0.769 + 1.08 \frac{Re_1}{Re_1 - 1} \right) R.
\]

Note that \( Re_1(t) = 11(t_0/t)^{0.3} \) and thus it decays, for example, to approximately 9 by \( x/M = 250 \). Note also that, unlike the stationary turbulence case, the sum \( \Sigma = R_1 + R_{a1} + R_{s2} + R \) does not approach a limiting value independent of \( R \). Instead, for decaying turbulence,

\[
\frac{1}{r_v} \frac{d\Sigma}{dt} = 1.08 Re_1 \left( R - (R - 0.769) \right) \tag{99}
\]

and thus \( \Sigma \) will depend on the initial scalar spectrum in a nontrivial manner.

For a decaying inert scalar field in decaying homogeneous turbulence, it is easily shown that \( R \) will have the form \( R = m/n \) where \( n \) and \( m \) are the power-law decay exponents of the velocity and scalar fields, respectively. Warhaft and Lumley have found experimentally that the value of \( m \) depends uniquely on the initial scalar-to-velocity integral-scale ratio and values between 0.6 and 2.4 have been observed with larger values corresponding to smaller initial ratios, and vice versa. For the present case, \( n = 1.3 \) and \( m \) can be found by solving the model equations. Three different sets of initial conditions have been examined corresponding to different initial scalar-to-velocity integral-scale ratios (see Table II).

The first set of initial conditions corresponds to an “equilibrium” scalar spectrum where the integral-scale ratio is near unity. The second corresponds to a scalar spectrum with all its energy at low wave numbers, and the third to a scalar spectrum skewed towards high wave numbers. The resultant scalar variances are shown in Fig. 11 where power-law decay is evident, but with a different limiting exponent for each case. This behavior is clearly seen in Fig. 12 where \( R \) is plotted up to \( x/M = 1000 \) and each initial condition results in a different limiting value for \( R \) (integration has been continued past \( x/M = 10000 \) and limiting values remain distinct). The model predicts limiting values for \( R \) in the range 1.0–1.5 which is considerably smaller than the range (0.6–2.4) seen by Warhaft and Lumley. Nevertheless, it should be noted that the limiting value is achieved relatively slowly and thus the experimental data (taken up to \( x/M = 140 \)) may still be evolving towards their final values. In any case, we can conclude that the spectral relaxation model exhibits the same sensitive dependence on the initial integral-scale ratio seen in experiments.

**D. Constant mean scalar gradient, decaying turbulence**

Experimental data is also available for an inert, passive scalar field with a constant mean gradient in decaying grid turbulence.
turbulence. We will again consider the case where the mean velocity is 3.4 m/s so that the turbulence fields are the same as those given in Figs. 9 and 10. With a constant scalar gradient, the source term $S_{\phi^2}$ becomes important and, since it involves $\langle u'_i \phi \rangle$, a model is required for the scalar flux. Here, we shall employ the gradient-diffusion model so that

$$S_{\phi^2} = \alpha_T \frac{k}{r_y} \left( \frac{d\langle \phi \rangle}{dy} \right)^2$$

(100)

with $\alpha_T=0.37$ found by fitting the predicted scalar flux to the experimental data.3

We will again take $n_2=2$ and solve the spectral relaxation model with nonzero source terms and $\langle \epsilon \rangle=0$. Since the square of the mean scalar gradient appears in both source terms, we will divide it out and include it in the definitions of the scalar variance and the scalar dissipations as done by Sirivat and Warhaft.3 Three sets of initial conditions have been employed in the simulations (Table III) and differ principally by their initial scalar variance. In case 1, the scalar variance production term is approximately equal to the dissipation term. In case 2, the dissipation term is dominant. In case 3, the initial scalar variance is null so that the production term dominates.

### Table III. Initial values ($\kappa/M=40$) of the scalar variance and the scalar dissipations employed in the spectral relaxation model for comparison with the constant mean scalar gradient, decaying, homogeneous grid turbulence data of Sirivat and Warhaft.3 The values have been divided by the square of the mean scalar gradient ($\beta^2$).

<table>
<thead>
<tr>
<th>Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\phi^2)/\beta^2$ m$^2$</td>
<td>0.0000 15</td>
<td>0.0000 06</td>
<td>0.0</td>
</tr>
<tr>
<td>$(\phi_d^2)/\beta^2$ m$^2$/s</td>
<td>0.0030</td>
<td>0.0080</td>
<td>0.0</td>
</tr>
<tr>
<td>$(\phi_s^2)/\beta^2$ m$^2$/s</td>
<td>0.0015</td>
<td>0.0060</td>
<td>0.0</td>
</tr>
<tr>
<td>$(\phi_d^2)/\beta^2$ m$^2$/s</td>
<td>0.0005</td>
<td>0.0030</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Results are shown in Figs. 13–16. The open circles are taken from the experimental data of Sirivat and Warhaft.3 In Fig. 13, the scalar variance is shown and agrees well with the experimental data; however, the good agreement is a direct result of fitting $\alpha_T$ to the data. Indeed, the slope of the scalar variance is directly proportional to $\alpha_T$. The scalar dissipation is shown in Fig. 14 and it also shows good agreement with the experimental data. This plot is a more stringent test of the spectral relaxation model. For example, the curve for case 2 exhibits the same slow approach to the limiting curve (case 1) that is seen in the experimental data for the mandoline (10,1).3

<FIG. 12 Mechanical-to-scalar time-scale ratio found from the spectral relaxation model for the decaying, homogeneous grid turbulence experiments of Warhaft and Lumley.2 Mean velocity=3.4 m/s. M=0.025 m. Lines (case from Table II): solid (1), dashed (2), dash–dot (3).>

<FIG. 13 Scalar variance $[(\phi^2)/\beta^2]\times 10^4$, m$^2$] found from the spectral relaxation model for the decaying, homogeneous grid turbulence experiments of Sirivat and Warhaft.3 Mean velocity=3.4 m/s. M=0.025 m. Lines (case from Table III): solid (1), dashed (2), dash–dot (3). Open circle: experimental data.>

<FIG. 14 Scalar dissipation $[(\epsilon)/(\beta^2)]\times 10^4$, m$^2$/s] found from the spectral relaxation model for the decaying, homogeneous grid turbulence experiments of Sirivat and Warhaft.3 Mean velocity=3.4 m/s. M=0.025 m. Lines (case from Table III): solid (1), dashed (2), dash–dot (3). Open circle: experimental data.>
FIG. 15. Scalar variance dissipation-to-production time-scale ratio ($\tau_d/\tau_p$) for the decaying, homogeneous grid found from the spectral relaxation model turbulence experiments of Sirivat and Warhaft. Mean velocity = 3.4 m/s, $M = 0.025$ m. Lines (case from Table III): solid (1), dashed (2), dash-dot (3). Open circle: experimental data.

The ratio of the predicted time scales for dissipation, production and turbulence relaxation can also be compared with the experimental data. These time scales are defined, respectively, by

$$\tau_1 = \frac{\langle \phi^2 \rangle}{\langle \varepsilon \phi \rangle}, \quad \tau_2 = \frac{r_p \langle \phi^2 \rangle}{\alpha_T k \beta^2},$$

and

$$\tau_3 = \frac{2k}{\varepsilon}. \quad (103)$$

The dissipation-to-production time-scale ratio is shown in Fig. 15. The limiting value found by Sirivat and Warhaft is approximately 1.5 and close to the model value of 1.4. The value of $\tau_2$ is sensitive to the scalar flux and the latter shows a definite dependence on the scalar-to-velocity integral-scale ratio that is not captured by the gradient-diffusion model; thus, an improved model for the scalar flux may improve the model predictions. Nevertheless, the dynamics for case 2 again agree well with the experimental data. The mechanical-to-scalar time-scale ratio is shown in Fig. 16. The experimental data show considerable scatter but the limiting value is in the range 1.2–1.6. The midrange value is thus in excellent agreement with the predicted value of 1.4. It is worth recalling that, except for $\alpha_T$ appearing in the scalar flux model, the excellent agreement between the experimental data and the spectral relaxation model is achieved without the introduction of any empirical or fitting parameters.

VI. CONCLUSIONS

A novel model for the scalar dissipation rate in a turbulent flow has been derived and verified against experimental and DNS data. The spectral relaxation model differs from earlier moment-closure models by the inclusion of multiple turbulence time scales and a simple description of the cascade of scalar energy from low to high wave numbers. All model constants are determined a priori and thus the excellent agreement with experimental and DNS data is achieved without fitting parameters. In its treatment of inertial-range spectral transport, the spectral relaxation model exhibits some similarities with the linear eddy model; however, the spectral relaxation model should have a much wider range of applicability due to its computational tractability for full three-dimensional inhomogeneous flows. Indeed, due to its mathematical form, the model can be immediately implemented in existing CFD codes to provide scalar time-scale information in place of the widely used eddy-breakup model. This step alone should greatly improve CFD predictions for low Reynolds number and high Schmidt number flows. The value of the new model is by no means limited to improving existing CFD codes. A Lagrangian version of the model is under development and should represent an important step forward in the closure of the scalar mixing term appearing in full PDF computations.

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4V. Eswaran and S. B. Pope, "Direct numerical simulations of the turbulent


K. Tsai and R. O. Fox, "Modeling multiple reactive scalar mixing with the generalized IEM model," submitted for publication.


